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S201 / 414 GARDENERS RD. ROSEBERY 2018
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YEAR 12 – EXT.2 MATHS

REVIEW TOPIC (SP1)
POLYNOMIALS II

(1) (a) The complex number z and its conjugate $\bar{z}$ satisfy the equation $z\bar{z} + 2iz = 12 + 6i$. Find the possible values of z.

$\boxed{3 - i, 3 + 3i}$

(b) $1 + i$ is a root of the equation $x^2 + (a + 2i)x + (5 + ib) = 0$, where a and b are real. Find the values of a and b.

$\boxed{a = -3, b = -1}$

(2) (a) $1-2i$ is one root of the equation $x^2+(1+i)x+k=0$.
Find the other root and the value of k.

$\boxed{k=5i, x=-2+i}$

(b) Find the zeros of $P(x)=x^4-4x^2+3=0$

(i) over **Q**;

$\boxed{\pm 1}$

(ii) over **R**;

$\boxed{\pm 1, \pm\sqrt{3}}$

(iii) over **C**,

$\boxed{\pm 1, \pm\sqrt{3}}$

(3) (a) Find $P(x)$, given that $P(x)$ is monic, of degree 3, with 5 as a single zero and -2 as a zero of multiplicity 2.

$$\boxed{P(x) = x^3 - x^2 - 16x - 20}$$

(b) $P(x)$ is an even monic polynomial of degree 4 with integer coefficients. If $\sqrt{2}$ is a zero, and the constant term is 6, factorise $P(x)$ fully over **R.**

$$\boxed{P(x) = (x - \sqrt{2})(x + \sqrt{2})(x - \sqrt{3})(x + \sqrt{3})}$$

(4) If $P(x) = x^3 - 3x^2 - 9x + c$ has a double zero, find c and factorise $P(x)$ over the real numbers.

$c = 27 \quad P(x) = (x-3)^2(x+3);$
$c = -5 \quad P(x) = (x+1)^2(x-5)$

(5) If $ax^3 + cx + d = 0$ has a double root, show that $4c^3 + 27ad^2 = 0$.

(6) (a) When $P(x) = x^4 + ax^2 + 2x$ is divided by $x^2 + 1$, the remainder is $2x + 3$.
Find the value of a.

$\boxed{a = -2}$

(b) When $P(x) = x^4 + ax^2 + bx + 2$ is divided by $x^2 + 1$, the remainder is $-x + 1$.
Find the values of a and b.

$\boxed{a = 2, b = -1}$

(7) (a) Two of the roots of $3x^3 + ax^2 + 23x - 6 = 0$ are reciprocals. Find the value of a and the three roots.

$a = -16$; roots are $3, \frac{1}{3}, 2$

(8) The equation $px^3 + qx^2 + rx + s = 0$ has roots $(a - c)$, a, $(a + c)$, which are in arithmetic progression. Show that the $a = \frac{-q}{3p}$ and hence show that $2q^3 - 9pqr + 27p^2 s = 0$.

(9) The equation $px^3 + qx^2 + rx + s = 0$ has the roots ac, a and $\frac{a}{c}$, which are in geometric progression. Show that $a = \sqrt[3]{(-s/p)}$ and hence show that $pr^3 - q^3 s = 0$.

(10) The equation $x^3 + x^2 - 2x - 3 = 0$ has roots α, β and γ. Find the equations with roots

(a) $\frac{\alpha}{2}, \frac{\beta}{2}$ and $\frac{\gamma}{2}$;

$\boxed{8x^3 + 4x^2 - 4x - 3 = 0}$

(b) $\alpha + 2, \beta + 2$ and $\gamma + 2$.

$\boxed{x^3 - 5x^2 + 6x - 3 = 0}$

Qu. (11) (a) The polynomial $\alpha x^{n+1}+\beta x^{n}+1$ is divisible by $(x-1)^2$.

Show that $\alpha = n$, and $\beta = -(1+n)$.

(b) Prove that $1+x+\dfrac{x^2}{2!}+....+\dfrac{x^n}{n!}$ has no multiple roots for any $n \geq 1$.

Qu. (12) (HSC 1994)

(4) (a) Find α and β, given that $z^3+3z+2i=(z-\alpha)^2(z-\beta)$.

$\boxed{\alpha=-i,\beta=2i}$

Qu. (13) (HSC 1994)

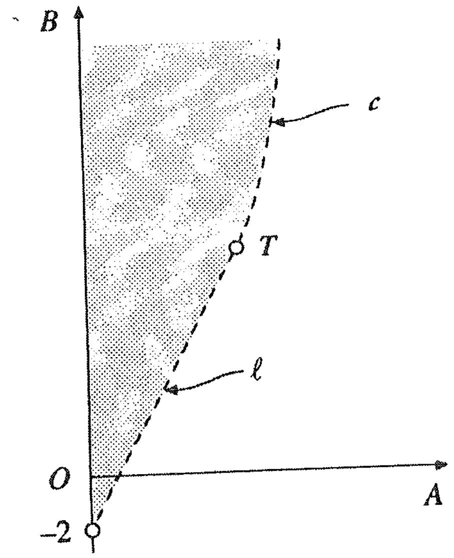
(8) (b) Let $x=\alpha$ be a root of the quartic polynomial $P(x)=x^4+Ax^3+Bx^2+Ax+1$, where A and B are real. Note that α may be complex.

(i) Show that $\alpha \neq 0$.

(ii) Show that $x = \alpha$ is also a root of $Q(x) = x^2 + \frac{1}{x^2} + A\left(x + \frac{1}{x}\right) + B$.

(iii) With $u = x + \frac{1}{x}$, show that $Q(x)$ becomes $R(u) = u^2 + Au + (B - 2)$.

(iv) For certain values of A and B, $P(x)$ has no real roots and $A \geq 0$.



The region **D** is shaded in the figure. Specify the bounding straight-line segment ℓ and curved segment C. Determine coordinates of T.

$T(4,6)$

Qu. (14) (HSC 1995)

(5) (b) Let $f(t) = t^3 + ct + d$, where c and d are constants. Suppose that the equation $f(t) = 0$ has three distinct real roots, t_1, t_2, and t_3.

(i) Find $t_1 + t_2 + t_3$.

0

(ii) Show that $t_1^2 + t_2^2 + t_3^2 = -2c$.

(iii) Since the roots are real and distinct, the graph of $y = f(t)$ has two turning points, at $t = u$ and $t = v$, and $f(u).f(v) < 0$.

Show that $27d^2 + 4c^3 < 0$.

Qu. (15) HSC 1996

(5) (b) Consider the polynomial equation

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

where *a, b, c,* and *d* are integers. Suppose the equation has a root of the form *ki*, where *k* is real, and $k \neq 0$.

(i) State why the conjugate $-ki$ is also a root.

(ii) Show that $c = k^2 a$.

(iii) Show that $c^2 + a^2 d = abc$.

(iv) If 2 is also a root of the equation and $b=0$, show that c is even.

Qu. (16) (HSC 1997)

(3) (b) Let $f(x) = 3x^5 - 10x^3 + 16x$.

(i) Show that $f'(x) \geq 1$ for all x.

(ii) For what values of x is $f''(x)$ positive?

$x>1$ and $-1 < x < 0$

(iii) Sketch the graph of $y = f(x)$, indicating any turning points and points of inflection.

Qu. (17) (HSC 1997)

(5) (c) Suppose that b and d are real numbers and $d \neq 0$. Consider the polynomial

$$P(z) = z^4 + bz^2 + d.$$

The polynomial has a double root α.

(i) Prove that $P'(z)$ is an odd function. (i.e. prove $P'(-z) = -P'(z)$)

(ii) Prove that $-\alpha$ is also a double root of $P(z)$.

(iii) Prove that $d = \dfrac{b^2}{4}$.

(iv) For what values of b does $P(z)$ have a double root equal to $\sqrt{3}i$?

$\boxed{b = 6}$

(v) For what values of b does $P(z)$ have real roots?

$\boxed{b < 0}$

Qu. (18) (HSC 2002)

(b) Let α, β, and γ be the roots of the equation $x^3 - 5x^2 + 5 = 0$.

(i) Find a polynomial equation with integer coefficients whose roots are $\alpha - 1$, $\beta - 1$, and $\gamma - 1$. **2**

$\boxed{x^3 - 2x^2 - 7x + 1 = 0}$

(ii) Find a polynomial equation with integer coefficients whose roots are α^2, β^2, and γ^2. **2**

$$\boxed{x^3 - 25x^2 + 50x - 25 = 0}$$

(iii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

Qu. (19) (HSC 1998)

(4) (a) (i) Suppose that k is a double root of the polynomial equation $f(x)=0$. **7**

Show that $f'(k)=0$.

(ii) What feature does the graph of a polynomial have at a root of multiplicity 2?

(iii) The polynomial $P(x)=ax^7+bx^6+1$ is divisible by $(x-1)^2$.

Find the coefficients a and b.

$\boxed{a=6, b=-7}$

(iv) Let $E(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$. Prove that $E(x)=0$ has no double roots.

(6) (a) Consider the following statements about a polynomial $Q(x)$. Indicate whether each of these statements is true or false. Give reasons for your answers.

(i) If $Q(x)$ is even, then $Q'(x)$ is odd.

TRUE

(ii) If $Q'(x)$ is even, then $Q(x)$ is odd.

FALSE

Qu. (20) (HSC 1999)

(2) (d) Consider the equation $2z^3 - 3z^2 + 18z + 10 = 0$ **2**

(i) Given that $1 - 3i$ is a root of the equation, explain why $1 + 3i$ is another root.

(ii) Find all roots of the equation.

$$\boxed{1-3i, 1+3i, -\frac{1}{2}}$$

Qu. (21) (HSC 1999)

(5) (a) The roots of $x^3 + 5x^2 + 11 = 0$, are α, β and γ. **3**

(i) Find the polynomial equation whose roots are α^2, β^2 and γ^2.

$$\boxed{y^3 - 25y^2 - 110y - 121 = 0}$$

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$.

$\boxed{25}$

Qu. (22) (HSC 2000)

(2) (b) **Consider the equation $z^2 + az + (1+i) = 0$.** **2**

Find the complex number *a*, given that i is a root of the equation.

$\boxed{a = -1}$

(5) (a) **Consider the polynomial $p(x) = ax^4 + bx^3 + cx^2 + dx + e$** **4**

where a, b, c, d and *e* are integers. Suppose α is an integer such that $p(\alpha) = 0$.

(i) **Prove that α divides *e*.**

(ii) Prove that the polynomial $q(x) = 4x^4 - x^3 + 3x^2 + 2x - 3$ does not have an integer root.

Qu. (23) HSC 2001

(3) (b) The numbers α, β and γ satisfy the equations

$$\alpha + \beta + \gamma = 3$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 2.$$

(i) Find the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. **3**

Explain why α, β and γ are the roots of the cubic equation

$$x^3 - 3x^2 + 4x - 2 = 0.$$

$\boxed{\sum \alpha\beta = 4,\ \alpha\beta\gamma = 2}$

(ii) Find the values of α, β and γ. **2**

$\boxed{1,\ 1+i,\ 1-i}$

Qu. (24) HSC 2001

7 * (b) Consider the equation $x^3 - 3x - 1 = 0$, which we denote by (*).

(i) Let $x = \frac{p}{q}$ where p and q are integers having no common divisors other than $+1$ and -1. Suppose that x is a root of the equation $ax^3 - 3x + b = 0$, where a and b are integers. **4**

Explain why p divides b and why q divides a. Deduce that (*) does not have a rational root.

(ii) Suppose that r, s and d are rational numbers and that $\sqrt{d}$ is irrational. **4**
Assume that $r+s\sqrt{d}$ is a root of (*).

Show that $3r^2s+s^3d-3s=0$ and show that $r-s\sqrt{d}$ must also be a root of (*).

Deduce from this result and part (i), that no root of (*) can be expressed in the form $r+s\sqrt{d}$ with r, s and d rational.

(iii) Show that one root of (*) is $2\cos\frac{\pi}{9}$. **1**

(You may assume the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.)

Qu. (25) HSC 2002

(5 (a) The equation $4x^3 - 27x + k = 0$ has a double root. Find the possible values of k. **2**

$\boxed{k = \pm 27}$

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YEAR 12 – EXT.2 MATHS

REVIEW TOPIC (SP1)
POLYNOMIALS II

See corrections on page 2, 6, 11, 19, 21, 24, 27

(1) (a) The complex number z and its conjugate $\overline{z}$ satisfy the equation $z\overline{z} + 2iz = 12 + 6i$. Find the possible values of z.

$(x^2+y^2) + 2i(x+iy) = 12+6i$ $\quad (x+iy)(x-iy)$ $\quad x^2+y^2$

$x^2+y^2 + 2ix - 2y = 12+6i$

$2x = 6$ ✓

$x = 3$

$3^2 + y^2 - 2y = 12$

$y^2 - 2y - 3 = 0$

$(y-3)(y+1) = 0$ ✓

$y = 3 \quad y = -1$

$\therefore z = 3 - i$ or $3 + 3i$ ✓

$3 - i, 3 + 3i$

(b) $1+i$ is a root of the equation $x^2 + (a+2i)x + (5+ib) = 0$, where a and b are real. Find the values of a and b.

$(1+i)^2 + (a+2i)(1+i) + 5 + ib = 0$ ✓

$1 + 2i - 1 + a + ai + 2i - 2 + 5 + ib = 0$

$3 + a = 0$ ✓ $\quad 2 + a + 2 + b = 0$

$a = -3 \quad 4 + a + b = 0$

$4 - 3 + b = 0$ ✓

$b = -1$

$\therefore a = -3$ ✓

$b = -1$

$a = -3, b = -1$

(2) (a) $1-2i$ is one root of the equation $x^2+(1+i)x+k=0$.
Find the other root and the value of k.

Sum of roots:

$1-2i + x+iy = -(1+i)$
$= -1-i$

$1+x=-1$ $\quad -2+y=-1$
$x=-2$ $\quad y=1$

$\therefore$ root is $-2+i$

product of roots:

$(1-2i)(-2+i) = k$
$-2+i+4i+2 = k$
$5i = k$ $\quad \therefore k=5i$

$k=5i, x=-2+i$

(b) Find the zeros of $P(x)=x^4-4x^2+3=0$

(i) over **Q**;

let $m=x^2$
$m^2-4m+3=0$
$(m-3)(m-1)=0$
$m=3 \quad m=1$

$x^2=3$ or $x^2=1$
$x=\pm1$

±1

(ii) over **R**;

$x=\pm1$
$x=\pm\sqrt{3}$

$\pm1, \pm\sqrt{3}$

(iii) over **C**,

$x=\pm1$
$x=\pm\sqrt{3}$

$\pm1, \pm\sqrt{3}$

(3) (a) Find $P(x)$, given that $P(x)$ is monic, of degree 3, with 5 as a single zero and -2 as a zero of multiplicity 2.

$P(x) = (x-5)(x+2)^2$
$= (x-5)(x^2+4x+4)$
$= x^3-5x^2+4x^2-20x+4x-20$
$= x^3-x^2-16x-20$

$P(x)=x^3-x^2-16x-20$

(b) $P(x)$ is an even monic polynomial of degree 4 with integer coefficients. If $\sqrt{2}$ is a zero, and the constant term is 6, factorise $P(x)$ fully over **R**.

$P(x) = x^4 + 0x^2 + 6$
$= (x+\sqrt{2})(x-\sqrt{2})(x^2+3)$
$= (x+\sqrt{2})(x-\sqrt{2})(x+\sqrt{3})(x-\sqrt{3})$

$P(x)=(x-\sqrt{2})(x+\sqrt{2})(x-\sqrt{3})(x+\sqrt{3})$

(4) If $P(x)=x^3-3x^2-9x+c$ has a double zero, find c and factorise $P(x)$ over the real numbers.

$P'(x) = 3x^2-6x-9$

$P'(x) = 0$

$x^2-2x-3=0$

$(x-3)(x+1)=0$ ✓

$x=3 \quad x=-1$ ✓

$P(3) = 3^3-3(3^2)-9(3)+c = 0$

$-27+c=0$

$c=27$ ✓

$\therefore P(x) = (x-3)^2(x+3)$

$P(-1) = (-1)^3-3(-1)^2-9(-1)+c = 0$

$5+c=0$ ✓

$c=-5$

$\therefore P(x) = (x+1)^2(x-5)$ ✓

$c=27$	$P(x)=(x-3)^2(x+3)$;
$c=-5$	$P(x)=(x+1)^2(x-5)$

(5) If $ax^3+cx+d=0$ has a double root, show that $4c^3+27ad^2=0$.

$P(x) = ax^3+cx+d$

$P'(x) = 3ax^2+c = 0$ ✓

$3ax^2 = -c$

$x^2 = -\frac{c}{3a}$

$x = \sqrt{-\frac{c}{3a}}$ ✓

$a\left(\sqrt{-\frac{c}{3a}}\right)^3 + c\sqrt{-\frac{c}{3a}} + d = 0$

$\sqrt{-\frac{c}{3a}}\left[a\left(-\frac{c}{3a}\right)+c\right] = -d$ ✓

$\sqrt{-\frac{c}{3a}}\left(\frac{2}{3}c\right) = -d$

$-\frac{c}{3a} = \frac{9d^2}{4c^2}$

$-4c^3 = 27ad^2$ ✓

$4c^3+27ad^2=0$

$P(x) = (x^2+1)(Qx) + 2x + 3.$

(6) (a) When $P(x) = x^4 + ax^2 + 2x$ is divided by x^2+1, the remainder is $2x+3$. Find the value of a.

or sub $x = \pm i$ & equate real + imaginary parts.

$$x^2+1 \overline{)\, x^4 + ax^2 + 2x} \quad \text{quotient } x^2 + (a-1)$$

$x^4 + x^2$

$(a-1)x^2 + 2x$

$(a-1)x^2 + (a-1)$

$2x - a + 1$

$2x - a + 1 = 2x + 3$

$-2 = a$

$\therefore a = -2.$

$a = -2$

(b) When $P(x) = x^4 + ax^2 + bx + 2$ is divided by x^2+1, the remainder is $-x+1$. Find the values of a and b.

Try substituting $x = i$

$$x^2+1 \overline{)\, x^4 + ax^2 + bx + 2} \quad \text{quotient } x^2 + (a-1)$$

$x^4 + x^2$

$(a-1)x^2 + bx + 2$

$(a-1)x^2 + 0 + a - 1$

$bx + 2 - a + 1$

$bx = -x$

$b = -1$

$2 - a + 1 = 1.$

$a = 2$

$a = 2, b = -1$

(7) (a) Two of the roots of $3x^3 + ax^2 + 23x - 6 = 0$ are reciprocals. Find the value of a and the three roots.

let roots be: $a, \frac{1}{a}, b$

$a \cdot \frac{1}{a} \cdot b = \frac{6}{3}$

$b = 2$

$a\frac{1}{a} + ab + \frac{b}{a} = \frac{23}{3}.$

$ab + \frac{b}{a} = \frac{20}{3}.$

$2a + \frac{2}{a} = \frac{20}{3}.$

$6a^2 + 6 = 20a.$

$6a^2 - 20a + 6 = 0$

$3a^2 - 10a + 3 = 0$

$3a^2 - 9a - a + 3 = 0$

$(3a-1)(a-3) = 0$

$a = \frac{1}{3}$ or $3.$

$\frac{1}{3} + 3 + 2 = -\frac{\alpha}{3}$

$1 + 9 + 6 = -\alpha.$

$\alpha = -16.$

$\therefore \alpha = -16,$ roots: $3, \frac{1}{3}, 2$

$a = -16$; roots are $3, \frac{1}{3}, 2$

(8) The equation $px^3+qx^2+rx+s=0$ has roots $(a-c)$, a, $(a+c)$, which are in arithmetic progression. Show that the $a=\frac{-q}{3p}$ and hence show that $2q^3-9pqr+27p^2s=0$.

sum of roots:

$$3a=-\frac{q}{p}$$

$$a=-\frac{q}{3p}$$

$$p\left(-\frac{q}{3p}\right)^3+q\left(-\frac{q}{3p}\right)^2+r\left(-\frac{q}{3p}\right)+s=0$$

$$\frac{-pq^3}{27p^3}+\frac{q^3}{9p^2}-\frac{rq}{3p}+s=0$$

$$-q^3+3q^3-9prq+27p^2s=0$$

$$2q^3-9prq+27p^2s=0$$

(9) The equation $px^3+qx^2+rx+s=0$ has the roots ac, a and $\frac{a}{c}$, which are in geometric progression. Show that $a=\sqrt[3]{(-s/p)}$ and hence show that $pr^3-q^3s=0$.

product of roots:

$$a^3=-\frac{s}{p}$$

$$a=\sqrt[3]{-\frac{s}{p}}$$

$$p\left(-\frac{s}{p}\right)+q\sqrt[3]{-\frac{s}{p}}^{\,2}+r\sqrt[3]{-\frac{s}{p}}+s=0$$

$$-s+q\sqrt[3]{-\frac{s}{p}}^{\,2}+r\sqrt[3]{-\frac{s}{p}}+s=0$$

$$q\sqrt[3]{-\frac{s}{p}}^{\,2}=-r\sqrt[3]{-\frac{s}{p}}$$

$$q^3\left(-\frac{s}{p}\right)^2=-r^3\left(-\frac{s}{p}\right)$$

$$\frac{q^3s^2}{p^2}=\frac{r^3s}{p}$$

$$q^3s=r^3p$$

$$r^3p-q^3s=0$$

(10) The equation $x^3+x^2-2x-3=0$ has roots α, β and γ. Find the equations with roots

(a) $\frac{\alpha}{2}, \frac{\beta}{2}$ and $\frac{\gamma}{2}$;

$x = \alpha$

$x = \frac{\alpha}{2}$

$2x = \alpha$

$(2x)^3 + (2x)^2 - 2(2x) - 3 = 0$

$8x^3 + 4x^2 - 4x - 3 = 0$

$\boxed{8x^3+4x^2-4x-3=0}$

(b) $\alpha+2, \beta+2$ and $\gamma+2$.

$x = \alpha + 2$

$\alpha = x - 2$

$(x-2)^3 + (x-2)^2 - 2(x-2) - 3 = 0$

$x^3 - 3x^2(2) + 3x(2^2) - 2^3 + x^2 - 4x + 4 - 2x + 4 - 3 = 0$

$x^3 - 6x^2 + 12x - 8 + x^2 - 4x + 4 - 2x + 4 - 3 = 0$

$x^3 - 5x^2 + 6x - 3 = 0$

$\boxed{x^3-5x^2+6x-3=0}$

Qu. (11) (a) The polynomial $\alpha x^{n+1} + \beta x^n + 1$ is divisible by $(x-1)^2$.

Show that $\alpha = n$, and $\beta = -(1+n)$.

$P'(x) = \alpha(n+1)x^n + \beta n x^{n-1} = 0$

$P'(1) = \alpha(n+1) + \beta n = 0$

$\alpha(n+1) = -\beta n$

$\therefore \alpha = n \quad \beta = -(n+1)$

(b) Prove that $1+x+\frac{x^2}{2!}+....+\frac{x^n}{n!}$ has no multiple roots for any $n \geq 1$.

$P'(x) = 1 + \frac{2x}{2!} + \cdots + \frac{nx^{n-1}}{n!}$

$= 1 + \frac{x}{1!} + \cdots + \frac{x^{n-1}}{(n-1)!}$

as $n \geq 1$, all x will be > 0 → Not necessarily true. Instead try proof by contradiction, please ask me!

let $\frac{x}{1!} + \cdots + \frac{x^{n-1}}{(n-1)!} = T$

if $T > 0$

$1 + T > 1$

$\therefore 1 + T \neq 0$

$\therefore P(x) \neq 0$

$\therefore$ no multiple roots

Qu. (12) (HSC 1994)

(4) (a) Find α and β, given that $z^3+3z+2i=(z-\alpha)^2(z-\beta)$.

roots : α, α, β.

P(z) = z³ + 3z + 2i

P'(z) = 3z² + 3 = 0.

3(z² + 1) = 0

z = −i ∴ α = i

∴ α × α × β = 2i.

−i × −i × β = 2i

−1 × β = 2i

β = −2i

∴ α = −i, β = −2i.

$\alpha=-i, \beta=2i$

Qu. (13) (HSC 1994)

(8) (b) Let $x=\alpha$ be a root of the quartic polynomial $P(x)=x^4+Ax^3+Bx^2+Ax+1$, where A and B are real. Note that α may be complex.

(i) Show that $\alpha \neq 0$.

P(α) = 0   α⁴ + Aα³ + Bα² + Aα + 1 = 0

if α = 0   P(0) = 1.

but α is a root, ∴ α ≠ 0

(ii) Show that $x=\alpha$ is also a root of $Q(x)=x^2+\frac{1}{x^2}+A\left(x+\frac{1}{x}\right)+B$.

Q(α) = α² + 1/α² + A(α + 1/α) + B

= α⁴ + 1 + Aα²(α + 1/α) + Bα²

= α⁴ + Aα³ + Bα² + Aα + 1

= 0

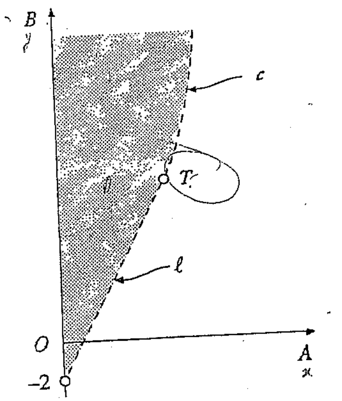
(iii) With $u=x+\frac{1}{x}$, show that $Q(x)$ becomes $R(u)=u^2+Au+(B-2)$.

Q(x) = x² + 1/x² + A(x + 1/x) + B.

= [(x + 1/x)² − 2] + A[x + 1/x] + B.

R(u) = u² + Au + B − 2.

(iv) For certain values of A and B, $P(x)$ has no real roots and $A \geq 0$.



A → x
B → y.

The region **D** is shaded in the figure. Specify the bounding straight-line segment l and curved segment C. Determine coordinates of T.

$R(u) = u^2 + Au + (B-2)$.

$\Delta = b^2 - 4ac$

$= A^2 - 4(B-2)$

$\Delta < 0$.

$\therefore A^2 - 4(B-2) < 0$

$u = x + \frac{1}{x}$.

$ux = x^2 + 1$

$x^2 - ux + 1 = 0$.

$\Delta = u^2 - 4$

$\Delta < 0$

$\therefore u^2 - 4 < 0$

$-2 < u < 2$

$u = -2$: $R(-2) = 4 - 2A + B - 2 = 0$.

$B = 2A - 2$.

$u = 2$ $R(2) = 4 + 2A + B - 2 = 0$.

$B = -2A - 2$.

l has +ve gradient $\therefore B = 2A - 2$.

$B = 2A - 2$

$A^2 - 4(B-2) = 0$.

$A^2 - 4(2A - 4) = 0$.

$A^2 - 8A + 16 = 0$.

$(A-4)^2 = 0$

$A = 4$.

$B = 2(4) - 2$

$= 6$.

$\therefore T(4, 6)$

$T(4,6)$

Qu. (14) (HSC 1995)

(5) (b) Let $f(t) = t^3 + ct + d$, where c and d are constants. Suppose that the equation $f(t) = 0$ has three distinct real roots, t_1, t_2, and t_3.

(i) Find $t_1 + t_2 + t_3$.

sum of roots = 0

0

(ii) Show that $t_1^2 + t_2^2 + t_3^2 = -2c$.

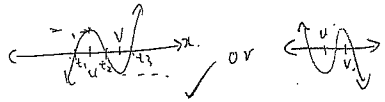
$(t_1 + t_2 + t_3)^2 - 2(\Sigma t_1 t_2)$

$= 0^2 - 2(c)$

$= -2c$

(iii) Since the roots are real and distinct, the graph of $y = f(t)$ has two turning points, at $t = u$ and $t = v$, and $f(u).f(v) < 0$.
Show that $27d^2 + 4c^3 < 0$.

$f(u) = u^3 + cu + d$
$f(v) = v^3 + cv + d$ ✓



~~$(u^3+cu+d)(v^3+cv+d) < 0$~~

$f'(u) = 3u^2 + c = 0$
$f'(v) = 3v^2 + c = 0$ ✓

$f(t) = t^3 + ct + d$
$f'(t) = 3t^2 + c$ ✓
$f'(t) = 0$
$3t^2 + c = 0$
$t^2 = -\frac{c}{3}$
$t = \pm\sqrt{-\frac{c}{3}}$ ✓

let: $u = \sqrt{-\frac{c}{3}}$
$v = -\sqrt{-\frac{c}{3}}$

$f(u)\cdot f(v) = \left[\left(\sqrt{-\frac{c}{3}}\right)^3 + c\sqrt{-\frac{c}{3}} + d\right]\left[\left(-\sqrt{-\frac{c}{3}}\right)^3 - c\sqrt{-\frac{c}{3}} + d\right] < 0$ ✓

$= \left[-\frac{c}{3}\sqrt{-\frac{c}{3}} + c\sqrt{-\frac{c}{3}} + d\right]\left[+\frac{c}{3}\sqrt{-\frac{c}{3}} - c\sqrt{-\frac{c}{3}} + d\right] < 0$

$= \left[\frac{c^2}{9} \times -\frac{c}{3}\right] + \frac{c^2}{3}\left(-\frac{c}{3}\right) - \frac{dc}{3}\sqrt{\frac{c}{3}} + \frac{c^2}{3}\left(-\frac{c}{3}\right) - c^2\left(-\frac{c}{3}\right) + dc\sqrt{\frac{c}{3}}$

$+ \frac{cd}{3}\left(\sqrt{-\frac{c}{3}}\right) - cd\sqrt{-\frac{c}{3}} + d^2 < 0$ ✓

$= \frac{c^3}{27} - \frac{c^3}{9} - \frac{c^3}{9} + \frac{c^3}{3} + d^2 < 0$

$\frac{4c^3}{27} + d^2 < 0$ ✓

$\therefore 4c^3 + 27d^2 < 0$

Qu. (15) HSC 1996

(5) (b) Consider the polynomial equation
$x^4 + ax^3 + bx^2 + cx + d = 0,$
where a, b, c, and d are integers. Suppose the equation has a root of the form ki, where k is real, and $k \neq 0$.

(i) State why the conjugate $-ki$ is also a root.

conjugate root theorum
(a, b, c, d are integers) ✓

(ii) Show that $c = k^2a$.

$P(x) = x^4 + ax^3 + bx^2 + cx + d = 0$
$P(ki) = (ki)^4 + a(ki)^3 + b(ki)^2 + c(ki) + d = 0$
$= k^4 - ak^3i - k^2b + cki + d$
$P(-ki) = (-ki)^4 + a(-ki)^3 + b(-ki)^2 + c(-ki) + d = 0$
$= k^4 + ak^3i - bk^2 - cki + d = 0.$

$P(ki) - P(-ki) = -2ak^3i + 2cki = 0.$
$2ki(c - ak^2) = 0.$
$c - ak^2 = 0.$ ✓
$\therefore c = ak^2$

(iii) Show that $c^2 + a^2d = abc$.

$P(ki) + P(-ki) = 0$ $\qquad k^2 = \frac{c}{a}$.

$2k^4 - 2k^2b + 2d = 0$.

$2\left(\frac{c}{a}\right)^2 - 2b\left(\frac{c}{a}\right) + 2d = 0$.

$\frac{c^2}{a^2} - \frac{bc}{a} + d = 0$.

$c^2 - abc + a^2d = 0$.

$\therefore c^2 + a^2d = abc$.

(iv) If 2 is also a root of the equation and $b=0$, show that c is even.

roots, $ki, -ki, 2, \alpha$.

$(ki)(-ki) + 2ki + \alpha ki - 2ki - \alpha ki + 2\alpha = b$

$k^2 + 2\alpha = 0$.

$\frac{c}{a} + 2a = 0$.

$\frac{c}{a} = -2a$

$c = -2a^2$.

$\therefore c$ is even.

Qu. (16) (HSC 1997)

(3) (b) Let $f(x) = 3x^5 - 10x^3 + 16x$.

-2 -1 0 1 2
-4 0 9

(i) Show that $f'(x) \geq 1$ for all x.

$f'(x) = 15x^4 - 30x^2 + 16$.

Prove $f'(x) \geq 1$

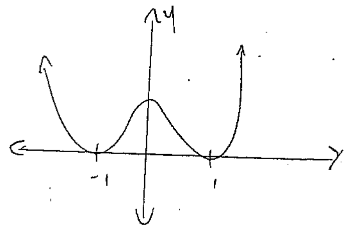
$f'(x) - 1 \geq 0$.

$15x^4 - 30x^2 + 15 \geq 0$.

$15(x^2-1)^2 \geq 0$.

from graph: $f'(x) - 1 \geq 0$

$\therefore f'(x) \geq 1$.



(ii) For what values of x is $f''(x)$ positive?

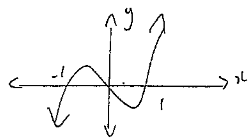
$f''(x) = 60x^3 - 60x$.

$f''(x) > 0$.

$60x(x^2-1) > 0$.

$f''(x) > 0$ when:

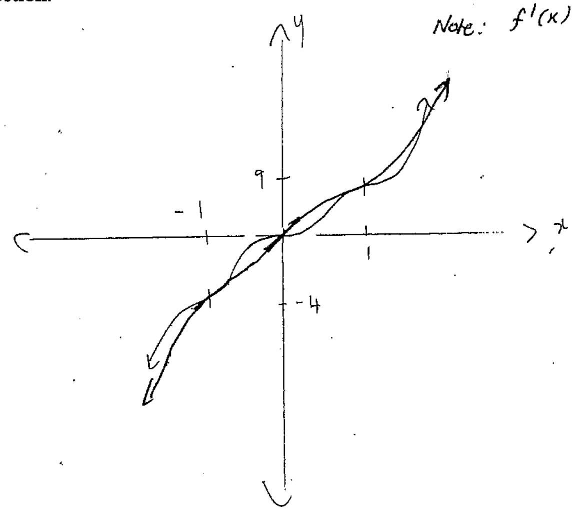
$x > 1, \quad -1 < x < 0$



$x > 1$ and $-1 < x < 0$

(iii) Sketch the graph of $y = f(x)$, indicating any turning points and points of inflection.

Note: $f'(x) \geq 1$ for all x



Qu. (17) (HSC 1997)

(5) (c) Suppose that b and d are real numbers and $d \neq 0$. Consider the polynomial

$$P(z) = z^4 + bz^2 + d.$$

The polynomial has a double root α.

(i) Prove that $P'(z)$ is an odd function. (i.e. prove $P'(-z) = -P'(z)$)

$P'(z) = 4z^3 - 2b(z)$

$P'(-z) = 4(-z)^3 - 2b(-z)$

$= -4z^3 + 2bz$

$-P'(z) = -(4z^3 - 2bz)$

$= -4z^3 + 2bz$

$= P'(-z)$

$\therefore$ odd fn

(ii) Prove that $-\alpha$ is also a double root of $P(z)$.

$P'(\alpha) = 0$.

$4a^3 - 2ba = 0$.

$P'(-\alpha) = -4a^3 + 2ba$.

$= -(4a^3 - 2ba)$

$= -(0)$

$= 0$

$\therefore -\alpha$, also double root.

(iii) Prove that $d = \frac{b^2}{4}$.

$\Delta = 0$ (double root)

$\therefore b^2 - 4d = 0$

$b^2 = 4d$

$d = \frac{b^2}{4}$

(iv) For what values of b does $P(z)$ have a double root equal to $\sqrt{3}i$?

$P'(z) = 4z^3 - 2bz$

$P'(\sqrt{3}i) = 0$.

$4(\sqrt{3}i)^3 - 2b(\sqrt{3}i) = 0$.

$(\sqrt{3}i)[-12 - 2b] = 0$.

$(-3)(-12-2b) = 0$.

$36 + 6b = 0$

$b = 6$

$b = 6$

(v) For what values of b does $P(z)$ have real roots?

Let $z^2 = m$. $P(z) = m^2 + bm + d$.

$\Delta = b^2 - 4d \rightarrow$ real roots: $\Delta > 0$

Using $\sum \alpha\beta = \alpha(-\alpha) + \alpha(0) + (-\alpha)\times 0 = \frac{b}{2}$

$\therefore \alpha^2 = -\frac{b}{2}$

$\therefore b < 0$ for real α.

$b < 0$

Qu. (18) (HSC 2002)

(b) Let α, β, and γ be the roots of the equation $x^3 - 5x^2 + 5 = 0$.

(i) Find a polynomial equation with integer coefficients whose roots are $\alpha - 1$, $\beta - 1$, and $\gamma - 1$. 2

$x = \alpha - 1$

$x + 1 = \alpha$

$(x+1)^3 - 5(x+1)^2 + 5 = 0$

$x^3 + 3x^2 + 3x + 1 - 5x^2 - 10x - 5 + 5 = 0$.

$x^3 - 2x^2 - 7x + 1 = 0$

$x^3 - 2x^2 - 7x + 1 = 0$

(ii) Find a polynomial equation with integer coefficients whose roots are α^2, β^2, and γ^2. **2**

$x = \alpha^2$

$\sqrt{x} = \alpha$.

$(\sqrt{x})^3 - 5(\sqrt{x})^2 + 5 = 0$ ✓

$x\sqrt{x} - 5x + 5 = 0$

$x\sqrt{x} = 5(x-1)$ ✓

$x^2 \cdot x = 25(x^2 - 2x + 1)$

$x^3 - 25x^2 + 50x - 25 = 0$ ✓

$\boxed{x^3 - 25x^2 + 50x - 25 = 0}$

(iii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

$\alpha^3 = 5\alpha^2 - 5$

$\alpha^3 + \beta^3 + \gamma^3 = 5(\alpha^2 + \beta^2 + \gamma^2) - 5(3)$ ✓

$= 5[25] - 15$

$= 110$. ✓

$x(Q(x)) - k(Q(x))$

Qu. (19) (HSC 1998)

(4) (a) (i) Suppose that k is a double root of the polynomial equation $f(x) = 0$. **7**

Show that $f'(k) = 0$.

if k is double root:

$f(x) = (x-k)^2 Q(x)$

$f'(x) = (x-k)^2 \cdot Q'(x) + Q(x) \times 2(x-k)$.

$f(k) = 0$.

$f'(k) = (k-k)^2 Q'(k) + 2Q(k)(k-k)$ ✓

$= 0$.

$\therefore f'(k) = 0$ is k is double root. ✓

(ii) What feature does the graph of a polynomial have at a root of multiplicity 2?

turning point on x-axis ✓

(iii) The polynomial $P(x) = ax^7 + bx^6 + 1$ is divisible by $(x-1)^2$.

Find the coefficients a and b.

$P(1) = 0$.

$a + b = -1$.

$P'(x) = 7ax^6 + 6bx^5 = 0$

$P'(1) = 7a + 6b = 0$.

$7a + 6(-1-a) = 0$.

$7a - 6 - 6a = 0$

$a = 6$. ✓

$6 + b = -1$

$b = -7$. ✓

$\therefore a = 6$

$b = -7$. ✓

$\boxed{a = 6, b = -7}$

(iv) Let $E(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$. Prove that $E(x)=0$ has no double roots.

Let α be the double root $\therefore 1+\alpha+\frac{\alpha^2}{2}+\frac{\alpha^3}{6}+\frac{\alpha^4}{24}=0$

$E'(x) = 1+\frac{2x}{2}+\frac{3x^2}{6}+\frac{4x^3}{24}$ ✓ $\quad E(x)=0$.

$0 = 1+x+\frac{x^2}{2}+\frac{x^3}{6}$ $\qquad 1+x+\frac{x^2}{2}+\frac{x^3}{6} = -\frac{x^4}{24}$

$0 = -\frac{x^4}{24} \Rightarrow x=0$ but $E(0)=1$ $\therefore \alpha$ cannot be a double root

~~$\therefore E(x)$ has no zeros.~~

(6) (a) Consider the following statements about a polynomial $Q(x)$. Indicate whether each of these statements is true or false. Give reasons for your answers.

(i) If $Q(x)$ is even, then $Q'(x)$ is odd.

True.

$Q(x) = ax^2+b$

$Q'(x) = 2ax$

$\Rightarrow Q'(-x) = 2a(-x)$

$= -2ax$

$= -Q'(x)$

$\therefore$ odd

TRUE

(ii) If $Q'(x)$ is even, then $Q(x)$ is odd.

False.

Similarly let

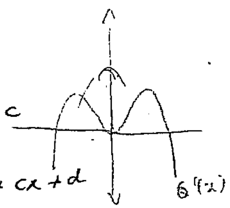
$Q'(x) = ax^4+bx^2+c$

$\therefore Q(x) = \frac{ax^5}{5}+\frac{bx^3}{3}+cx+d$

$Q(-x) = -\frac{ax^5}{5}-\frac{bx^3}{3}-cx+d$

$\neq Q(x)$

$\therefore$ False.



FALSE

Qu. (20) (HSC 1999)

(2) (d) Consider the equation $2z^3-3z^2+18z+10=0$ **2**

(i) Given that $1-3i$ is a root of the equation, explain why $1+3i$ is another root.

conjugate roots ✓

(ii) Find all roots of the equation.

$(1-3i)(1+3i)\alpha = -\frac{10}{2}$ ✓

$(1+9)\alpha = -5$

$10\alpha = -5$ ✓

$\alpha = -\frac{1}{2}$

$+ \; 1-3i, \; 1+3i$

$1-3i, 1+3i, -\frac{1}{2}$

Qu. (21) (HSC 1999)

(5) (a) The roots of $x^3+5x^2+11=0$, are α, β and γ. **3**

(i) Find the polynomial equation whose roots are α^2, β^2 and γ^2.

$x = \alpha^2$

$\sqrt{x} = \alpha$.

$(\sqrt{x})^3 + 5\sqrt{x}^2 + 11 = 0$ ✓

$x\sqrt{x} = -5x-11$

$= -(5x+11)$

$x^3 = 25x^2+110x+121$

$x^3-25x^2-110x-121 = 0$ ✓

$y^3-25y^2-110y-121=0$

(ii) Find the value of $\alpha^2+\beta^2+\gamma^2$.

sum of roots $= -\left(\frac{a}{b}\right)$ ✓
$= -(-25)$
$= 25$. ✓

25

Qu. (22) (HSC 2000)

(2) (b) Consider the equation $z^2+az+(1+i)=0$. **2**

Find the complex number a, given that i is a root of the equation.

Let other root be x.

$ix = 1+i$.
$x = \frac{1}{i}+1$.
$= i^3+1$
$= -i+1$.

∴ sum of roots;
$x+1+i = -a$
$a=-1$

$a=-1$

(5) (a) Consider the polynomial $p(x)=ax^4+bx^3+cx^2+dx+e$ **4**

where a,b,c,d and e are integers. Suppose α is an integer such that $p(\alpha)=0$.

(i) Prove that α divides e.

$P(\alpha)=0$ ∴
$a\alpha^4+b\alpha^3+c\alpha^2+d\alpha+e=0$
$\alpha(a\alpha^3+b\alpha^2+c\alpha+d)=-e$
↓ all integers ✓
∴ α divides e.

$5 \div 20 \rightarrow 5\times 4 = 20$ (4 ↓ integer)
$\alpha \div e \rightarrow \alpha \cdot c = e$. (c ↓ integer.)

(ii) Prove that the polynomial $q(x)=4x^4-x^3+3x^2+2x-3$ does not have an integer root.

$q(x) = 4x^4-x^3+3x^2+2x-3=0$.
If α is an integer root then it must divide -3
then try $q(\pm1)$ & $q(\pm3)$ to show that $\neq 0$.

Qu. (23) HSC 2001

(3) (b) The numbers α,β and γ satisfy the equations

$$\alpha+\beta+\gamma=3$$

$$\alpha^2+\beta^2+\gamma^2=1$$

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=2.$$

(i) Find the values of $\alpha\beta+\beta\gamma+\gamma\alpha$ and $\alpha\beta\gamma$. **3**

Explain why α,β and γ are the roots of the cubic equation

$$x^3-3x^2+4x-2=0.$$

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$
$1 = 9 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$
$\alpha\beta+\beta\gamma+\gamma\alpha = 4$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$
$2 = \frac{4}{\alpha\beta\gamma}$
$\alpha\beta\gamma = 2$

sum: $\sum\alpha = 3$.

$\sum\alpha\beta = 4$ } fits calculations

$\alpha\beta\gamma = 2$

$\therefore$ α, β, γ are roots of eqn

$\sum\alpha\beta = 4,\ \alpha\beta\gamma = 2$

(ii) Find the values of α, β and γ. **2**

$P(x) = x^3 - 3x^2 + 4x - 2 = 0$.

$P(1) = 0$.

$\therefore x = 1$ is a root

$$x - 1 \overline{)\, x^3 - 3x^2 + 4x - 2} = x^2 - 2x + 2$$

$x^3 - x^2$

$-2x^2 + 4x$

$-2x^2 + 2x$

$2x - 2$

$\therefore x^2 - 2x + 2 = 0$

$x = \dfrac{2 \pm \sqrt{4 - 4(2)}}{2}$

$= \dfrac{2 \pm \sqrt{-4}}{2}$

$= 1 \pm i$

$\therefore$ roots are $1 \pm i$, 1.

$1,\ 1+i,\ 1-i$

Qu. (24) HSC 2001

7 (b) Consider the equation $x^3 - 3x - 1 = 0$, which we denote by (*).

(i) Let $x = \dfrac{p}{q}$ where p and q are integers having no common divisors other than +1 and −1. Suppose that x is a root of the equation $ax^3 - 3x + b = 0$, where a and b are integers. **4**

Explain why p divides b and why q divides a. Deduce that (*) does not have a rational root.

$p \cdot n = b$ $\quad q \cdot m = a$.

if x is a root of *

$a\left(\dfrac{p}{q}\right)^3 - 3\left(\dfrac{p}{q}\right) + b = 0$

$\dfrac{ap^3}{q^3} - \dfrac{3p}{q} + b = 0$

$p\left(\dfrac{ap^2}{q^3} - \dfrac{3p}{q}\right) = -b$.

all integers $\therefore$ $p \div b$.

$ap^3 - 3pq^2 + bq^3 = 0$.

$a - \dfrac{3q^2}{p^3} + \dfrac{bq^3}{p^3} = 0$.

$q\left(\dfrac{bq^2}{p^3} - \dfrac{3q}{p}\right) = -a$.

integer

$q \div a$

$\therefore p / -1 \to p = \pm 1$

$q / 1 \to q = \pm 1$.

$x = \dfrac{p}{q} = \pm 1$

$P(1) = 1 - 3 - 1$

$= -3 \neq 0$

$P(-1) = -1 + 3 - 1$

$= 1 \neq 0$

$\therefore$ no rational roots

(ii) Suppose that r, s and d are rational numbers and that $\sqrt{d}$ is irrational. Assume that $r+s\sqrt{d}$ is a root of (*). **4**

Show that $3r^2s+s^3d-3s=0$ and show that $r-s\sqrt{d}$ must also be a root of (*).

Deduce from this result and part (i), that no root of (*) can be expressed in the form $r+s\sqrt{d}$ with r, s and d rational.

$P(x) = x^3 - 3x - 1$

if $r+s\sqrt{d}$ is a root

$r-s\sqrt{d}$ also a root (conjugate root theorem).

$P(r+s\sqrt{d}) = (r+s\sqrt{d})^3 - 3(r+s\sqrt{d}) - 1 = 0.$

$r^3 + 3r^2s\sqrt{d} + 3rs^2d + s^3d\sqrt{d} - 3r - 3s\sqrt{d} - 1 = 0$

$\sqrt{d}(3r^2s + s^3d - 3s) = 1 + 3r - r^3$

$= -(r^3 - 3r - 1)$

$\sqrt{d}(3r^2s + s^3d - 3s) = 0$

$\therefore 3r^2s + s^3d - 3s = 0$

if $r+s\sqrt{d}$ is a root; if $\sqrt{d}$ is rational

$r+s\sqrt{d} \neq$ root (no rational roots).

if $\sqrt{d}$ is irrational:

let roots be, $r+s\sqrt{d}$, $r-s\sqrt{d}$, γ.

$r+s\sqrt{d} + r - s\sqrt{d} + \gamma = 0.$

$2r = -\gamma \Rightarrow$ rational roots

$\therefore r+s\sqrt{d} \neq$ root.

$\therefore$ no roots can be expressed in form $r+s\sqrt{d}$

(iii) Show that one root of (*) is $2\cos\frac{\pi}{9}$. **1**

(You may assume the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.)

$x^3 - 3x - 1 = 0$

$4x^3 - 3x - 3x^3 - 1 = 0.$

let $x = \cos\theta$. $\cos 3\theta - (3\cos^3\theta + 1) = 0$

$\cos 3\theta = 1 + 3\cos^3\theta$

$\left(2\cos\frac{\pi}{9}\right)^3 - 3\left(2\cos\frac{\pi}{9}\right) - 1$

$= 8\cos^3\frac{\pi}{9} - 6\cos\frac{\pi}{9} - 1$

$= 2\left(\cos 3\left(\frac{\pi}{9}\right)\right) - 1$

$= 2\left(\cos\frac{\pi}{3}\right) - 1$

$= 2\left(\frac{1}{2}\right) - 1$

$= 0.$

Qu. (25) HSC 2002

(5 (a) The equation $4x^3 - 27x + k = 0$ has a double root. Find the possible values of k. **2**

$P(x) = 4x^3 - 27x + k$

$P'(x) = 12x^2 - 27 = 0$

$x^2 = \frac{27}{12}$

$x = \pm\frac{3\sqrt{3}}{2\sqrt{3}}$

$= \pm\frac{3}{2}$

$P(\frac{3}{2}) = 0$

$4(\frac{3}{2})^3 - 27(\frac{3}{2}) + k = 0$

$\frac{27}{2} - \frac{81}{2} + k = 0$

$27 - 81 + 2k = 0$

$2k = 54$

$k = 27$

$P(-\frac{3}{2}) = 0$

$-\frac{27}{2} + \frac{81}{2} + k = 0$

$k = -27$

$k = \pm 27$