NAME :

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YEAR 12 – MATHS EXT.2

REVIEW TOPIC (PAPER 1): REDUCTION FORMULA

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CSSA 2000 Q7

(b) (i) If $I_n = \int_0^1 (x^2-1)^n \, dx$, $n = 0, 1, 2, \ldots$, show that 8

$I_n = \frac{-2n}{2n+1} I_{n-1}$, $n = 1, 2, 3, \ldots$

(ii) Hence use the method of Mathematical Induction to show that $I_n = \dfrac{(-1)^n\, 2^{2n}\, (n!)^2}{(2n+1)\,!}$

for all positive integers n.

CSSA 2001 Q2

(d) (i) If $I_n = \int_0^1 (1+x^2)^n \, dx$, $n = 0, 1, 2, \ldots$ show that $(2n+1) I_n = 2^n + 2n I_{n-1}$ 3

for $n = 1, 2, 3, \ldots$

(ii) Hence find a reduction formula for $J_m = \int_0^{\frac{\pi}{4}} \sec^{2m} x \, dx$ 2

NEAP 2001 Q7

(a) (i) Show that $\frac{t^n}{1+t^2} = t^{n-2} - \frac{t^{n-2}}{1+t^2}$. 1

(ii) Let $I_n = \int \frac{t^n}{1+t^2} dt$. 1

Show that $I_n = \frac{t^{n-1}}{n-1} - I_{n-2}$, $n \geq 2$.

(iii) Show that $\int_0^1 \frac{t^6}{1+t^2}\,dt = \frac{13}{15} - \frac{\pi}{4}$. **3**

RC 2002 Q1

b) i) If $C_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$ then show that $C_n = \frac{n-1}{n} C_{n-2}$. 4

ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^5 dx$ 2

S&G 2001 Q7

b) Let $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \text{cosec}^n x \, dx$, where n is a positive integer.

i) Using integration, show that 4

$$(n-1)\, I_n = 2^{n-2}\sqrt{3} + (n-2)\, I_{n-2}$$

ii) Evaluate $J = \int_0^{\frac{\pi}{3}} \sec^4 x \, dx$ 2

SBHS 2002 Q1

(c) Given that $I_n = \int_1^e (\ln x)^n \, dx, \quad n = 0, 1, 2, \ldots,$ **3**

show that $I_n = e - nI_{n-1}$.

SOLUTIONS

CSSA 2000 Q7

(b)(i) For $n = 1,\ 2,\ 3, \ldots$

$$I_n = \int_0^1 \left(x^2 - 1\right)^n \; dx$$

$$= \left[x\left(x^2-1\right)^n\right]_0^1 - 2n\int_0^1 x^2\left(x^2-1\right)^{n-1} \; dx$$

$$= 0 \; - 2n\int_0^1 \left(x^2-1+1\right)\left(x^2-1\right)^{n-1} \; dx$$

$$= -2n\int_0^1 \left(x^2-1\right)^n + \left(x^2-1\right)^{n-1} \; dx$$

$$= -2n\left(I_n + I_{n-1}\right)$$

$$(2n+1)\, I_n = -2n\, I_{n-1}$$

$$I_n = \frac{-2n\, I_{n-1}}{2n+1}$$

(ii) $I_0 = \int_0^1 1 \; dx = 1 \Rightarrow I_1 = \frac{-2}{2+1}\; I_0 = -\frac{2}{3}$

For $n = 1,\ 2,\ 3, \ldots$

let $S(n)$ be the statement

$$I_n = \frac{(-1)^n\, 2^{2n}\,(n!)^2}{(2n+1)!}$$

$$\frac{(-1)^1\, 2^2\,(1!)^2}{(2+1)!} = \frac{-4}{3\times 2} = -\frac{2}{3} = I_1$$

$\therefore\ S(1)$ is true.

(b) (ii) (continued)

If $S(k)$ is true, $I_k = \frac{(-1)^k\, 2^{2k}\,(k!)^2}{(2k+1)!}$ **

Consider $S(k+1)$, k some positive integer.

$$I_{k+1} = \frac{-2(k+1)}{2(k+1)+1}\; I_{n-1}$$

$$= \frac{-2(k+1)}{2(k+1)+1}\cdot\frac{(-1)^k 2^{2k}(k!)^2}{(2k+1)!}, \quad \text{if } S(k) \text{ is true using ** }$$

$$= \frac{(-1)^{k+1}2^{2k+1}(k+1)(k!)^2}{(2k+3)(2k+1)!}$$

$$= \frac{(-1)^{k+1}2^{2k+1}(2k+2)(k+1)(k!)^2}{(2k+3)(2k+2)(2k+1)!}$$

$$= \frac{(-1)^{k+1}2^{2(k+1)}(k+1)^2(k!)^2}{(2k+3)!}$$

$$= \frac{(-1)^{k+1}2^{2(k+1)}\{(k+1)!\}^2}{\{2(k+1)+1\}!}$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true and then $S(3)$ is true, and so on. By Mathematical Induction, $S(n)$ is true for $n = 1,\ 2,\ 3, \ldots$

Hence $I_n = \frac{(-1)^n\, 2^{2n}\,(n!)^2}{(2n+1)!}$ for all positive integers n.

CSSA 2001 Q2

(i)

$$I_n = \int_0^1 \left(1+x^2\right)^n \, dx$$

$$= \left[x\left(1+x^2\right)^n\right]_0^1 - \int_0^1 x \, . \, n\left(1+x^2\right)^{n-1} . \, 2x \, dx$$

$$= 2^n - 2n\int_0^1 x^2\left(1+x^2\right)^{n-1} \, dx$$

$$= 2^n - 2n\int_0^1 \left(1+x^2-1\right)\left(1+x^2\right)^{n-1} \, dx$$

$$= 2^n - 2n\left\{\int_0^1 \left(1+x^2\right)^n \, dx - \int_0^1 \left(1+x^2\right)^{n-1} \, dx\right\}$$

$$I_n = 2^n - 2n\, I_n + 2n\, I_{n-1}$$

$$\therefore \left(2n+1\right) I_n = 2^n + 2n\, I_{n-1} \, , \quad n = 1, 2, 3, \ldots$$

(ii)

$u = \tan x \qquad x = 0 \Rightarrow u = 0$

$du = \sec^2 x \, dx \qquad x = \frac{\pi}{4} \Rightarrow u = 1$

$$J_m = \int_0^{\frac{\pi}{4}} \sec^{2m} x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\sec^2 x\right)^{m-1} . \, \sec^2 x \, dx$$

$$= \int_0^1 \left(1+u^2\right)^{m-1} \, du$$

$$\therefore \; J_m = I_{m-1} \, , \quad m = 1, 2, 3, \ldots$$

$$\left\{2\left(m-1\right)+1\right\} J_m = 2^{m-1} + 2\left(m-1\right) I_{m-2}$$

$$\therefore \quad \left(2m-1\right) J_m = 2^{m-1} + 2\left(m-1\right) J_{m-1}$$

$$m = 2, \; 3, \; 4, \ldots$$

NEAP 2001 Q7

(a) (i) $\text{RHS} = t^{n-2} - \frac{t^{n-2}}{1+t^2}$

$= \frac{(1+t^2)t^{n-2} - t^{n-2}}{1+t^2}$

$= \frac{t^{n-2} + t^n - t^{n-2}}{1+t^2}$

$= \frac{t^n}{1+t^2}$

$= \text{LHS}$ ✓

(ii) $I_n = \int \frac{t^n}{1+t^2}dt$

$= \int \left(t^{n-2} - \frac{t^{n-2}}{1+t^2}\right)dt$

$= \frac{t^{n-1}}{n-1} - \int \frac{t^{n-2}}{1+t^2}dt$

$= \frac{t^{n-1}}{n-1} - I_{n-2}$ ✓

(iii) Let $J_n = \int_0^1 \frac{t^n}{1+t^2}dt$

Then $J_n = \left[\frac{t^{n-1}}{n-1}\right]_0^1 - J_{n-2}$

$= \frac{1}{n-1} - J_{n-2}$ ✓

Hence $J_6 = \frac{1}{5} - J_4$

$= \frac{1}{5} - \frac{1}{3} + J_2$

$= \frac{1}{5} - \frac{1}{3} + 1 - J_0$ ✓

But $J_0 = \int_0^1 \frac{1}{1+t^2}dt$

$= \left[\tan^{-1} t\right]_0^1 = \frac{\pi}{4}$

Hence $J_6 = \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}$

$= \frac{13}{15} - \frac{\pi}{4}$ ✓

RC 2002 Q1

b) i)

$$C_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x\,dx \qquad \text{Let } u = \cos^{n-1} x, \quad v' = \cos x \qquad \therefore v = \sin x$$

$$= \left[\sin x \cos^{n-1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x (n-1)\cos^{n-2} x(-\sin x)\,dx \qquad 1$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x\,dx$$

1

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x\left(1-\cos^2 x\right)dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx \qquad 1$$

$$= \sin x \cos^{n-1} x + (n-1)C_{n-2} - (n-1)C_n$$

$$nC_n = (n-1)C_{n-2}$$

$$C_n = \frac{n-1}{n} C_{n-2} \qquad 1$$

ii)

$$C_5 = \frac{4}{5} C_3$$

$$= \frac{4}{5}\left(\frac{2}{3} C_1\right) \qquad 1$$

$$= \frac{8}{15}\int_0^{\frac{\pi}{2}} \cos x\,dx$$

$$= \frac{8}{15}\left[\sin x\right]_0^{\frac{\pi}{2}} \qquad 1$$

$$= \frac{8}{15}$$

S&G 2001 Q7

b) $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^n x \, dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx$

N.B: $y = \cot x$

$\frac{dy}{dx} = \frac{\cos x}{\sin x}$

$\therefore \frac{dy}{dx} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$

$= \frac{-1}{\sin^2 x}$

$= -\operatorname{cosec}^2 x$

Using by parts:

let $u = \operatorname{cosec}^{n-2} x$

$du = -(n-2) \operatorname{cosec}^{n-3} x \cot x \operatorname{cosec} x \, dx$

$dv = \operatorname{cosec}^2 x \, dx \quad \therefore v = -\cot x$

$\therefore I_n = [-\cot x \operatorname{cosec}^{n-2} x]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$- (n-2) \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^{n-2} x \cot^2 x \, dx$

$= 2^{n-2}\sqrt{3} - (n-2) \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^n x \, dx - \int \operatorname{cosec}^{n-2} x \, dx \right]$ (since $\cot^2 x = \operatorname{cosec}^2 x - 1$)

$\therefore I_n = 2^{n-2}\sqrt{3} - (n-2)(I_n - I_{n-2})$

$= 2^{n-2}\sqrt{3} - (n-2) I_n + (n-2) I_{n-2}$

$\therefore (n-1) I_n = 2^{n-2}\sqrt{3} + (n-2) I_{n-2}$.

(4 marks)

ii) $\int_0^{\frac{\pi}{3}} \sec^4 x \, dx$

let $u = \frac{\pi}{2} - x$, $\sec(\frac{\pi}{2} - u)$

$-du = dx$ $= \operatorname{cosec} u$

$\therefore J = \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} -\operatorname{cosec}^4 u \, du$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^4 u \, du = I_4$

From i) $(n-1) I_n = 2^{n-2}\sqrt{3} + (n-2) I_{n-2}$

For $n=4$, $3I_4 = 4\sqrt{3} + 2I_2$.

For $n=2$, $I_2 = \sqrt{3}$

$\therefore 3I_4 = 4\sqrt{3} + 2\sqrt{3} = 6\sqrt{3}$

$\therefore I_4 = 2\sqrt{3}$

$\therefore \int_0^{\frac{\pi}{3}} \sec^4 x \, dx = 2\sqrt{3}$ (2 marks)

SBHS 2002 Q1

(c) $I_n = \int_1^e (\ln x)^n \, dx$

$= \int_1^e (\ln x)^n \cdot \frac{d(x)}{dx} dx$ ($\frac{1}{2}$)

$= \left[x \ln x\right]_1^e - \int_1^e x \cdot n(\ln x)^{n-1} \cdot \frac{1}{x}$

$= e - n\int_1^e (\ln x)^{n-1} dx$

$\therefore I_n = e - nI_{n-1}$