NAME :

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YEAR 12 – MATHS EXT.2

REVIEW TOPIC (PAPER 1): RESISTED & CIRCULAR MOTION

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CSSA 2000

6.

A toy of mass m kg has a parachute device attached. It is released from rest at the top of a vertical cliff 40 m high. During its fall, the forces acting are gravity and, owing to the parachute, a resistance force of magnitude $\frac{1}{10}mv^2$ when the speed of the toy is v ms^{-1}. After $2\ln 2$ seconds, the parachute disintegrates, and then the only force acting on the toy is gravity. The acceleration due to gravity is taken as $g = 10$ ms^{-2}. At time t seconds, the toy has fallen a distance x metres from the top of the cliff, and its speed is v ms^{-1}. 15

(i) Show that while the parachute is operating, $10\ddot{x} = 100 - v^2$. Hence show that

$$v = 10\left(\frac{e^{2t}-1}{e^{2t}+1}\right) \quad \text{and} \quad x = -5\ln\left\{1-\left(\frac{v}{10}\right)^2\right\}.$$

(ii) Find the exact speed of the toy and the exact distance fallen just before the parachute disintegrates.

(iii) After the parachute disintegrates, find an expression for $\ddot{x}$ and use integration to find the speed of the toy just before it reaches the base of the cliff. Give your answer correct to 2 significant figures.

CSSA 2001

6.

An object of mass m kg is dropped from rest from the top of a cliff 40 m above the water. Before the object reaches the water, the resistance to its motion has magnitude $\frac{1}{10}mv$ when the object has speed v ms^{-1}. After the object enters the water, the resistance to its motion has magnitude $\frac{1}{10}mv^2$. Take $g = 10\text{ms}^{-2}$.

(a) (i) Write an expression for $\ddot{x}$ before the object enters the water, where x metres is the distance the object has fallen in t seconds. **1**

(ii) Show $10\frac{dv}{dx} = \frac{100 - v}{v}$, and show that the speed of the object as it enters the water is V ms^{-1} where V satisfies $\frac{V}{100} + \ln\left(1 - \frac{V}{100}\right) + 0 \cdot 04 = 0$. **3**

(iii) Show this equation has a solution for V between 20 and 30, and taking 25 as a first approximation, use Newton's Method to show that $V \approx 25 \cdot 7$ to one decimal place. 3

(b) (i) Write an expression for $\ddot{x}$ after the object enters the water. Deduce the object slows on entry to the water, and find its terminal velocity in the water. 3

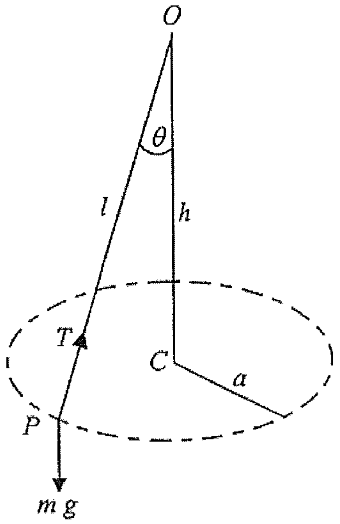
(ii) Show that t seconds after entering the water $10\frac{dv}{dt} = 100 - v^2$, and the velocity v ms $^{-1}$ of the object is given by $2t = \ln\left\{\frac{(v+10)(V-10)}{(v-10)(V+10)}\right\}$, where V is the velocity on entry to the water calculated in (a). 3

(iii) How long after it enters the water will the body slow to 105% of its terminal velocity ? 2

HEFFERNAN 2002

4.

(b)



A particle P is attached to one end of a string of length l metres and moves in a horizontal circle of radius a and centre C at ω radians per second where $\omega = 60\pi$. The other end of the string is attached at point O which is vertically above point C. The angle COP is θ, the tension in the string is T newtons and the particle is subject to a gravitational force of mg newtons. When the particle is moving at 30 revolutions per second, the height of O above C is h metres.

(i) Show that $h = \dfrac{g}{\omega^2}$ 3

(ii) If the particle starts to move at 60 revolutions per second, find the height, h_1, of O above C in terms of h. **2**

HEFFERNAN 2002

6.

(b) A particle of mass 4kg is projected vertically upwards from a platform attached to the side of a city skyscraper with an initial speed of 10 metres per second.
The particle is subjected to a downwards gravitational force of 40 newtons and air resistance of $\frac{v^2}{10}$ newtons in the opposite direction to the velocity, v, metres per second. The height of the particle at time t seconds is y metres.

(i) Explain why the equation of motion of the particle, until it reaches its maximum height, is given by $\ddot{y} = -10 - \frac{v^2}{40}$. 1

(ii) Given that $v^2 = 100\left(5e^{\frac{-y}{20}} - 4\right)$, $t \geq 0$ until the particle reaches its maximum height, find the maximum height. 1

(iii) What was the speed of the particle as it returned to the platform? 3

(iv) If the platform were to be removed by the time the particle returned to its point of projection, find the terminal velocity of the particle. 3

(v) How long did it take the particle to fall from its maximum height to the position where the platform had been? **1**

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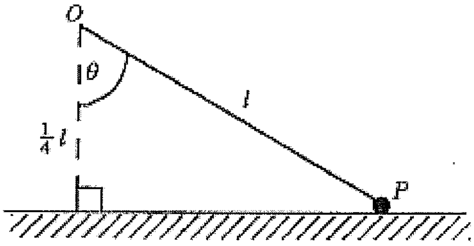
6.

(b)



One end of a light inextensible string of length l is attached to a fixed point O which is at a height $\frac{1}{4}l$ above a smooth horizontal table. A particle P of mass m is attached to the other end of the string and rests on the table with the string taut. The particle is projected so that it moves in a circle on the table with constant speed v.

(i) Copy the diagram. Show the forces acting on the particle P. 1

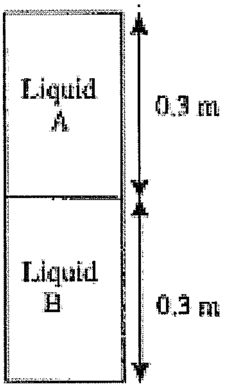
(ii) Show that the tension in the string has magnitude $T = \dfrac{16mv^2}{15l}$. 2

(iii) Show that the reaction R exerted by the table on P has magnitude $R = m\left(g - \dfrac{4v^2}{15l}\right)$. **2**

(iv) Hence show that $v \leq \sqrt{\dfrac{15gl}{4}}$. Explain what would happen at a higher speed. **3**

JAMES RUSE 2000

6.



A container is filled with liquid A of height 0.3 m on top of liquid B of height 0.3 m.

A steel ball of mass 10 grams is released from rest at the top of liquid A.

It falls experiencing a resistive force in liquid A of $0.04v^2$ Newtons and a resistive force of $0.05v$ Newtons in liquid B, where v is the velocity (m/s) of the steel ball.

Assuming that no mixing of the liquids occurs, and the acceleration due to gravity is 10 m/s^2 then

(i) show that the velocity of the steel ball when it passes from liquid A to liquid B is 1.51 m/s.

(ii) show that the final velocity of the steel ball satisfies the equation: $v + 2\ln(2 - v) + 1.42 = 0$

(iii) show that the final velocity is approximately 1.80 m/s

(iv) find the total time to reach the bottom of liquid B.

SOLUTIONS

CSSA 2000 Q6

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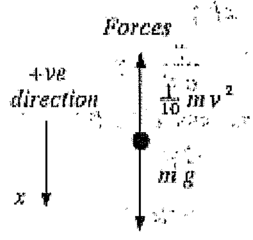
) While parachute is operating .



Initial conditions

$$t = 0 \Rightarrow \begin{cases} x = 0, \\ v = 0 \end{cases}$$

Equation of motion

$$m \ddot{x} = m g - \tfrac{1}{10} m v^2$$
$$\ddot{x} = 10 - \tfrac{1}{10} v^2$$
$$10 \ddot{x} = 100 - v^2$$

o find x in terms of v :

$$5 \frac{dv^2}{dx} = 100 - v^2$$
$$\frac{1}{5} \frac{dx}{d\left(v^2\right)} = \frac{1}{100 - v^2}$$
$$-\tfrac{1}{5} x = \ln\left\{\left(100 - v^2\right) B\right\}, \quad B \text{ constant}$$
$$\left.\begin{matrix} x = 0 \\ v = 0 \end{matrix}\right\} \Rightarrow \ln(100 B) = 0 \Rightarrow B = \tfrac{1}{100}$$
$$-\tfrac{1}{5} x = \ln\left\{\tfrac{1}{100}\left(100 - v^2\right)\right\}$$
$$x = -5 \ln\left\{1 - \left(\tfrac{v}{10}\right)^2\right\}$$

To find v in terms of t :

$$10 \frac{dv}{dt} = 100 - v^2$$
$$\frac{1}{10} \frac{dt}{dv} = \frac{1}{(10 + v)(10 - v)}$$
$$2 \frac{dt}{dv} = \frac{1}{(10 + v)} + \frac{1}{(10 - v)}$$
$$2t = \ln\left\{\frac{10 + v}{10 - v} A\right\}, \quad A \text{ constant.}$$
$$\left.\begin{matrix} t = 0 \\ v = 0 \end{matrix}\right\} \Rightarrow \ln A = 0 \Rightarrow A = 1$$
$$2t = \ln\left\{\frac{10 + v}{10 - v}\right\}$$
$$e^{2t} = \frac{10 + v}{10 - v}$$
$$e^{2t}(10 - v) = 10 + v$$
$$10\left(e^{2t} - 1\right) = v\left(e^{2t} + 1\right)$$
$$v = 10\left(\frac{e^{2t} - 1}{e^{2t} + 1}\right)$$

) Parachute disintegrates when $t = 2 \ln 2$

$$v = 10\left(\frac{e^{2t} - 1}{e^{2t} + 1}\right)$$
$$t = 2 \ln 2 \Rightarrow v = 10\left(\frac{2^4 - 1}{2^4 + 1}\right) = \tfrac{150}{17}$$
$$x = -5 \ln\left\{1 - \left(\tfrac{v}{10}\right)^2\right\}$$
$$t = 2 \ln 2 \Rightarrow x = -5 \ln\left\{1 - \left(\tfrac{15}{17}\right)^2\right\} = 10 \ln \tfrac{17}{8}$$

Just before parachute disintegrates, speed is $\frac{150}{17}$ ms^{-1}, and distance fallen is $10 \ln \frac{17}{8}$ m.

(iii) After parachute disintegrates, $\ddot{x} = 10$.

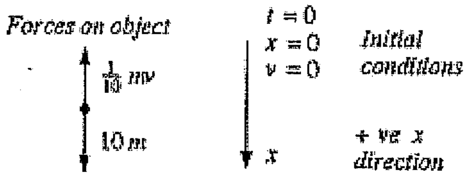
$$\frac{1}{2} \frac{dv^2}{dx} = 10$$
$$v^2 = 20x + c, \quad c \text{ constant}$$
$$\left.\begin{matrix} v = \tfrac{150}{17} \\ x = 10 \ln \tfrac{17}{8} \end{matrix}\right\} \Rightarrow \left(\tfrac{150}{17}\right)^2 = 200 \ln \tfrac{17}{8} + c$$
$$v^2 - \left(\tfrac{150}{17}\right)^2 = 20x - 200 \ln \tfrac{17}{8}$$
$$x = 40 \Rightarrow v^2 = \left(\tfrac{150}{17}\right)^2 + 800 - 200 \ln \tfrac{17}{8}$$
$$v = 26 \cdot 96$$

At base of cliff, speed is 27ms^{-1} (to 2 sig.fig).

CSSA 2001 Q6

(a)

i)



$m\ddot{x} = 10m - \frac{1}{10}mv \qquad \therefore\ \ddot{x} = 10 - \frac{1}{10}v$

ii) $\ddot{x} = v\frac{dv}{dx} = 10 - \frac{1}{10}v \Rightarrow 10\frac{dv}{dx} = \frac{100-v}{v}$

$\frac{-1}{10}\frac{dx}{dv} = \frac{-v}{100-v} = 1 + \frac{-100}{100-v}$

$-\frac{1}{10}x = v + 100\ln(100-v) + c$, c constant

$t=0, x=0, v=0 \Rightarrow c = -100\ \ln 100$

$\therefore -\frac{1}{10}x = v + 100\ln\left(1 - \frac{v}{100}\right)$

$\left.\begin{matrix} t=40 \\ v=V \end{matrix}\right\} \Rightarrow \begin{matrix} -4 = V + 100\ln\left(1-\frac{V}{100}\right) \\ -0\cdot 04 = \frac{V}{100} + \ln\left(1-\frac{V}{100}\right) \end{matrix}$

$\therefore$ Speed V ms^{-1} just before entering water satisfies

$\frac{V}{100} + \ln\left(1-\frac{V}{100}\right) + 0\cdot 04 = 0 \quad$ **

(iii)

Let $\lambda = \frac{V}{100}$, $f(\lambda) = \lambda + \ln(1-\lambda) + 0\cdot 04$

$f(0\cdot 2) = 0\cdot 02 > 0 \qquad f(0\cdot 3) = -0\cdot 02 < 0$

and $f(\lambda)$ is a continuous function. Hence $f(\lambda) = 0$ has a solution for λ between $0\cdot 2$ and $0\cdot 3$, and ** has a solution for V between 20 and 30. Using Newton's Method with a first approximation $\lambda = 0\cdot 25$ $(V = 25)$

$$f(\lambda) = \lambda + \ln(1-\lambda) + 0\cdot 04$$

$$f'(\lambda) = 1 - \frac{1}{1-\lambda} = \frac{-\lambda}{1-\lambda}$$

$$\frac{f(\lambda)}{f'(\lambda)} = \{\lambda + \ln(1-\lambda) + 0\cdot 04]\left(\frac{1-\lambda}{-\lambda}\right)$$

$$= \lambda - 1 - \frac{(1-\lambda)\{\ln(1-\lambda) + 0\cdot 04]}{\lambda}$$

$$\lambda - \frac{f(\lambda)}{f'(\lambda)} = 1 + \frac{(1-\lambda)\{\ln(1-\lambda) + 0\cdot 04]}{\lambda}$$

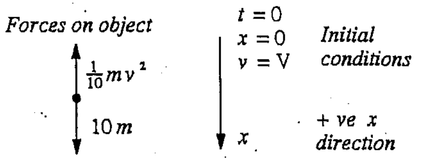
λ	$1 + \frac{1-\lambda}{\lambda}\{\ln(1-\lambda) + 0\cdot 04\}$
$0\cdot 25$	$1 + 3(\ln 0\cdot 75 + 0\cdot 04) = 0\cdot 257$
$0\cdot 257$	$1 + \frac{0\cdot 743}{0\cdot 257}(\ln 0\cdot 743 + 0\cdot 04) = 0\cdot 257$

Hence $\lambda = 0\cdot 257 \Rightarrow V = 25\cdot 7$ to one decimal place.

(b)

Answer

(i) After entering the water,



$m\ddot{x} = 10m - \frac{1}{10}mv^2 \qquad \therefore\ \ddot{x} = 10 - \frac{1}{10}v^2$

$\ddot{x} = 10 - \frac{1}{10}V^2 < 0$ and $\dot{x} = V > 0$

Hence object slows on entry to the water.

$\ddot{x} \to 0$ as $v \to 10$

Hence terminal velocity in the water is 10 ms^{-1}.

(ii) $\ddot{x} = \frac{dv}{dt} = 10 - \frac{1}{10}v^2 \Rightarrow 10\frac{dv}{dt} = 100 - v^2$

$$\frac{1}{10}\frac{dt}{dv} = \frac{1}{(10+v)(10-v)}$$

$$= \frac{1}{20}\left\{\frac{1}{(10+v)} + \frac{1}{(10-v)}\right\}$$

$$2\frac{dt}{dv} = \frac{1}{(v+10)} - \frac{1}{(v-10)}$$

$$2t = \ln\left\{\frac{(v+10)}{(v-10)}A\right\}, \ A \text{ constant}$$

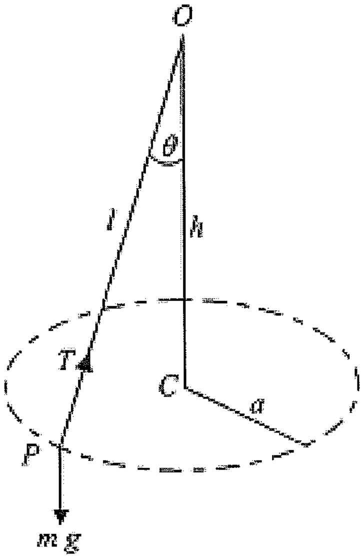
$$\left.\begin{matrix} t=0 \\ v=V \end{matrix}\right\} \Rightarrow \frac{(V+10)}{(V-10)}A = 1 \Rightarrow A = \frac{(V-10)}{(V+10)}$$

$$\therefore\ 2t = \ln\left\{\frac{(v+10)(V-10)}{(v-10)(V+10)}\right\}$$

(iii) $v = 105\%$ of $10 \Rightarrow v = 10\cdot 5$ and $2t = \ln\left\{\frac{(20\cdot 5)(15\cdot 7)}{(0\cdot 5)(35\cdot 7)}\right\} \Rightarrow t = 1\cdot 4.$

Hence particle slows to 105% of its terminal velocity $1\cdot 4$ seconds after entering the water.

HEFFERNAN 2002 Q4



(b) (i) By resolving forces we obtain

$$T\cos\theta = mg \qquad \text{(vertical)}$$

$$T\sin\theta = ma\omega^2 \qquad \text{(radial)}$$

(1 mark)

Now $$\frac{T\sin\theta}{T\cos\theta} = \frac{ma\omega^2}{mg}$$

$$\tan\theta = \frac{a\omega^2}{g}$$

Now, from the diagram,

we have $\tan\theta = \frac{a}{h}$. **(1 mark)**

So, $$\frac{a}{h} = \frac{a\omega^2}{g}$$

$$h = \frac{g}{\omega^2}$$

as required. **(1 mark)**

(ii) When $n = 30\,rps$

$$h = \frac{g}{(2\pi \times 30)^2}$$

$$= \frac{g}{3600\pi^2}$$

(1 mark)

Let h_1 be the height of O above C when $n = 60\,rps$

$$h_1 = \frac{g}{(2\pi \times 60)^2}$$

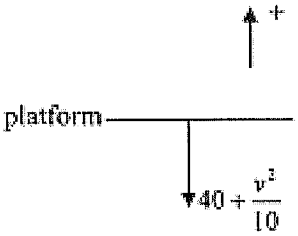
$$= \frac{g}{14400\pi^2}$$

$$= \frac{g}{4 \times 3600\pi^2}$$

So $h_1 = \frac{1}{4} \times h$ **(1 mark)**

HEFFERNAN 2002 Q6

(b) (i) Take the origin to be that part on the platform from which the particle is projected. The downward forces acting on the particle are the gravitational and air resistance forces. Taking the positive direction as upwards, and using the equation of motion, $F = ma$ we obtain



$-\left(40 + \frac{v^2}{10}\right) = 4\ddot{y}$ where $\ddot{y}$ is vertical acceleration.

So, $\ddot{y} = -10 - \frac{v^2}{40}$ as required. **(1 mark)**

(ii) The maximum height is reached when $v = 0$.

Now, $v^2 = 100\left(5e^{\frac{-y}{20}} - 4\right)$

becomes $0 = 100\left(5e^{\frac{-y}{20}} - 4\right)$

So, $e^{\frac{-y}{20}} = \frac{4}{5}$

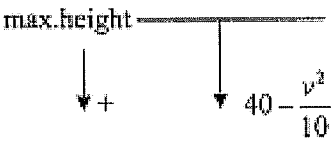
$e^{\frac{y}{20}} = \frac{5}{4}$

$\ln\left(\frac{5}{4}\right) = \frac{y}{20}$

$y = 20\ln\left(\frac{5}{4}\right)$

So the maximum height reached is $20\ln\left(\frac{5}{4}\right)$ metres. **(1 mark)**

(iii) We need to consider the downward part of the particle's journey. Taking its maximum height as the origin, the downward direction as positive, and using the equation of motion $F = ma$,



we have $40 - \frac{v^2}{10} = 4\ddot{y}$ where $\ddot{y}$ is the vertical acceleration.

So, $\ddot{y} = 10 - \frac{v^2}{40}$ **(1 mark)**

Now, given that $\ddot{y} = v\dfrac{dv}{dy}$

We have,

$$\begin{aligned} v\frac{dv}{dy} &= 10 - \frac{v^2}{40} \\ \frac{dv}{dy} &= \frac{10}{v} - \frac{v}{40} \\ &= \frac{400 - v^2}{40v} \\ \frac{dy}{dv} &= \frac{40v}{400 - v^2} \\ \int \frac{dy}{dv}\,dv &= 40\int \frac{v}{400 - v^2}\,dv \\ &= 40\int -\frac{1}{2}\frac{du}{dv}\cdot u^{-1}dv \\ &= -20\int u^{-1}\,du \\ y &= -20\ln(400 - v^2) + c \end{aligned}$$

let $u = 400 - v^2$

$\dfrac{du}{dv} = -2v$

When $y = 0$, $v = 0$

$$\begin{aligned} 0 &= -20\ln 400 + c \\ c &= 20\ln 400 \\ y &= -20\ln(400 - v^2) + 20\ln 400 \\ &= 20\ln\left(\frac{400}{400 - v^2}\right) \end{aligned}$$

(1 mark)

From part (ii), the particle returns to the platform when it has travelled $20\ln\left(\dfrac{5}{4}\right)$ metres. So, when $y = 20\ln\left(\dfrac{5}{4}\right)$

We have $$20\ln\left(\frac{5}{4}\right) = 20\ln\left(\frac{400}{400 - v^2}\right)$$

So, $$\begin{aligned} \frac{5}{4} &= \frac{400}{400 - v^2} \\ 5(400 - v^2) &= 1600 \\ 2000 - 5v^2 &= 1600 \\ v^2 &= 80 \\ v &= \pm\sqrt{80} \end{aligned}$$

So, the particle is travelling at a speed of $4\sqrt{5}$ metres per second when it returns to the platform since for this portion of the journey we have designated the downwards direction as positive. **(1 mark)**

(iv) We need an expression for velocity as a function of time. Looking at the motion of the particle from when it reached its maximum height, we have, from part (iii)

$$\ddot{y} = 10 - \frac{v^2}{40}$$

Now, $$\frac{dv}{dt} = \frac{400 - v^2}{40}$$

So, $$\frac{dt}{dv} = \frac{40}{400 - v^2}$$

Now, $$\frac{40}{400 - v^2} = \frac{40}{(20 - v)(20 + v)}$$

Let $$\frac{40}{(20 - v)(20 + v)} = \frac{A}{20 - v} + \frac{B}{20 + v}$$

$$= \frac{A(20 + v) + B(20 - v)}{(20 - v)(20 + v)}$$

True iff $40 \equiv A(20 + v) + B(20 - v)$

Put $v = -20$, $40 = 40B$, $B = 1$

Put $v = 20$, $40 = 40A$, $A = 1$

So, $$\frac{40}{400 - v^2} = \frac{1}{20 - v} + \frac{1}{20 + v}$$

So, $$\frac{dt}{dv} = \frac{1}{20 - v} + \frac{1}{20 + v}$$ **(1 mark)**

$$\int \frac{dt}{dv}\,dv = \int \frac{1}{20 - v}\,dv + \int \frac{1}{20 + v}\,dv$$

$$t = -\ln(20 - v) + \ln(20 + v) + c$$

When $t = 0$, $v = 0$ $$0 = -\ln 20 + \ln 20 + c$$

$$0 = \ln 1 + c$$

$$c = 0$$

So $$t = \ln \frac{20 + v}{20 - v}$$

So, $$e^t = \frac{20 + v}{20 - v}$$

$$20e^t - ve^t = 20 + v$$

$$-ve^t - v = 20 - 20e^t$$

$$v(e^t + 1) = 20e^t - 20$$

$$v = \frac{20e^t - 20}{e^t + 1}$$

$$= \frac{20(e^t - 1)}{e^t + 1}$$

So, $$v = 20\left(1 - \frac{2}{e^t + 1}\right)$$ **(1 mark)**

$$\begin{array}{r} 1 \\ e^t + 1\overline{)\,e^t - 1} \\ \underline{e^t + 1} \\ -2 \end{array}$$

As, $t \to \infty, \dfrac{2}{e^t+1} \to 0$

So, $v \to 20$ from below.

So the terminal speed of the particle is 20 metres per second.

(1 mark)

(v) For that part of the particle's journey where it is falling downwards, we have from part (iv) that $t = \ln\dfrac{20+v}{20-v}$.

From part (iii) the speed of the particle when it returned to the platform was $4\sqrt{5}$ metres per second.

So when $v = 4\sqrt{5}$, (since we assume that downwards is positive).

$$t = \ln\frac{20+4\sqrt{5}}{20-4\sqrt{5}}$$

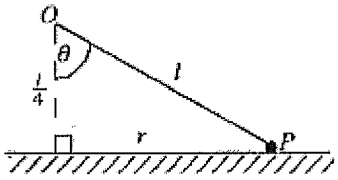
$$= 0.96 \text{ seconds (correct to 2 decimal places.)}$$

(1 mark)

INDEPENDENT 2001 Q6

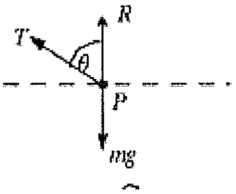
Answer

(i)



$\cos\theta = \frac{1}{4}$, $\sin\theta = \frac{\sqrt{15}}{4}$, $r = l\sin\theta$

Forces on P



Resultant force on P has magnitude $\dfrac{mv^2}{r}$ *and is directed horizontally towards the centre of the circle of motion.*

6(b)

(ii) Horizontal component of resultant force on P has magnitude $\dfrac{mv^2}{r}$.

$$T\sin\theta = \frac{mv^2}{r}$$

$$T = \frac{mv^2}{l\sin^2\theta} = \frac{16mv^2}{15l}$$

(iii) Vertical component of resultant force on P is zero.

$$T\cos\theta + R = mg$$

$$R = mg - \tfrac{1}{4}T = m\left(g - \frac{4v^2}{15l}\right)$$

(iv) Particle in contact with table provided $R \ge 0$

$$\therefore \frac{4v^2}{15l} \le g \Rightarrow v \le \sqrt{\frac{15gl}{4}}$$

Particle would lift off table for higher speeds.

JAMES RUSE 2000 Q6

6. (a).

$m\ddot{x} = mg - r$

$10\ddot{x} = mg - 0.04v^2$

$10\ddot{x} = 10g - 0.04v^2$

$\ddot{x} = g - 0.004v^2$

$v\frac{dv}{dx} = g - 0.004v^2$

$\int \frac{dx}{dv} dv = \int \frac{v}{g - 0.004v^2} dv$

$\int_0^{x} dx = \frac{-1}{0.008}\int_0^{v} \frac{-0.008v}{g - 0.004v^2}$

$x = \frac{-1}{0.008}[\ln(g - 0.004v^2)] + c.$

At x=0, v=0.

$0 = \frac{-1}{0.008}[\ln(10)] + c.$

$c = \frac{1}{0.008}\ln 10.$

$\therefore x = \frac{1}{0.008}\ln\left|\frac{10}{10 - 0.004v^2}\right|$

when x = 0.3,

$0.3 = \frac{1}{0.008}\ln\left|\frac{10}{10 - 0.004v^2}\right|$

$\therefore 1.0024 = \frac{10}{10 - 0.004v^2}$

$\therefore 10 - 0.004v^2 = \frac{10}{1.0024}.$

$10 - \frac{10}{1.0024} = 0.004v^2$

$v^2 = \frac{10 - \frac{10}{1.0024}}{0.004}.$

$v = 2.41.$