NAME :

Centre of Excellence in Mathematics
S201 / 414 GARDENERS RD. ROSEBERY 2018
www.cemtuition.com.au

MOBILE 0412 880 475

PHONE 96663331

YEAR 12 – EXT.2 MATHS

REVIEW TOPIC (SP2)
POLYNOMIALS II

Factors and Zeroes

They depend on the field we are concerned with. Example: $P(x)=x^4-2x^2-3$

Over Q (rational): $P(x)=(x^2-3)(x^2+1)$, zeroes are none

Over R (real): $P(x)=(x-\sqrt{3})(x+\sqrt{3})(x^2+1)$, zeroes are $\sqrt{3},-\sqrt{3}$

Over C (complex): $P(x)=(x-\sqrt{3})(x+\sqrt{3})(x-i)(x+i)$, zeroes are $\sqrt{3},-\sqrt{3},i,-i$

Long Division Involving Complex Numbers

Example:
$$\begin{array}{r l} & x+(1+i) \\ \cline{2-2} x-i \big) & x^2+x+3 \\ & x^2-ix \\ \cline{2-2} & (1+i)x+3 \\ & (1+i)x+(1-i) \\ \cline{2-2} & 2+i \end{array}$$

Factoring a Quadratic by Completing the Squares

Example: $x^2-6x+7=x^2-6x+9-2=(x-3)^2-(\sqrt{2})^2=(x-3-\sqrt{2})(x-3+\sqrt{2})$

Multiplicity

- If α is a zero of multiplicity (or order) r of $P(x)$, $r>1$, then α is a zero of multiplicity $r-1$ of $P'(x)$. Proof: differentiate $P(x)$, then show that $P'(x)=(x-\alpha)^{r-1}A(x)$ where $A(\alpha)\neq 0$.
- As an implication, α is a zero of multiplicity r of $P(x)$ iff α is a zero of $P^{(r-1)}(\alpha)$, that is, $P(\alpha)=P'(\alpha)=P''(\alpha)=\ldots=P^{(r-1)}(\alpha)=0$.

The Division Transformation

This refers to the writing of $P(x)$ as $D(x)\cdot Q(x)+R(x)$, where $\deg R<\deg D$.

The Fundamental Theorem of Algebra

It states that every polynomial of degree greater than or equal to one has at least one zero over C. Implication: by mathematical induction, every polynomial of degree n has n zeroes over C (not necessarily distinct); it can be factorised into n linear factors.

Polynomial with Integer Coefficients

- Any integer zero α of the polynomial is a divisor of the constant term.

  Proof: let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0=x(a_nx^{n-1}+a_{n-1}x^{n-2}+\ldots+a_1)+a_0$

  Since $P(\alpha)=0$, $0=\alpha(a_n\alpha^{n-1}+a_{n-1}\alpha^{n-2}+\ldots+a_1)+a_0$

  Rearranging, $a_0=\alpha(-a_n\alpha^{n-1}-a_{n-1}\alpha^{n-2}-\ldots-a_1)$

  But: α *and all* a'*s* are integral, so $(-a_n\alpha^{n-1}-a_{n-1}\alpha^{n-2}-\ldots-a_1)$ is integral.

  Hence, α is a divisor of the constant term a_0.
- The numerator of any rational zero is a divisor of the constant term, and its denominator is a divisor of the leading coefficient.
- Any non-rational zeroes occur in pairs of conjugate surds.

Polynomial with Real Coefficients (note: real includes integers)

- If $P(x)$ has odd degree, then it has at least one real zero.
- For any $P(x)$, the non-real zeroes occur in complex conjugate pairs.

Hence, any $P(x)$ can be factorised into real linear and real quadratic factors. Note: $(x-\alpha)(x-\overline{\alpha}) = x^2 - (\alpha+\overline{\alpha})x + \alpha\overline{\alpha} = x^2 - 2\operatorname{Re}(\alpha)x + |\alpha|^2$, which is a real quadratic.

Polynomial with Non-Real Coefficients

No pattern; zeroes need not be conjugate pairs. Example: $x^2 + ix + 2 = (x-i)(x+2i)$.

Sum of the Products of Roots

For $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$,

Sum of the products of roots taken r at a time $= \boxed{(-1)^r \frac{a_{n-r}}{a_n}}$

Notations for the sums: let's take $P(x)$ of degree 4: $\sum\alpha, \sum\alpha\beta, \sum\alpha\beta\gamma, \alpha\beta\gamma\delta$

Some Questions

1. Suppose $x^3 - 2x^2 + a \equiv (x+2)Q(x) + 3$ where $Q(x)$ is a polynomial. Find the value of a. SOLN: Since LHS has x^3, Q(x) must have x^2. $x^3 - 2x^2 + a \equiv (x+2)(x^2+bx+c)+3$. Equating coefficient of x^2, $b+2=-2 \Rightarrow b=-4$. Equating coefficient of x, $c+2b=0 \Rightarrow c=8$. $a=19$.
2. a and b are real numbers such that the sum of the squares of the roots of the equation $x^2 + (a+ib)x + 3i = 0$ is 8. Find all possible pairs of a and b. SOLN: Let the roots be α, β. $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = [-(a+ib)]^2 - 2[3i] = a^2 - b^2 + (2ab-6)i = 8$. $2ab-6=0 \Rightarrow a = 3/b$ and $a^2 - b^2 = (3/b)^2 - b^2 = 8$. Pairs (a,b): (3,1) and (-3,-1).
3. $1+i$ is a root of the equation $x^2 + (a+2i)x + (5+ib) = 0$, where a and b are real. Find the values of a and b. SOLN: Put $x = 1+i$ into the equation and expand.

Changing the Variable to Find a New Polynomial

Let $ax^3 + bx^2 + cx + d = 0$ has three roots represented by α, that is, $a\alpha^3 + b\alpha^2 + c\alpha + d = 0$. To find an equation with roots 3α, we can replace x with $x/3$ so that when 2α is substituted as x, we still get $a\alpha^3 + b\alpha^2 + c\alpha + d = 0$. By writing the new equation in y instead of x, we can find the substitution systematically. Examples:

1. $x^3 + x^2 - 2x - 3 = 0$ has roots α, β, γ. Find the equation with roots $\alpha^2\beta\gamma, \alpha\beta^2\gamma, \alpha\beta\gamma^2$. SOLN: $\alpha\beta\gamma = 3$. New roots are $3\alpha, 3\beta, 3\gamma$. Let $y = 3x \Rightarrow x = y/3$. New equation in y:
$$\left(\frac{y}{3}\right)^3 + \left(\frac{y}{3}\right)^2 - 2\left(\frac{y}{3}\right) - 3 = 0 \text{ or } y^3 + 3y^2 - 18y - 81 = 0.$$
2. $x^3 + 4x^2 + 2x - 1 = 0$ has roots α, β, γ. Find the equation with roots $\boxed{\alpha^2, \beta^2, \gamma^2}$. SOLN: Let $y = x^2 \Rightarrow x = \sqrt{y}$. The equation in y with roots $\alpha^2, \beta^2, \gamma^2$ is given by: $\sqrt{y}^3 + 4\sqrt{y}^2 + 2\sqrt{y} - 1 = 0 \Rightarrow \sqrt{y}(y+2) = -(4y-1)$ ($\boxed{\text{separate the terms with } \sqrt{y}}$). $\boxed{\text{Squaring}}$, $y(y+2)^2 = (4y-1)^2 \Rightarrow y^3 - 12y^2 + 12y - 1 = 0$.

Not everything can be done by directly $\boxed{\text{letting } y = \ldots x \text{ and } x = \ldots y}$. For example, to find the equation with roots $\alpha^4, \beta^4, \gamma^4$ in the above two examples, we need to first find those with roots $\alpha^2, \beta^2, \gamma^2$, and after making it a cubic, repeat the process for $(\alpha^2)^2, (\beta^2)^2, (\gamma^2)^2$.

$\boxed{\textit{Evaluating } \alpha^3 + \beta^3 + \gamma^3}$

We don't have to find the new cubic by replacing x with its cube root. We use the fact that $a\alpha^3 + b\alpha^2 + c\alpha + d = 0$, $a\beta^3 + b\beta^2 + c\beta + d = 0$ and $a\gamma^3 + b\gamma^2 + c\gamma + d = 0$, and then sum them to get: $a(\alpha^3+\beta^3+\gamma^3)+b(\alpha^2+\beta^2+\gamma^2)+c(\alpha+\beta+\gamma)+3d=0$ [$(\alpha^2+\beta^2+\gamma^2)=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$ if you haven't found the equation with roots $\alpha^2,\beta^2,\gamma^2$].

A quartic with symmetric coefficients $P(x)=ax^4+bx^3+cx^2\pm bx+a$ can be converted to quadratic in $\left(x\pm {}^{1}\!/_{x}\right)$. Method: write it as $x^2\left(ax^2+bx+c\pm bx^{-1}+ax^{-2}\right)=x^2\left[a\left(x^2+x^{-2}\right)+b\left(x\pm x^{-1}\right)+c\right]$, then use the fact that $\left(x+x^{-1}\right)^2=\left(x^2+x^{-2}\right)+2$. Also note that $x=0$ won't be a zero because then x^{-1} is not defined.

Solving Equations over C Using Complex Number Technique

Solving $x^n\pm1=0$***:*** find the complex *n*th roots of ∓1. Use De Moivre's theorem to find the roots in modulus-argument form. Alternatively: $x^n-1=0$, *n even* can be solved by treating it as a difference of two squares; $0=x^4+1=\left(x^4+2x^2+1\right)-2x^2$ then complete the square and treat the result as a difference of two squares; etc.

Solving $x^{n-1}+x^{n-2}+\ldots+x+1=0$***:*** multiply by $(x-1)$ to give $x^n-1=0$

Solving $x^{n-1}-x^{n-2}+\ldots+(-1)^{n-1}=0$***:*** multiply by $(x+1)$ to give $x^n\pm1=0$

Note that $x=1$ won't be a solution to the last two problems above; $x=-1$ might be.

Solving $x^n=c$***:*** use De Moivre's theorem

Solving $x^n+cx=0$***:*** write it as $x\left(x^{n-1}+c\right)=0$; etc.

New Expressions for $\sin n\theta$ *and* $\cos n\theta$

Let $z=\cos\theta+i\sin\theta$. We can equate the imaginary and real parts of the expressions of z^n using De Moivre's theorem and Binomial expansion. From the imaginary part, we get $\sin n\theta$ equals some summation of products in the form of $\sin^a\theta\cos^b\theta$; similarly we get an alternative expression for $\cos n\theta$ from the real part.

Sample questions with answers:

(i) Express $\tan 5\theta$ in terms of powers of $\tan\theta$, hence show that $x^4-10x^2+5=0$ has roots $\pm\tan{}^{\pi}\!/_{5}, \pm\tan{}^{2\pi}\!/_{5}$. SOLN:

$$\tan 5\theta=\frac{\sin 5\theta}{\cos 5\theta}=\frac{5\sin\theta\cos^4\theta-10\sin^3\theta\cos^2\theta+\sin^5\theta}{\cos^5\theta-10\sin^2\theta\cos^3\theta+5\sin^4\theta\cos\theta}=\frac{5\tan\theta-10\tan^3\theta+\tan^5\theta}{1-10\tan^2\theta+5\tan^4\theta},$$

after dividing top and bottom by $\cos^5\theta$.

Then, let $x=\tan\theta$. Pay attention to the top part of the fraction above and notice that it resembles $x\left(x^4-10x^2+5\right)$, hence $\tan 5\theta=0$ if x is a root of $x^4-10x^2+5=0$.

(ii) Deduce that $\tan{}^{\pi}\!/_{5}\tan{}^{2\pi}\!/_{5}\tan{}^{3\pi}\!/_{5}\tan{}^{4\pi}\!/_{5}=5$. SOLN:

Notice that $-\tan{}^{\pi}\!/_{5}=\tan{}^{4\pi}\!/_{5}, -\tan{}^{2\pi}\!/_{5}=\tan{}^{3\pi}\!/_{5}$. From $x^4-10x^2+5=0$, the product of the four roots is 5.

(iii) By solving $x^4-10x^2+5=0$ another way, find the value of $\tan{}^{\pi}\!/_{5}$ as a surd. SOLN:

Consider $x^4-10x^2+5=0$ as a quadratic in x^2 and use the quadratic formula.

Partial Fractions

Method: Set up an identity, $\dfrac{P(x)}{Q(x)}\equiv\dfrac{c_1}{x-\alpha_1}+\dfrac{c_2}{x-\alpha_2}+\ldots+\dfrac{c_n}{x-\alpha_n}$, where $Q(x)=k(x-\alpha_1)(x-\alpha_2)(\ldots)(x-\alpha_n)$, all zeroes of Q(x) distinct, and $\deg P<\deg Q$.

We can then rearrange this identity so that the fractions disappear by multiplying both sides by Q(x). Now, substitute selected values of x by which we can equate coefficients of x-terms to determine the constants, c's. Useful values to substitute are $\alpha_1, \alpha_2, \ldots$, and if the factors are irreducible quadratics, zero and ki (for factors like x^2+1, x^2+k^2) [$i=\sqrt{-1}$]

An alternative approach to equating coefficients in the identity:

[Note: This approach is mentioned in the syllabus but is very rarely asked.]

$\boxed{c_i = \dfrac{P(\alpha_i)}{Q'(\alpha_i)}}$, where $i = 1, 2, 3, \ldots, n$

Proof: $\dfrac{P(x)}{Q(x)} \equiv \dfrac{c_1}{x-\alpha_1} + \dfrac{c_2}{x-\alpha_2} + \ldots + \dfrac{c_n}{x-\alpha_n}$; let's investigate the case when $i = 1$

× $(x-\alpha_i)$, i.e. $(x-\alpha_1)$ in this case: $P(x)\cdot\dfrac{x-\alpha_1}{Q(x)} \equiv c_1 + c_2\dfrac{x-\alpha_1}{x-\alpha_2} + \ldots + c_n\dfrac{x-\alpha_1}{x-\alpha_n}$

If we take the limit of the RHS as $x \to \alpha_1$: RHS = c_1

Taking the same limit of the LHS will give $\dfrac{P(\alpha_1)}{Q'(\alpha_1)}$; how does it work?

Notice, in the LHS, that $\dfrac{x-\alpha_1}{Q(x)} = \dfrac{x-\alpha_1}{Q(x)-Q(\alpha_1)}$, since $Q(\alpha_1)=0$

Also, $\lim\limits_{x\to\alpha_1} \dfrac{Q(x)-Q(\alpha_1)}{x-\alpha_1} = Q'(\alpha_1)$ by the first principle of differentiation

A Case Where $\deg P \geq \deg Q$

Perform division transformation with $D(x) = Q(x)$ before setting up the identity. For example, express $\dfrac{x^3-2x^2-2x-1}{x^2-2x-3}$ as a sum of partial fractions.

Using division transformation, $x^3-2x^2-2x-1 = (x^2-2x-3)x + (x-1)$. Hence, $\dfrac{x^3-2x^2-2x-1}{x^2-2x-3} = x + \dfrac{x-1}{x^2-2x-3}$, then $\dfrac{x-1}{x^2-2x-3} \equiv \dfrac{c_1}{x+1} + \dfrac{c_2}{x-3}$ and so on.

A Case Where Q(x) Has Irreducible Quadratic Factor(s) Over Real

Example: $\dfrac{x^2+1}{(x+2)(x-1)(x^2+x+1)} \equiv \dfrac{c_1}{x+2} + \dfrac{c_2}{x-1} + \dfrac{ax+b}{x^2+x+1}$

[Notice: either a or b may turn out to equal zero.]

[Remember that zeroes, factors and partial fractions depend on the number field asked. For example, there's no such thing as irreducible cubic factor over R, but there is over Q or Z. In that case, let the numerator be that cubic.]

[In general, let the degree of the numerator (of each fraction on the RHS of the identity) be one less than that of the corresponding denominator.]

A Case Where Q(x) Has Multiple Zero(es)

Example: $\dfrac{x^2-7x+4}{(x+1)(x-1)^2} \equiv \dfrac{a}{x+1} + \dfrac{bx+c}{(x-1)^2}$ (same zeroes must be grouped together) which then gives $\dfrac{3}{x+1} + \dfrac{-2x+1}{(x-1)^2} = \dfrac{3}{x+1} + \dfrac{(-2x+2)-(1)}{(x-1)^2} = \dfrac{3}{x+1} + \dfrac{-2}{(x-1)} + \dfrac{-1}{(x-1)^2}$.

Generally, $\dfrac{P(x)}{(x-\alpha_1)^n(x-\alpha_2)} \equiv \dfrac{c}{(x-\alpha_1)} + \dfrac{d}{(x-\alpha_1)^2} + \ldots + \dfrac{b}{(x-\alpha_1)^n} + \dfrac{a}{(x-\alpha_2)}$

[The syllabus says not to discuss this case but there have been related questions on it.]

(1) (i) Prove that if α is a double root of the equation $P(x)=0$ then $P'(\alpha)=0$. **2**

(ii) Prove that $P(x)=x^3-3px+q$ has a double root if $q^2=4p^3$. **3**

(iii) Hence or otherwise solve the equation **2**

$$x^3+3x+2i=0.$$

$\boxed{-i,-i,2i}$

(2) If a and b are two roots of the equation $x^3 + 4x - 2 = 0$, show that ab is a root of the equation $x^3 - 4x^2 - 4 = 0$. **4**

(3) (a) The complex number z and its conjugate $\bar{z}$ satisfy the equation $z\bar{z} + 2iz = 12 + 6i$. Find the possible values of z.

$\boxed{3-i, 3+3i}$

(b) $1+i$ is a root of the equation $x^2 + (a+2i)x + (5+ib) = 0$, where a and b are real. Find the values of a and b.

$\boxed{a=-3, b=-1}$

(4) (a) $1-2i$ is one root of the equation $x^2+(1+i)x+k=0$. Find the other root and the value of *k*.

$\boxed{k=5i, x=-2+i}$

(b) Find the zeros of $P(x)=x^4-4x^2+3=0$

(i) over **Q**;

$\boxed{\pm 1}$

(ii) over **R**;

$\boxed{\pm 1, \pm\sqrt{3}}$

(iii) over **C**,

$\boxed{\pm 1, \pm\sqrt{3}}$

(5) (a) Find $P(x)$, given that $P(x)$ is monic, of degree 3, with 5 as a single zero and -2 as a zero of multiplicity 2.

$$\boxed{P(x) = x^3 - x^2 - 16x - 20}$$

(b) $P(x)$ is an even monic polynomial of degree 4 with integer coefficients. If $\sqrt{2}$ is a zero, and the constant term is 6, factorise $P(x)$ fully over **R.**

$$\boxed{P(x) = (x - \sqrt{2})(x + \sqrt{2})(x - \sqrt{3})(x + \sqrt{3})}$$

(6) If $P(x) = x^3 - 3x^2 - 9x + c$ has a double zero, find two possible values of c and factorise $P(x)$ over the real numbers.

$c = 27 \quad P(x) = (x-3)^2(x+3)$

$c = -5 \quad P(x) = (x+1)^2(x-5)$

(7) If $ax^3 + cx + d = 0$ has a double root, show that $4c^3 + 27ad^2 = 0$.

(8) (a) When $P(x)=x^4+ax^2+2x$ is divided by x^2+1, the remainder is $2x+3$. Find the value of a.

$\boxed{a=-2}$

(b) When $P(x)=x^4+ax^2+bx+2$ is divided by x^2+1, the remainder is $-x+1$. Find the values of a and b.

$\boxed{a=2, b=-1}$

(9) (a) Two of the roots of $3x^3 + ax^2 + 23x - 6 = 0$ are reciprocals. Find the value of a and the three roots.

$$\boxed{a = -16;\ \text{roots are } 3, \frac{1}{3}, 2}$$

(10) The equation $px^3 + qx^2 + rx + s = 0$ has roots $(a - c)$, a, $(a + c)$, which are in arithmetic progression. Show that the $a = \frac{-q}{3p}$ and hence show that $2q^3 - 9pqr + 27p^2 s = 0$.

(11) The equation $px^3 + qx^2 + rx + s = 0$ has the roots *ac*, *a* and $\frac{a}{c}$, which are in geometric progression. Show that $a = \sqrt[3]{(-s/p)}$ and hence show that $pr^3 - q^3s = 0$.

(12) The equation $x^3 + x^2 - 2x - 3 = 0$ has roots α, β and γ. Find the equations with roots

(a) $\frac{\alpha}{2}, \frac{\beta}{2}$ and $\frac{\gamma}{2}$;

$$\boxed{8x^3 + 4x^2 - 4x - 3 = 0}$$

(b) $\alpha + 2, \beta + 2$ and $\gamma + 2$.

$$\boxed{x^3 - 5x^2 + 6x - 3 = 0}$$

(1) (i) Prove that if α is a double root of the equation $P(x)=0$ then $P'(\alpha)=0$. 2

$P(x) = \overset{u}{(x-\alpha)^2}\,\overset{v}{Q(x)}$. $vu'+uv'$.

$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x)$, ✓

$= (x-\alpha)[\,2Q(x) + (x-\alpha)\,Q'(x)\,]$. ✓

(ii) Prove that $P(x)=x^3-3px+q$ has a double root if $q^2=4p^3$. 3

$P'(x) = 3x^2 - 3p$.

$x = \sqrt{p}$ ✓

$q^2 = 4p^3$ ✓

$P(\sqrt{p}) = (\sqrt{p})^3 - 3p(\sqrt{p}) + q$.

$-q\,(= p\sqrt{p} - 3p\sqrt{p}$. ✓

$-a = -2p\sqrt{p}$.

(iii) Hence or otherwise solve the equation 2

$x^3+3x+2i=0$.

$3x^2+3=0$

$x = \pm i$.

$(x+i)(x+i) = x^2-1$.

$-i + -i + \alpha = 0$ ✓

$-2i + \alpha = 0$

$\alpha = 2i$ ✓

$\boxed{-i,-i,2i}$

---

(2) If a and b are two roots of the equation $x^3+4x-2=0$, show that ab is a root of the equation $x^3-4x^2-4=0$. 4

$3x^2+4=0$

$x =$

a, b, α

$ab\alpha = 2$.

$ab = \frac{2}{\alpha}$.

$\left(\frac{2}{x}\right)^3 - 4\left(\frac{2}{x}\right)^2 - 4 = 0$

$= \frac{8}{x^3} - \frac{16}{x^2} - 4$

$= \frac{8-16x-4x^3}{x^3}$.

$= \frac{2-4x-x^3}{x^3}$. $= \frac{-(\overset{\nearrow 0}{x^3+4x-2})}{x^3} = 0$

x^3. $x^3+4x-2=0$.

(3) (a) The complex number z and its conjugate $\bar{z}$ satisfy the equation $z\bar{z} + 2iz = 12 + 6i$. Find the possible values of z.

$(x+iy)(x-iy)$

x^2+y^2

$x^2+y^2+2i(x+iy) = 12+6i$

$x^2+y^2+2ix-2y = 12+6i$

$x^2+y^2-2y+2xi = 12+6i$

$2x = 6$

$x = 3$

$y^2-2y+9 = 12$

$y^2-2y-3 = 0$

$(y-3)(y+1) = 0$

$y = 3$ or -1

$\underline{3+3i}$ or $\underline{3-i}$

$\boxed{3-i,3+3i}$

(b) $1+i$ is a root of the equation $x^2+(a+2i)x+(5+ib)=0$, where a and b are real. Find the values of a and b.

$(1+i)^2+(a+2i)(1+i)+5+ib=0$

$1+2i-1+a(1+i)+2i(1+i)+5+ib=0$

$2i+a+ai+2i-2+5+ib=0$

$4i+a+ai+3+ib=0$

$i(4+a+b)+3+a=0$

$a=-3$

$4-3+b=0$

$1+b=0$

$b=-1$

$\boxed{a=-3,b=-1}$

(4) (a) $1-2i$ is one root of the equation $x^2+(1+i)x+k=0$. Find the other root and the value of k.

$1-2i,\ \beta$

$1-2i+\beta = -1-i$

$2-i+\beta$

$\beta = i-2$

$= \underline{-2+i}$

$(-2+i)(1-2i)$

$= -2(1-2i)+i(1-2i)$

$= -2+4i+i+2$

$= \underline{\underline{5i}}$

$\boxed{k=5i,x=-2+i}$

(b) Find the zeros of $P(x)=x^4-4x^2+3=0$

(i) over **Q**;

let $u=x^2$

u^2-4u+3

$(u-3)(u-1)=0$

$(x^2-3)(x^2-1)=0$

$(x^2-3)(x-1)(x+1)=0$

$\boxed{\pm 1}$

(ii) over **R**;

$(x-\sqrt{3})(x+\sqrt{3})(x-1)(x+1)=0$

$\boxed{\pm 1, \pm\sqrt{3}}$

(iii) over **C**;

$\boxed{\pm 1, \pm\sqrt{3}}$

(5) (a) Find $P(x)$, given that $P(x)$ is monic, of degree 3, with 5 as a single zero and -2 as a zero of multiplicity 2.

$ax^3+bx^2+cx+d=0$.

$-b = 5+-2+-2$

$= 1$ ✓

$b=-1$.

$c = 5(-2)+5(-2)+(-2)(-2)$

$= -16$.

$d = 5(-2)(-2)$

$= 20$. ✓

$x3-x^2-16x+20 = P(x)$. ✓

$\boxed{P(x)=x^3-x^2-16x-20}$

(b) $P(x)$ is an even monic polynomial of degree 4 with integer coefficients. If $\sqrt{2}$ is a zero, and the constant term is 6, factorise $P(x)$ fully over **R.**

$(x-\sqrt{2})(x+\sqrt{2}) = \quad ax^4+bx^3+cx^2+dx+e$.

$\alpha\beta\gamma\delta = 6$

$2\gamma\delta = 6$. ✓

$\gamma\delta = 3$.

$(x-\sqrt{2})(x+\sqrt{2})(x-\sqrt{3})(x+\sqrt{3}]$. ✓

$\boxed{P(x)=(x-\sqrt{2})(x+\sqrt{2})(x-\sqrt{3})(x+\sqrt{3})}$

(6) If $P(x)=x^3-3x^2-9x+c$ has a double zero, find two possible values of c and factorise $P(x)$ over the real numbers.

$P'(x)=3x^2-6x-9$.

x^2-2x-3.

$(x-3)(x+1)$. ✓

$x=3$ or -1.

when $x=3$:

$3=3+3+B$.

$3=6+B$. ✓

$B=-3$

$-c = 3\cdot 3\cdot -3$

$-c = -27$ ✓

$c=27$.

when $x=-1$.

$3=-1+-1+B$

$3=-2+B$

$B=5$.

$-c=(-1)(-1)B$ ✓

$-c = -5$

$c=-5$.

$P(x)=(x-3)^2(x+3)$. $c=27$

$P(x)=(x+1)^2(x-5)$ ✓ $c=-5$.

$$\boxed{\begin{array}{ll} c=27 & P(x)=(x-3)^2(x+3) \\ c=-5 & P(x)=(x+1)^2(x-5) \end{array}}$$

(7) If $ax^3+cx+d=0$ has a double root, show that $4c^3+27ad^2=0$.

$P(x) = ax^3 + cx + d$.

$P'(x) = 3ax^2 + c$.

$3ax^2 + c = 0$.

$x = \sqrt{\frac{-c}{3a}}$ ✓

$P\left(\sqrt{\frac{-c}{3a}}\right) = a\left(\sqrt{\frac{-c}{3a}}\right)^3 + c\left(\sqrt{\frac{-c}{3a}}\right) + d = 0.$

$-\frac{\cancel{a}c}{3\cancel{a}}\left(\sqrt{\frac{-c}{3a}}\right) + c\left(\sqrt{\frac{-c}{3a}}\right) = -d.$

$\sqrt{\frac{-c}{3\cancel{a}}}\left(\frac{-c}{3} + c\right) = -d.$ ✓

$\sqrt{\frac{-c}{3\cancel{a}}}\left(\frac{2c}{3}\right) = -d.$

$-\frac{c}{3\cancel{a}}\left(\frac{4c^2}{9}\right) = d^2.$

$-\frac{4c^3}{27} = d^2.$

$-4c^3 = 27d^2$ ✓

$27d^2 + 4c^3 = 0$

---

(8) (a) When $P(x)=x^4+ax^2+2x$ is divided by x^2+1, the remainder is $2x+3$. Find the value of a.

1
1 2 1
1 3 3 1
1 4 6 4 1

$$x^2+1 \overline{\smash{)}\, x^4 + ax^2 + 2x} \quad \text{quotient: } x^2+(a-1)x$$

$x^4 + x^2$

$(a-1)x^2$ ✓

$(a-1)x^2 + a - 1$

$-(a-1) + 2x$

$-a + 1 + 2x = 2x + 3$

$-a = 2.$

$a = -2$ ✓

$\boxed{a=-2}$

(b) When $P(x)=x^4+ax^2+bx+2$ is divided by x^2+1, the remainder is $-x+1$. Find the values of a and b.

$$x^2+1 \overline{\smash{)}\, x^4 + ax^2 + bx + 2} \quad \text{quotient: } x^2+(a-1)+(1-(a-1))$$

$x^4 + x^2$

$(a-1)x^2$ ✓

$(a-1)x^2 + (a-1)$

$2-(2+(a+1)) - bx$

$2 - a + 1 - bx.$

$3 - a - bx = 1 - x.$ ✗

$3 - a = 1.$ $b = 1$

$a = 2.$

Try this quicker method

$P(x) = Q(x).(x^2+1) \underbrace{-x+1}_{R(x)}$

$\therefore (i)^4 + a(i^2) + bi + 2 = -i + 1$

$1 - a + bi + 2 = -i + 1$

Equating Re & Im

$1 - a + 2 = 1$

$\therefore \underline{a = 2}$

$\underline{b = -1}$

$\boxed{a=2, b=-1}$

(9) (a) Two of the roots of $3x^3+ax^2+23x-6=0$ are reciprocals. Find the value of a and the three roots.

$\alpha, \frac{1}{\alpha}, \beta.$

Sum : $\frac{\alpha^2+1+\alpha\beta}{\alpha}.$

$=$

prod: $\beta = 2$:

~~$-\alpha$~~

$\alpha(\frac{1}{\alpha})+\alpha\beta+\frac{1}{\alpha}\beta = \frac{23}{3}$

$1+2\alpha+\frac{2}{\alpha}=\frac{23}{3}$

$2\alpha+\frac{2}{\alpha}=\frac{20}{3}.$

$\frac{2\alpha^2+2}{\alpha}=\frac{20}{3}.$

$6\alpha^2+6=20\alpha$

$6\alpha^2-20\alpha+6=0$

$\alpha=\frac{20\pm\sqrt{20^2-4(6)(6)}}{12}$

$=\frac{20\pm\sqrt{16}}{12}$

$=3$ or. $1/3$.

∴ The roots are $3, 1/3, 2$.

sum $= 16/3$.

$a = -16$.

$\boxed{a=-16;\text{ roots are }3,\frac{1}{3},2}$

---

(10) The equation $px^3+qx^2+rx+s=0$ has roots $(a-c)$, a, $(a+c)$, which are in arithmetic progression. Show that the $a=\frac{-q}{3p}$ and hence show that $2q^3-9pqr+27p^2s=0$.

$a-c+a+a+c$

$3a = -q/p$.

$a = -q/3p$.

$p\left(\frac{-q}{3p}\right)^3+q\left(\frac{-q}{3p}\right)^2+r\left(\frac{-q}{3p}\right)+s=0.$

$\frac{-q^3p}{27p^3}+\frac{q^3}{9p^2}+\frac{-qr}{3p}+s=0.$

$\frac{-q^3}{27p^2}+\frac{q^3}{9p^2}-\frac{qr}{3p}+s=0.$

$-q^3+3q^3-9pqr+27p^2s=0$

$2q^3-9pqr+27p^2s=0$.

(11) The equation $px^3+qx^2+rx+s=0$ has the roots ac, a and $\frac{a}{c}$, which are in geometric progression. Show that $a=\sqrt[3]{(-s/p)}$ and hence show that $pr^3-q^3s=0$.

~~a+t~~

product of roots = a^3.

$a^3=-s/p$.

$a=\sqrt[3]{-s/p}$ ✓

$p(-s/p)+q\left(\sqrt[3]{-s/p}\right)^2+r\left(\sqrt[3]{-s/p}\right)+s=0$.

$q(-s/p)^{2/3}+r(-s/p)^{1/3}=0$.

$(-s/p)^{1/3}\left[q(-s/p)^{1/3}+r\right]=0$ ✓

$\therefore q(-s/p)^{1/3}=-r$.

$q^3(-s/p)=-r^3$.

$\frac{-sq^3}{p}=-r^3$.

$-sq^3=-pr^3$ ✓

$pr^3-sq^3=0$.

(12) The equation $x^3+x^2-2x-3=0$ has roots α,β and γ. Find the equations with roots

(a) $\frac{\alpha}{2},\frac{\beta}{2}$ and $\frac{\gamma}{2}$;

$\frac{X}{2}=x$.

$X=2x$. ✓

$(2x)^3+(2x)^2-2(2x)-3$

$=8x^3+4x^2-4x-3$ ✓.

$\boxed{8x^3+4x^2-4x-3=0}$

(b) $\alpha+2,\beta+2$ and $\gamma+2$.

$X+2=x$.

$X=x-2$. ✓

$(x-2)^3+(x-2)^2-2(x-2)-3=0$.

$x^3-3x^2(2)+3(x)(4)-(8)+(x^2-4x+4)-2x+4-3=0$

$x^3-6x^2+12x-8+x^2$ ✓ $-4x+4-2x+4-3=0$.

$x^3-5x^2+6x-3=0$.

$\boxed{x^3-5x^2+6x-3=0}$