SPECIMEN PAPER 3

EXT 2 (CORONEOS)

Question 1

(i) Decompose $\frac{8}{(x+2)(x^2+4)}$ into partial fractions, and use this result to evaluate $\int_0^2 \frac{8\,dx}{(x+2)(x^2+4)}$.

(ii) Evaluate, giving numerical answers to 3 significant figures,

(a) $\int_0^{1/3} \frac{dx}{\sqrt{1-9x^2}}$ (b) $\int_{-1}^{1} 2^x\,dx$ (c) $\int_1^e x \log_e x\,dx$

(iii) Show that $\frac{d}{dx}\log(x+\sqrt{x^2+4}) = \frac{1}{\sqrt{x^2+4}}$

Hence, or otherwise, prove that $\int_{-\pi/4}^{\pi/4} \frac{\sec^2 x\,dx}{\sqrt{\tan^2 x+4}} = 2\log\{(\sqrt{5}+1)/2\}$

Question 2

(i) Show that the curve $y = \frac{x^3+4}{x^2}$ has one stationary point, but no point of inflexion and is always concave up.

For large x, show that the equation of the curve approximates to that of a straight line. What does this suggest is an asymptote?

Sketch the curve.

Use the graph to show that the equation $x^3 - kx^2 + 4 = 0$ may have 1, 2 or 3 real roots. State the values of k corresponding to each case.

(ii) By means of substitution $y = a - x$ or otherwise, prove that

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Hence evaluate $\int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x}$

Question 3

The ellipse E has cartesian equation $\frac{x^2}{25} + \frac{y^2}{16} = 1$. Write down the eccentricity, the coordinates of the foci S and S', and the equations of the directrices.

Sketch the curve, and indicate the foci and directrices on the diagram.

(a) Show that any point P on E can be represented by the coordinates $(5\cos\theta, 4\sin\theta)$ and prove that PS + PS' is independent of the position of P on the curve.

(b) Find the equation of the normal n at the point P on the ellipse. If this normal meets the major axis of the ellipse in M and the minor axis in N, prove that PM/PN = 16/25.

Also show that the line n bisects the angle S'PS.

Question 4

(i) If $Z = (4+3i)/(1-2i)$ determine

(a) $|Z|$ (b) $\text{Re}(Z)$ (c) $\text{Im}(Z)$ (d) $\overline{Z}$

(e) $Z + \overline{Z}$ (f) $Z\overline{Z}$ (g) $\arg Z$ (to the nearest minute)

(ii) If the complex numbers z, w are related by $w = z^2$. Find the locus of w if z describes

(a) the locus $|z| = 3$ (b) the locus $\arg z = \pi/3$

(c) the y-axis (d) the line $x = 3$

Sketch the loci of z and w on an Argand diagram, in each case.

Question 5

Show that $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

If m, n are unequal integers, prove that

(a) $\int_0^{\pi} \sin mx \sin nx\,dx = 0$ (b) $\int_0^{\pi} \sin^2 mx\,dx = \frac{\pi}{2}$

From graphical considerations, or otherwise, prove that if $0 \le x \le \pi$, then $\sin x + \frac{1}{3}\sin 3x \ge 0$.

Find the area A between the curve $y = a(\sin x + \frac{1}{3}\sin 3x)$ and the x-axis between the limits 0 and π, $(a > 0)$.

Find also the volume V of the solid obtained by rotating this area about the x-axis, and prove that $V = \frac{1}{4}\pi^2 aA$.

Question 6

A particle is projected from O with initial speed V at an angle α with the horizontal, where $\alpha = \tan^{-1}(3/4)$. The particle passes through a point P whose coordinates relative to horizontal and vertical axes through O are $(8a, 4a)$. Show that $V = 5\sqrt{ag}$ (where g is the acceleration due to gravity, assumed constant), and find the time of flight from O to P.

It is possible for this particle to be projected from O, at a different angle β to the horizontal, with the same speed V so as to still pass through P. Find the cartesian equation of its path, and hence determine $\tan\beta$.

Question 7.

(i) Solve the equation $4x^3 - 24x^2 + 23x + 18 = 0$, given that the roots are in arithmetic progression.

(ii) If α, β, γ are the roots of the equation $x^3 - 7x^2 + 18x - 7 = 0$ find the value of $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)$.

(iii) By considering the value of $\int_0^1 \sqrt{x(1 - x)}$ as the limit of a sum, show that $\lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{n^2}\sqrt{r(n - r)} = \frac{\pi}{8}$.

Question 8

(i) Using the letters A, A, B, C, D, E, E find how many 'code-words' can be formed, if the code-word contains

(a) all 7 letters

(b) exactly 4 of the 7 letters.

(Solutions to (b) should contain sufficient explanation to make the method clear.)

(ii) Find (as trigonometrical quantities) the five fifth roots of unity, and represent them on an Argand diagram.

If α is one of the complex roots, show that the other complex roots can be expressed as α^2, α^{-1}, α^{-2} and find the value of $\alpha^2 + \alpha^{-2}$.

Factorise $x^5 - 1$ as the product of a linear factor and two real quadratic factors.

WORKED SOLUTIONS TO SPECIMEN PAPER 3

1.(i) Let $\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$, then $8 = A(x^2+4) + (Bx+C)(x+2)$.

When $x = -2$, $8 = A.8$, i.e. $A = 1$

Equating coeffts x^2, $0 = A + B$, i.e. $B = -1$

Equating constants, $8 = 4A + 2C$, i.e. $C = 2$

Thus $\frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{-x+2}{x^2+4}$ #

$$\int_0^2 \frac{8dx}{(x+2)(x^2+4)} = \int_0^2 \left\{\frac{1}{x+2} - \frac{x}{x^2+4} + \frac{2}{x^2+4}\right\} dx$$

$$= \left[\log(x+2) - \tfrac{1}{2}\log(x^2+4) + \tfrac{2}{2}\tan^{-1}\left(\tfrac{x}{2}\right)\right]_0^2$$

$$= \{\log 4 - \tfrac{1}{2}\log 8 + \tfrac{\pi}{4}\} - \{\log 2 - \tfrac{1}{2}\log 4 + 0\} = \tfrac{1}{2}\log 2 + \tfrac{\pi}{4} \; \#$$

(ii)(a) $\int_0^{1/3} \frac{dx}{\sqrt{1-9x^2}} = \frac{1}{3}\left[\sin^{-1}(3x)\right]_0^{1/3} = \frac{\pi}{6} \doteqdot 0.524$ #

{OR let $u = 3x$, then $I = \int_0^1 \frac{du/3}{\sqrt{1-u^2}} = \frac{1}{3}\left[\sin^{-1}u\right]_0^1$ etc}

(b) Noting $\frac{d}{dx}(a^x) = a^x \log_e a$, then $\frac{d}{dx}(a^x) = 2^x \ln 2$

Thus $\int_{-1}^1 2^x dx = \frac{1}{\ln 2}\left[2^x\right]_{-1}^1 = \frac{1}{\ln 2}\left\{\frac{3}{2}\right\} \doteqdot 2.16$ #

(c) Integrating by parts, $I = \int_1^e x \log x \, dx = \int_1^e \log x . \frac{d}{dx}\left(\frac{1}{2}x^2\right) dx$

$\therefore I = \left[\log x . \frac{1}{2}x^2\right]_1^e - \int_1^e \frac{1}{2}x^2 . \frac{d}{dx}(\log x)\, dx = \frac{1}{2}e^2 - \frac{1}{2}\int_1^e x^2 . \frac{1}{x} dx$

$= \frac{1}{2}e^2 - \frac{1}{2}\left[\frac{1}{2}x^2\right]_1^e = \frac{1}{4}(e^2 + 1) \doteqdot 2.10$ #

(iii) $\frac{d}{dx}(x + \sqrt{x^2+4}) = 1 + \frac{1}{2}(x^2+4)^{-\frac{1}{2}}.2x = 1 + \frac{x}{\sqrt{x^2+4}} = \frac{\sqrt{x^2+4} + x}{\sqrt{x^2+4}}$

$\frac{d}{dx}\log(x + \sqrt{x^2+4}) = \frac{1}{x+\sqrt{x^2+4}} . \frac{d}{dx}(x + \sqrt{x^2+4})$

$= \frac{1}{x+\sqrt{x^2+4}} . \left\{\frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}}\right\} = \frac{1}{\sqrt{x^2+4}}$ #

In $I = \int_{-\pi/4}^{\pi/4} \frac{\sec^2 x \, dx}{\sqrt{\tan^2 x + 4}}$ let $u = \tan x$, $\therefore du = \sec^2 x \, dx$

$\therefore I = \int_{-1}^1 \frac{du}{\sqrt{u^2+4}} = \left[\log(u + \sqrt{u^2+4})\right]_{-1}^1 = \log(1+\sqrt{5}) - \log(-1+\sqrt{5})$

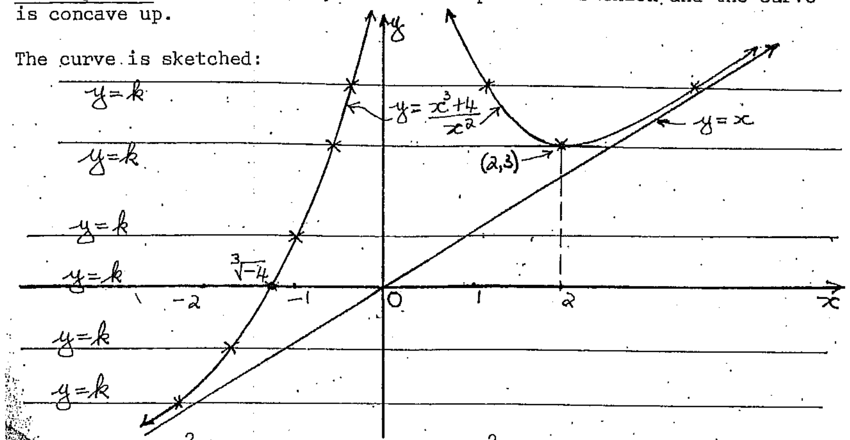
$= \log\left(\frac{\sqrt{5}+1}{\sqrt{5}-1} \times \frac{\sqrt{5}+1}{\sqrt{5}+1}\right) = \log\left\{\frac{(\sqrt{5}+1)^2}{4}\right\} = 2\log\{(\sqrt{5}+1)/2\}$

2.(i) $y = \frac{x^3+4}{x^2} = x + \frac{4}{x^2}$, $\frac{dy}{dx} = 1 - \frac{8}{x^3}$, $\frac{d^2y}{dx^2} = \frac{24}{x^4}$.

Now $y' = 0$ when $x = 2$; then $y = 3$ and $y'' > 0$. There is only one st.pt., it is at $(2,3)$ and is a rel. min.#

Since $y'' > 0$ for all x ($\neq 0$), there is no pt. of inflexion and the curve is concave up.

The curve is sketched:



As $x \to \pm\infty$, $4/x^2 \to 0$, and thus $y \doteqdot x + 4/x^2 \to x$. The line $y = x$ is thus an asymptote to the curve. Since $4/x^2 > 0$ for all $x \neq 0$, the curve is above the line $y = x$.

Domain is all $x \neq 0$; the line $x = 0$ (the y-axis) is a vertical asymptote to the curve.

Curve crosses the x-axis, where $x^3 + 4 = 0$, i.e. $x = \sqrt[3]{-4} \doteqdot -1.6$.
{When $\sqrt[3]{-4} < x < 0$, $y > 0$; when $x < \sqrt[3]{-4}$, $y < 0$; when $x > 0$, $y > 0$}.

Eqn. $x^3 - kx^2 + 4 = 0$ can be written $x^3 + 4 = kx^2$, i.e. $\frac{x^3+4}{x^2} = k$.

The roots of the given eqn. are the abscissae of the pts. of int. of the curve sketched and the line $y = k$. This line cuts the curve once when $k < 3$, twice when $k = 3$ and thrice when $k > 3$.

That is, the eqn. $x^3 - kx^2 + 4 = 0$ has 1 real root if $k < 3$, 2 real roots if $k = 3$ (in this case, one of the roots is repeated) and 3 real (and distinct) roots if $k > 3$ #

(ii) In $I = \int_0^a f(x)dx$, put $y = a-x$, i.e. $x = a-y$, ∴ $dx = -dy$

When $x = 0$, $y = a$ and when $x = a$, $y = 0$

$I = \int_a^0 f(a-y)\cdot -dy = \int_0^a f(a-y)\,dy = \int_0^a f(a-x)\,dx$ # since the value of the definite integral is independent of the variable used.

{Thus $\int_0^a f(a-y)\,dy = \int_0^a f(a-p)\,dp = \int_0^a f(a-z)\,dz$ etc}

Now $J = \int_0^\pi \frac{x \sin x\, dx}{1+\cos^2 x} = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)\,dx}{1+\cos^2(\pi-x)}$, by above

$= \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx = \int_0^\pi \frac{\pi \sin x\, dx}{1+\cos^2 x} - \int_0^\pi \frac{x \sin x\, dx}{1+\cos^2 x}$

∴ $2J = \pi \int_0^\pi \frac{\sin x\, dx}{1+\cos^2 x}$, let $u = \cos x$, ∴ $du = -\sin x\, dx$

$= \pi \int_1^{-1} \frac{-du}{1+u^2} = \pi \int_{-1}^{1} \frac{du}{1+u^2} = \pi\left[\tan^{-1} u\right]_{-1}^{1} = \frac{\pi^2}{2}$

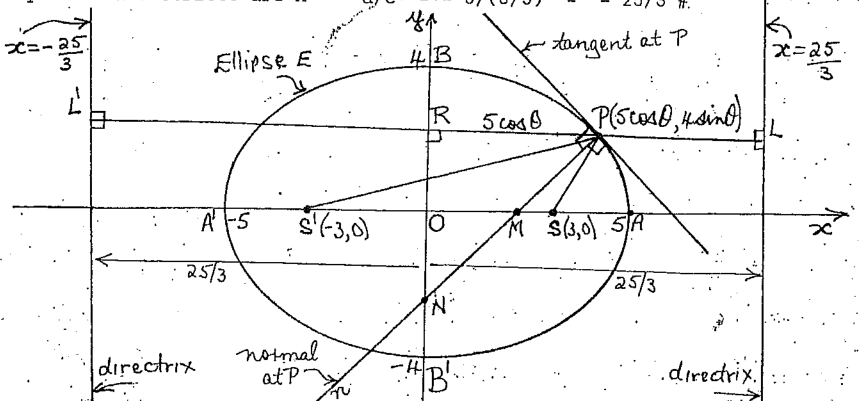
∴ $J = \pi^2/4$ #

3. In the eqn. $\frac{x^2}{25} + \frac{y^2}{16} = 1$, $a = 5$, $b = 4$

From $b^2 = a^2(1-e^2)$, ∴ $16 = 25(1-e^2)$, i.e. $e^2 = 1 - \frac{16}{25}$, $e = \frac{3}{5}$ #

Coords. of foci S, S' are $(\pm ae, 0)$, i.e. $(\pm 5 \cdot \frac{3}{5}, 0)$, i.e. $(\pm 3, 0)$ #

Eqns. of directrices are $x = \pm a/e = \pm 5/(3/5) = \pm 25/3$ #



(a) Subst. coords. of P $(5\cos\theta, 4\sin\theta)$ in the eqn. of E, then

$\frac{x^2}{25} + \frac{y^2}{16} = \frac{(5\cos\theta)^2}{25} + \frac{(4\sin\theta)^2}{16} = \cos^2\theta + \sin^2\theta = 1$, and thus P lies on E #

By definition of an ellipse, $\frac{\text{dist. of P from focus S}}{\text{dist. of P from corresp. directrix}} = e$

i.e. $\frac{PS}{PL} = e$, i.e. $PS = e.PL = \frac{3}{5}(\frac{25}{3} - 5\cos\theta) = 5 - 3\cos\theta$

Also $\frac{PS'}{PL'} = e$, i.e. $PS' = ePL' = \frac{3}{5}(\frac{25}{3} + 5\cos\theta) = 5 + 3\cos\theta$

Thus $PS + PS' = (5 - 3\cos\theta) + (5 + 3\cos\theta) = 10$, which is indep. of the position of P on E #

{OR $PS = \sqrt{(5\cos\theta - 3)^2 + (4\sin\theta)^2} = \sqrt{25\cos^2\theta - 30\cos\theta + 9 + 16\sin^2\theta}$

$= \sqrt{16 + 9\cos^2\theta - 30\cos\theta + 9}$, *noting* $16\cos^2\theta + 16\sin^2\theta = 16$

$= \sqrt{25 - 30\cos\theta + 9\cos^2\theta} = \sqrt{(5 - 3\cos\theta)^2} = 5 - 3\cos\theta$

Simy. $PS' = 5 + 3\cos\theta$, *etc}*

(b) From $\frac{x^2}{25} + \frac{y^2}{16} = 1$, then $\frac{2x}{25} + \frac{2y}{16}\frac{dy}{dx} = 0$, i.e. $\frac{dy}{dx} = \frac{-16x}{25y}$

At P $(5\cos\theta, 4\sin\theta)$, $\frac{dy}{dx} = \frac{-16.5\cos\theta}{25.4\sin\theta} = \frac{-4\cos\theta}{5\sin\theta}$

{OR let $x = 5\cos\theta$, $y = 4\sin\theta$.

Then $\frac{dx}{d\theta} = -5\sin\theta$, $\frac{dy}{d\theta} = 4\cos\theta$ *and* ∴ $\frac{dy}{dx} = \frac{4\cos\theta}{-5\sin\theta}$ *}*

Grad. of normal n is $5\sin\theta/4\cos\theta$, and eqn. of n is

$y - 4\sin\theta = \frac{5\sin\theta}{4\cos\theta}(x - 5\cos\theta)$, i.e. $5\sin\theta x - 4\cos\theta y = 9\sin\theta\cos\theta$ #

This line meets the major axis A'A ($y = 0$) where $x = 9\cos\theta/5$ i.e. M is $(9\cos\theta/5, 0)$ and meets the minor axis B'B ($x = 0$) where $y = -9\sin\theta/4$, i.e. N is $(0, -9\sin\theta/4)$.

In the fig, OM is parallel to RP, and thus $\frac{PM}{PN} = \frac{RO}{RN}$

Now $RO = 4\sin\theta$ and $RN = 4\sin\theta + \frac{9}{4}\sin\theta = \frac{25\sin\theta}{4}$

Thus $\frac{RO}{RN} = \frac{4\sin\theta}{25\sin\theta/4} = \frac{16}{25}$ and thus $\frac{PM}{PN} = \frac{16}{25}$ #

{OR ΔS OMN, RPN are similar and hence $\frac{NM}{NP} = \frac{NO}{NR} = \frac{OM}{RP}$

This gives $\frac{NM}{NP} = \frac{9\cos\theta/5}{5\cos\theta} = \frac{9}{25}$ *and thus* $\frac{PM}{PN} = \frac{25-9}{25} = \frac{16}{25}$

Note for ratios along a line, it is better to avoid the distance formula, for then, if $c = \cos\theta$, $s = \sin\theta$

$PM = \sqrt{(5c - \frac{9c}{5})^2 + (4s)^2} = \sqrt{(\frac{16c}{5})^2 + 16s^2} = \sqrt{\frac{16}{25}(16c^2 + 25s^2)}$

$PN = \sqrt{(5c)^2 + (4s + \frac{9s}{4})^2} = \sqrt{25c^2 + (\frac{25s}{4})^2} = \sqrt{\frac{25}{16}(16c^2 + 25s^2)}$

$PM/PN = 4/5 \div 5/4 = 16/25$ # }

Grad. of PS = $\frac{4 \sin\theta}{5\cos\theta - 3} = \frac{4s}{5c-3}$ and grad. of n = $\frac{5\sin\theta}{4\cos\theta} = \frac{5s}{4c}$

$\tan M\hat{P}S = \left|\frac{\frac{5s}{4c} - \frac{4s}{5c-3}}{1 + \frac{5s}{4c}\cdot\frac{4s}{5c-3}}\right| = \left|\frac{25sc - 15s - 16sc}{20c^2 - 12c + 20s^2}\right|$

$= \left|\frac{9sc - 15s}{20 - 12c}\right| = \left|\frac{3s\,(3c-5)}{4\,(5-3c)}\right| = \frac{3s}{4} = \frac{3\sin\theta}{4}$

$\tan S'\hat{P}M = \left|\frac{\frac{5s}{4c} - \frac{4s}{5c+3}}{1 + \frac{5s}{4c}\cdot\frac{4s}{5c+3}}\right| = \left|\frac{25sc + 15s - 16sc}{20c^2 + 12c + 20s^2}\right|$

$= \left|\frac{9sc + 15s}{20 + 12c}\right| = \left|\frac{3s\,(3c+5)}{4\,(5+3c)}\right| = \frac{3s}{4} = \frac{3\sin\theta}{4}$

Thus $M\hat{P}S = S'\hat{P}M$; hence normal n bisects angle S'PS.#

4.(i) $Z = \frac{4+3i}{1-2i} = \frac{4+3i}{1-2i} \times \frac{1+2i}{1+2i} = \frac{-2+11i}{1+4} = \frac{-2}{5} + \frac{11i}{5}$

(a) $|Z| = \left|\frac{4+3i}{1-2i}\right| = \frac{|4+3i|}{|1-2i|} = \frac{\sqrt{4^2+3^2}}{\sqrt{1^2+(-2)^2}} = \frac{\sqrt{25}}{\sqrt{5}} = \sqrt{5}$ #

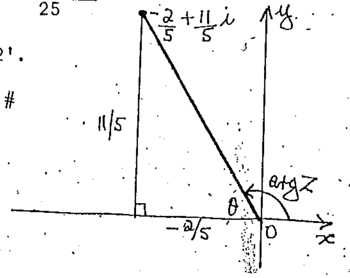
or $|Z| = \sqrt{(\frac{-2}{5})^2 + (\frac{11}{5})^2} = \sqrt{\frac{4 + 121}{25}} = \sqrt{5}$ #

(b) Re $(Z) = \frac{-2}{5}$ # (c) Im $(Z) = \frac{11}{5}$ # (d) $\bar{Z} = \frac{-2}{5} - \frac{11}{5} i$ #

(e) $Z + \bar{Z} = \frac{-4}{5}$ # (f) $Z\bar{Z} = (\frac{-2}{5})^2 + (\frac{11}{5})^2 = \frac{4 + 121}{25} = 5$ #

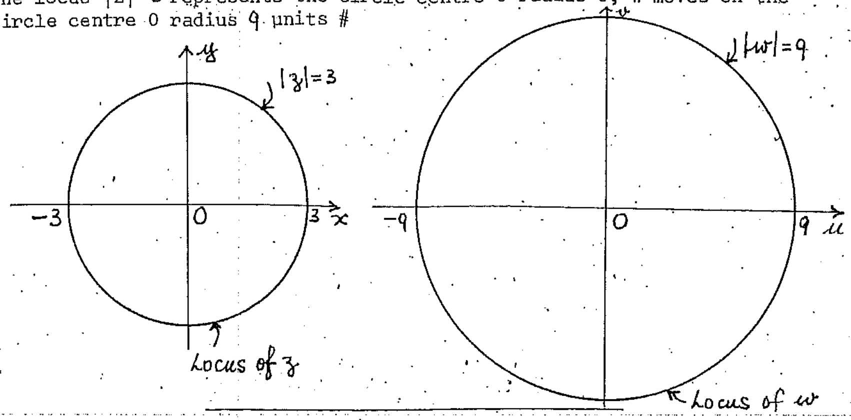
(g) In fig. $\tan\theta = 11/2$, and $\therefore\ \theta \doteqdot 79^\circ 42'$.

Thus $\arg Z \doteqdot (180^\circ - 79^\circ 42') = 100^\circ 18'$ #



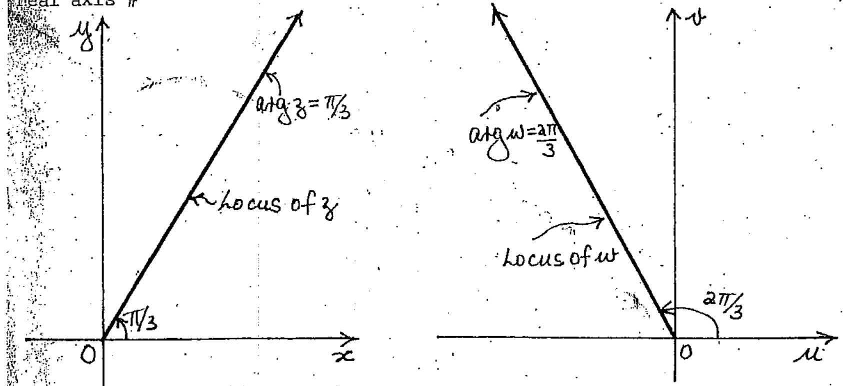
(ii)(a) If $w = z^2$, then $|w| = |z^2| = |z|^2 = 3^2 = 9$

The locus $|z| = 3$ represents the circle centre O radius 3; w moves on the circle centre O radius 9 units #



(b) If $w = z^2$, then arg w = arg z^2 = 2 arg z = $2\pi/3$

The locus arg z = $\pi/3$ represents the ray from O inclined at $\pi/3$ to the positive x-axis; w moves on a corresponding ray inclined at $2\pi/3$ to the real axis #

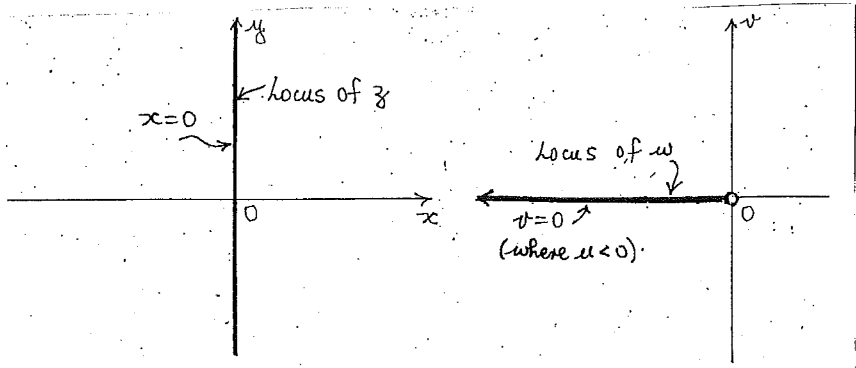


(c) Let $w = u + iv$, $z = x + iy$ (where u, v, x, y are real)

Since $w = z^2$, then $u + iv = (x + iy)^2 = (x^2 - y^2) + 2ixy$

Thus $u = x^2 - y^2$, $v = 2xy$

When z moves on the y-axis, i.e. $x = 0$, then $u = -y^2$ and $v = 0$. Thus, since $v = 0$ is independent of variables, the locus of w is the line $v = 0$, i.e. the u-axis; however, $u = -y^2 \le 0$ and hence the exact locus is the real axis for $u \le 0$.



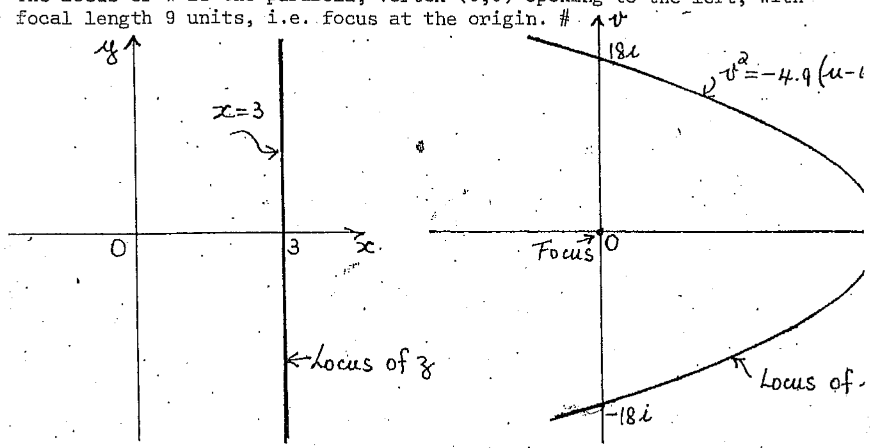
(d) From (c), if $w = z^2$, then $u = x^2 - y^2$, $v = 2xy$

When z describes the line $x = 3$, then $u = 9 - y^2$, $v = 6y$

Eliminating y, then $y^2 = 9 - u$ and $y^2 = (v/6)^2$

Thus $\frac{v^2}{36} = 9 - u$, i.e. $v^2 = 36(9 - u) = -4.9(u - 9)$

The locus of w is the parabola, vertex (9,0) opening to the left, with focal length 9 units, i.e. focus at the origin. #



5. $\cos(A-B) = \cos A \cos B + \sin A \sin B$
$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$ #, where $A>B$

(a) $\int_0^\pi \sin mx \sin nx = \frac{1}{2}\int_0^\pi \{\cos(mx-nx) - \cos(mx+nx)\}\,dx$

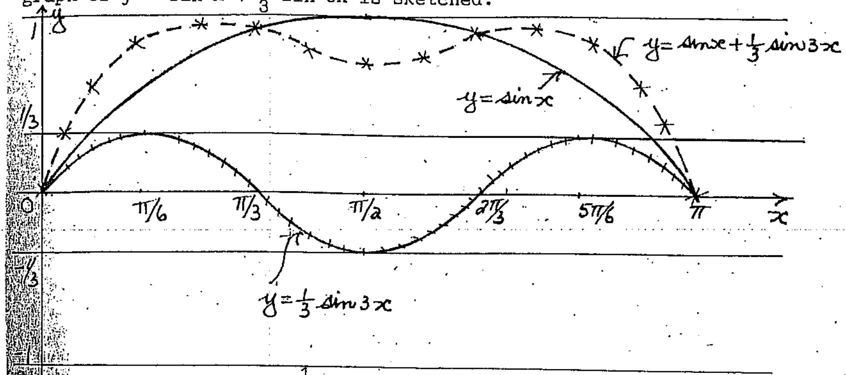
$= \frac{1}{2}\int_0^\pi \{\cos(m-n)x - \cos(m+n)x\}\,dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^\pi$

$= 0$, # *since* $\sin 0 = 0$ *and* $\sin k\pi = 0$ *for integral* k

(b) *Using result* $\cos 2\theta = 1 - 2\sin^2\theta$, *i.e.* $2\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ then

$\int_0^\pi \sin^2 mx\,dx = \int_0^\pi \frac{1}{2}(1 - \cos 2mx)\,dx = \frac{1}{2}\left[x - \frac{\sin 2mx}{2m}\right]_0^\pi = \frac{\pi}{2}$ #.

sin x has period 2π and amplitude 1; $\frac{1}{3}\sin 3x$ has period $2\pi/3$ and amplitude 1/3. These are sketched on the same diagram; from these the graph of $y = \sin x + \frac{1}{3}\sin 3x$ is sketched.



From the sketch, $\sin x + \frac{1}{3}\sin 3x \geq 0$ for $0 \leq x \leq \pi$ #

Area below curve $y = a(\sin x + \frac{1}{3}\sin 3x)$ for $x = 0$ to $x = \pi$ is given by

$A = \int_0^\pi a(\sin x + \frac{1}{3}\sin 3x)\,dx = a\left[-\cos x - \frac{1}{9}\cos 3x\right]_0^\pi$

$= -a\{(\cos\pi + \frac{1}{9}\cos 3\pi) - (\cos 0 + \frac{1}{9}\cos 0)\} = -a\{(-1 - \frac{1}{9}) - (1 + \frac{1}{9})\} = \frac{20a}{9}$ units^2 #

Vol. generated about the x-axis is given by

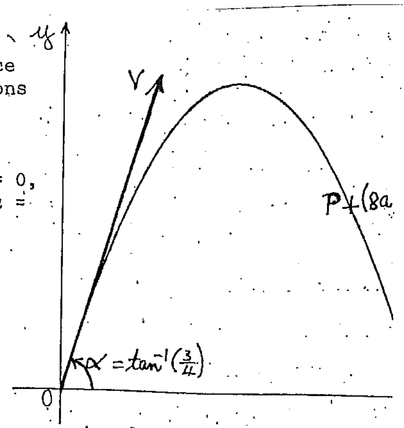
$V = \int_0^\pi \pi a^2(\sin x + \frac{1}{3}\sin 3x)^2\,dx$

$= \pi a^2 \int_0^\pi (\sin^2 x + \frac{2}{3}\sin x \sin 3x + \frac{1}{9}\sin^2 3x)\,dx$

$= \pi a^2\left[\frac{\pi}{2} + \frac{2}{3}\cdot 0 + \frac{1}{9}\cdot\frac{\pi}{2}\right]$, *using the results of (a), (b) above*

$= \frac{5\pi^2 a^2}{9}$ units # $= \frac{1}{4}\pi^2 a.\frac{20a}{9} = \frac{1}{4}\pi^2 aA$ #

6. The only force acting on the particle is its weight; air resistance is neglected. The eqns. of its motions in the x, y direction are $m\ddot{x} = 0$ and $m\ddot{y} = -mg$.



The initial conditions are $t = 0$, $x = 0$, $y = 0$, $\dot{x} = V\cos\alpha = \frac{4}{5}V$, $\dot{y} = V\sin\alpha = \frac{3}{5}V$.

Integrating the eqns. of motion and using the initial conditions to find the constants of integration we have

$\ddot{x} = 0$; $\dot{x} = V\cos\alpha = \frac{4}{5}V$; $x = \frac{4}{5}Vt$

$\ddot{y} = -g$; $\dot{y} = -gt + V\sin\alpha = -gt + \frac{3}{5}V$; $y = -\frac{1}{2}gt^2 + \frac{3}{5}Vt$

Since the particle passes through P (8a,4a) then

$8a = \frac{4}{5}Vt$ .. (1) and $4a = \frac{-1}{2}gt^2 + \frac{3}{5}Vt$ .. (2)

From (1), $t = \frac{10a}{V}$ and in (2), $4a = -\frac{1}{2}g\left(\frac{100a^2}{V^2}\right) + \frac{3}{5}V.\left(\frac{10a}{V}\right)$

Simplifying, $\frac{50ga^2}{V^2} = 6a - 4a$, i.e. $V^2 = \frac{50ga^2}{2a}$, i.e. $V = 5\sqrt{ga}$ #

Time of flight from 0 to P is given by $t = \frac{10a}{5\sqrt{ga}} = 2\sqrt{\frac{a}{g}}$ #.

If the particle is projected at an angle β(with the same speed $V = 5\sqrt{ag}$ to pass through the same point P (8a,4a), the corresponding equations to those above are

$\ddot{x} = 0$; $\dot{x} = V\cos\beta$; $x = Vt\cos\beta$

$\ddot{y} = -g$; $\dot{y} = -gt + V\sin\beta$; $y = -\frac{1}{2}gt^2 + Vt\sin\beta$

From $x = Vt\cos\beta$, $t = x/(V\cos\beta)$ and thus the cartesian eqn. of its pat

$$y = -\frac{1}{2}g\left(\frac{x^2}{V^2\cos^2\beta}\right) + V\left(\frac{x}{V\cos\beta}\right)\sin\beta = -\frac{1}{2}g\frac{x^2\sec^2\beta}{V^2} + x\tan\beta \#$$

Since the particle passes through P (8a,4a) and noting $V = 5\sqrt{ag}$ still,

$\therefore 4a = -\frac{1}{2}g.\frac{(64a^2)}{25ag}(1 + \tan^2\beta) + 8a\tan\beta$,

Simplifying, $8\tan^2\beta - 50\tan\beta + 33 = 0$, i.e. $(4\tan\beta - 3)(2\tan\beta - 11)$:

Thus $\tan\beta = 11/2$ # (*$\tan\beta = 3/4$ when $\beta = \alpha$*)

7.(i) Let the roots be a - d, a, a + d (these are in A.P.)

Now (a-d) + a + (a+d) = +24/4 = 6 --(1), i.e. 3a = 6, i.e. a = 2

and (a-d)a + a(a+d) + (a+d)(a-d) = 23/4 --(2)

and (a-d)(a)(a+d) = -18/4 --(3)

Subst. a = 2 in (3) gives (2-d).2.(2+d) = -9/2, and $4-d^2 = -9/4$, i.e. $d^2 = 25/4$, i.e. $d = \pm 5/2$

When a = 2, d = 5/2 roots are $-\frac{1}{2}$, 2, $4\frac{1}{2}$

When a = 2, d = -5/2 roots are $4\frac{1}{2}$, 2, $-\frac{1}{2}$

Thus the given eqn. has roots $-\frac{1}{2}$, 2, $4\frac{1}{2}$ #

[OR If a = 2, then (x-2) is a factor of $p(x) = 4x^3 - 24x^2 + 23x + 18$

By division or at sight, the other factor is $(4x^2 - 16x - 9) = (2x-9)(2x+1)$

Thus $p(x) = (x-2)(2x-9)(2x+1)$ and the reqd. roots are $-\frac{1}{2}$, 2, 9/2 #]

(ii) If α, β, γ are the roots of $x^3 - 7x^2 + 18x - 7 = 0$, then

α + β + γ = 7, αβ + βγ + γα = 18, αβγ = 7

$(1+\alpha^2)(1+\beta^2)(1+\gamma^2) = 1 + (\alpha^2+\beta^2+\gamma^2) + (\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2) + \alpha^2\beta^2\gamma^2$

Now $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = (7)^2 - 2(18) = 13$

and $\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 = (\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2(\alpha\beta.\beta\gamma+\beta\gamma.\gamma\alpha+\gamma\alpha.\alpha\beta)$

$= (\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma) = (18)^2 - 2.7.7 = 226$

Thus $(1+\alpha^2)(1+\beta^2)(1+\gamma^2) = 1+13+226+(7)^2 = 289$ #

(iii) The eqn. $y = \sqrt{x(1-x)}$ represents the 'top half' of the curve $y = \pm\sqrt{x(1-x)}$, i.e. $y^2 = x(1-x)$ i.e. $x^2+y^2-x = 0$, i.e. $(x-\frac{1}{2})^2+y^2 = \frac{1}{4}$, i.e. the circle centre $(\frac{1}{2},0)$ radius $\frac{1}{2}$ unit.

$\int_0^1 \sqrt{x(1-x)}\,dx$ represents the area of a semi-circle and its value is

$$\tfrac{1}{2}.\pi(\tfrac{1}{2})^2 = \frac{\pi}{8}.$$

Divide the interval [0,1] into n equal sub-divisions, each of width h = 1/n and construct the rectangles as shown. Sum S of the areas of these rectangles is given by

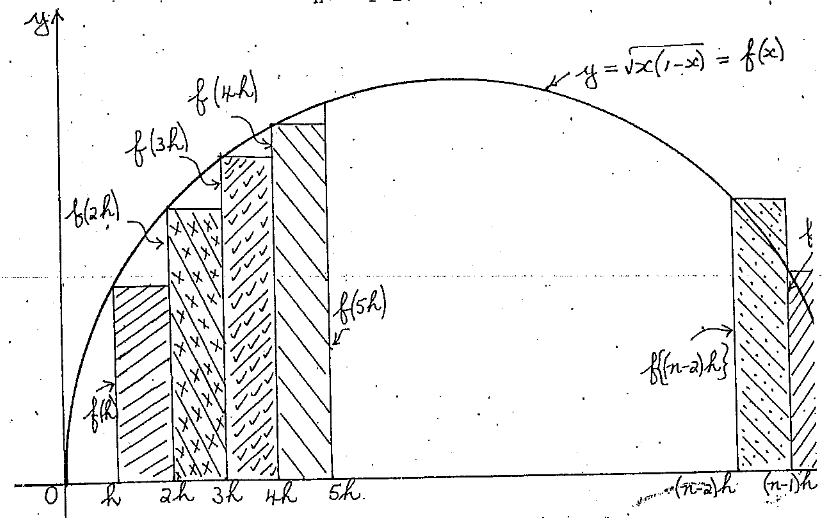
$$S = h\{f(h) + f(2h) + f(3h) + \ldots\ldots + f(\overline{n-1}h)\}$$

$$= h\sum_{r=1}^{n-1} f(rh) = \frac{1}{n}\sum_{r=1}^{n-1} f(\tfrac{r}{n}) = \frac{1}{n}\sum_{r=1}^{n-1}\sqrt{\tfrac{r}{n}(1-\tfrac{r}{n})}$$

$$= \frac{1}{n^2}\sum_{r=1}^{n-1}\sqrt{r(n-r)}$$

$$\therefore \text{ Area 'under' curve } = \lim_{n\to\infty}\frac{1}{n^2}\sum_{r=1}^{n-1}\sqrt{r(n-r)}$$

Since this area = $\pi/8$, then $\lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{n^2}\sqrt{r(n-r)} = \frac{\pi}{8}$ #



8.(i)(a) No. of codewords using all 7 letters A A B C D E E, noting ther are 2 A's and 2 E's is $\frac{7!}{2!2!} = 1260$ #

(b) No of codewords using only 4 letters, these letters being

all different (i.e. 4 chosen from A B C D E) is ${}^5C_4 \times 4! = 120$

2A's and no E (i.e. AA and 2 of B C D) is ${}^3C_2 \times \frac{4!}{2!} = 36$

2A's and 1E (i.e. AAE and 1 of B C D) is ${}^3C_1 \times \frac{4!}{2!} = 36$

2A's and 2E's (i.e. AAEE) is $\frac{4!}{2!2!} = 6$

2E's and no A (i.e. EE and 2 of B C D) is ${}^3C_2 \times \frac{4!}{2!} = 36$

2E's and 1A (i.e. EEA and 1 of B C D) is ${}^3C_1 \times \frac{4!}{2!} = 36$

These cover all the possible 4-letter codewords, and the total no. is 27

(ii) The 5 fifth roots of unity are the 5 roots of the eqn. $x^5 = 1$. Let these roots be of the form $r(\cos\theta + i\sin\theta)$; at sight $r = 1$.

Thus $(\cos\theta + i\sin\theta)^5 = 1$, i.e. $\cos 5\theta + i\sin 5\theta = 1 + 0i$

This gives $\cos 5\theta = 1$, $\sin 5\theta = 0$

whence $5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$ and

$\therefore\ \theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

The 5 fifth roots of unity are

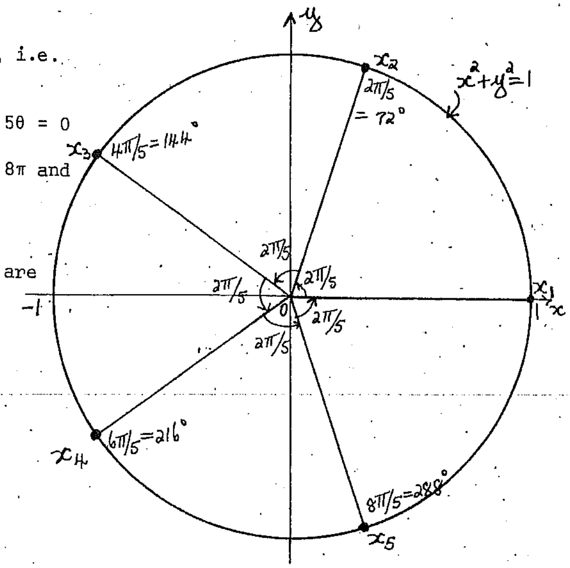
$x_1 = \cos 0 + i\sin 0 = 1$

$x_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$x_3 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$x_4 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$

$x_5 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$ #



The 5 roots lie on the circle centre 0, radius 1 unit, and angular distance $2\pi/5 = 72^\circ$ apart. #

If $x_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha$

then $x_3 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = (\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5})^2 = \alpha^2$

and $x_4 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = \cos(2\pi - \frac{4\pi}{5}) + i\sin(2\pi - \frac{4\pi}{5})$

$= \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5} = \cos(\frac{-4\pi}{5}) + i\sin(\frac{-4\pi}{5}) = \alpha^{-2}$

and $x_5 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \cos(2\pi - \frac{2\pi}{5}) + i\sin(2\pi - \frac{2\pi}{5})$

$= \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} = \cos(\frac{-2\pi}{5}) + i\sin(\frac{-2\pi}{5}) = \alpha^{-1}$ #

$\alpha^2 + \alpha^{-2} = (\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}) + (\cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}) = 2\cos\frac{4\pi}{5}$ #

$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^{-2})(x-\alpha^{-1})$

$= (x-1)\{x^2 - (\alpha+\alpha^{-1})x + \alpha\alpha^{-1}\}\{x^2 - (\alpha^2+\alpha^{-2})x + \alpha^2\alpha^{-2}\}$

$= (x-1)(x^2 - 2\cos\frac{2\pi}{5}x + 1)(x^2 - 2\cos\frac{4\pi}{5}x + 1)$, as the product of a linear factor and two real quadratic factors #