SPECIMEN PAPER 1

Instructions. Time allowed: three hours.

All questions may be attempted. All questions are of equal value.

Note. The questions are not necessarily arranged in order of difficulty.

Candidates are advised to read the whole paper carefully at the start.

Mathematical tables will be provided; approved slide-rules or calculators may be used.

Question 1

(i) Find the derivatives, in simplest form, of

(a) $\sqrt{\dfrac{1+x^2}{1-x^2}}$ (b) $(\tan^{-1}x)^x$ (c) $\tan^3(\sqrt{6x+7})$

(ii) Find dy/dx if

(a) $x = e^{-t}\cos t$, $y = e^{-t}\sin t$, and $t = 0$.

(b) $2x^2 - 3xy + y^2 = 6$ and $x = 1$.

(iii) If $y = \sin(\log x)$, show that $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + ky = 0$, where k is a constant, and find k.

Question 2

(i) Find the indefinite integrals

(a) $\int\dfrac{2x\,dx}{(x+1)(x+3)}$ (b) $\int x^2e^{x^3+1}\,dx$ (c) $\int\dfrac{x+1}{\sqrt{4-9x^2}}\,dx$

(ii) Evaluate (a) $\int_0^{\pi/2}\sin^2 x\cos^2 x\,dx$ (b) $\int_0^{\pi/2}\dfrac{d\theta}{2+\cos\theta}$

(iii) If $u_n = \int_0^{\pi/2} x^n\sin x\,dx$, prove that, for $n \geq 2$,

$$u_n + n(n-1)\,u_{n-2} = n(\pi/2)^{n-1}$$

Calculate u_1 directly, and deduce that $u_3 = \frac{3}{4}\pi^2 - 6$

Question 3

(i) If z represents a variable complex number, show the region of the Argand diagram in which $2 < |z| < 3$ and $\frac{\pi}{3} \leq \arg z \leq \frac{2\pi}{3}$.

(ii) State and prove De Moivre's Theorem for an integral index (both positive and negative).

(a) Simplify $\dfrac{(\cos\theta + i\sin\theta)^9.(\cos 3\theta + i\sin 3\theta)^{-5}}{(\cos 2\theta - i\sin 2\theta)^4}$

(b) Express $-1 + i\sqrt{3}$ in mod-arg form and hence evaluate $(-1+i\sqrt{3})^{-6}$

(iii) On an Argand Diagram, the points P, Q represent the complex numbers z, 1/z respectively. If P moves on the straight line $x = 1$, show that Q lies on a certain circle, and find its centre and radius.

Question 4

(i) Show that the circle on diameter the join of (x_1, y_1) and (x_2, y_2) has equation $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

(ii) Prove that the tangent at the point P $(a \cos \theta, b \sin \theta)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ has equation $\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$.

(a) This ellipse meets the y-axis at B, B'. The tangents at B, B' to the ellipse meet the tangent at P at the points Q, Q' respectively. Prove that $BQ.B'Q' = a^2$.

(b) The circle on QQ' as diameter meets the x-axis at the points R, R'. Prove that $OR.OR' = a^2 - b^2$, where O is the origin.

Question 5

(i) Sketch the curve $9y^2 = x(3 - x)^2$ and show that it forms a loop. Find

(a) the maximum width of loop measured parallel to the y-axis;

(b) the area enclosed by the loop.

(ii)(a) The base of a certain solid S_1 is the region bounded by the parabola $y^2 = 4ax$ and the latus rectum. Each section of the solid by planes parallel to the y-axis is an equilateral triangle. Find the volume of S_1.

(b) The area bounded by the parabola $y^2 = 4ax$ and the latus rectum is rotated through four right angles about the latus rectum. Calculate the volume of the solid S_2 so generated.

Question 6

Prove, from first principles, that the acceleration of a point moving with constant angular velocity ω in a circle of radius r is $r\omega^2$ towards the centre.

(a) A particle is fastened to one end of a light inextensible string of length ℓ, the other end of which is fastened to a fixed point O. The particle rotates with uniform angular velocity ω about the vertical through O. Show that if α is the inclination of the string to the downward vertical, then $\alpha = \cos^{-1}(g/\ell\omega^2)$ and deduce that steady circular motion is impossible if $\ell\omega^2 < g$.

What is the effect on the inclination of the string to the vertical by an increase in ω?

(b) If the point O, instead of being fixed, is descending with uniform acceleration f, the particle still rotating with uniform angular velocity ω, find f in order that the string may make an angle α with the upward vertical.

Question 7

(*i*) Show that $1 + i$ is a root of the polynomial $p(x) = x^3 + x^2 - 4x + 6$ and hence resolve $p(x)$ into irreducible factors over the field of

(a) complex numbers (b) real numbers.

(*ii*) If α, β, γ are the roots of the cubic equation $x^3 + qx + r = 0$, find the value of $\Sigma\alpha$, $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$ in terms of q, r.

Hence prove that $(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2 = -6q$.

(*iii*) Verify, without use of tables, calculator or slide rule, that $\theta = 18^0$ satisfies the equation $\sin 2\theta = \cos 3\theta$.

Deduce that $\sin 18^0$ is a root of the equation $4x^2 + 2x - 1 = 0$, and hence find the value of $\sin 18^0$ in simplest surd form. What does the other root represent?

Question 8

(*i*) Assuming a, b, c, d to be real, write down the condition that the roots of the equation $(a^2 + b^2)x^2 + 2(ac + bd)x + (c^2 + d^2) = 0$ should be real.

Then prove that if this condition holds, the roots must be equal; and show that they are equal to $-c/a$.

(*ii*) State the binomial theorem for $(1 + x)^n$ where n is a positive integer.

(a) If k is a positive integer, show that $(1 + \frac{1}{n})^k \to 1$ as $n \to \infty$.

(b) Prove that $(1 + \frac{1}{n})^n$ approaches the sum of an infinite series as $n \to \infty$.

(c) Show that $\frac{1}{n!} < \frac{1}{2^{n-1}}$ for all positive integral $n \geq 3$ and use this to deduce that $\lim_{n\to\infty}(1 + \frac{1}{n})^n = \ell$ where $2 < \ell < 3$.

SPECIMEN PAPER 1

1.(i)(a) $\frac{d}{dx}\sqrt{\frac{1+x^2}{1-x^2}} = \frac{1}{2}\left(\frac{1+x^2}{1-x^2}\right)^{-\frac{1}{2}}.\left\{\frac{(1-x^2).2x-(1+x^2).-2x}{(1-x^2)^2}\right\}$

$= \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{\frac{1}{2}}.\frac{4x}{(1-x^2)^2} = \frac{2x}{(1+x^2)^{\frac{1}{2}}(1-x^2)^{3/2}}$ #

(b) Let $y = (\tan^{-1}x)^x$; *taking natural logarithms, then*

$\log y = x\log(\tan^{-1}x)$, *and differentiating with respect to x*

$\therefore \frac{1}{y}\frac{dy}{dx} = x.\left\{\frac{1}{\tan^{-1}x}.\frac{1}{1+x^2}\right\} + \log(\tan^{-1}x).1$, *as a product*

$\therefore \frac{dy}{dx} = (\tan^{-1}x)^x\left\{\frac{x}{\tan^{-1}x(1+x^2)} + \log(\tan^{-1}x)\right\}$ #

(c) $\frac{d}{dx}(\tan\sqrt{6x+7})^3 = 3(\tan\sqrt{6x+7})^2.\frac{d}{dx}(\tan\sqrt{6x+7})$

$= 3\tan^2\sqrt{6x+7}.\sec^2\sqrt{6x+7}.\frac{d}{dx}(\sqrt{6x+7}) = \frac{9\tan^2\sqrt{6x+7}\sec^2\sqrt{6x+7}}{\sqrt{6x+7}}$ #

(ii)(a) If $x = e^{-t}\cos t$, then $\frac{dx}{dt} = e^{-t}.-\sin t + \cos t.-e^{-t} = -e^{-t}(\sin t + \cos t)$

and $y = e^{-t}\sin t$, then $\frac{dy}{dt} = e^{-t}.\cos t + \sin t.-e^{-t} = -e^{-t}(\cos t - \sin t)$?

Thus $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t - \sin t}{-(\cos t + \sin t)} = \frac{1-0}{-(1+0)} = -1$ at $t = 0$ #

(b) Differentiate $2x^2 - 3xy + y^2 = 6$ implicitly w.r.t.x,

then $4x - 3(x\frac{dy}{dx} + y.1) + 2y\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{3y-4x}{2y-3x}$

Subst. $x = 1$ in given eqn. gives $2 - 3y + y^2 = 6$, i.e. $y = 4$ or -1

At (1, 4); $\frac{dy}{dx} = \frac{12-4}{8-3} = \frac{8}{5}$ # and at (1, -1); $\frac{dy}{dx} = \frac{-3-4}{-2-3} = \frac{7}{5}$ #

(iii) If $y = \sin(\log x)$, then $\frac{dy}{dx} = \cos(\log x).\frac{1}{x}$, i.e. $x\frac{dy}{dx} = \cos(\log x)$

Differentiating again, $\{x.\frac{d^2y}{dx^2} + \frac{dy}{dx}.1\} = -\sin(\log x).\frac{1}{x}$ $(= -y.\frac{1}{x})$

Thus, $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + ky = 0$, where $k = 1$ #

2.(i)(a) Let $\frac{2x}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$, $\therefore 2x = A(x+3) + B(x+1)$

Subst. $x = -1, -3$ resp. gives $A = -1$, $B = 3$

$\int\frac{2x\,dx}{(x+1)(x+3)} = \int\left(\frac{-1}{x+1} + \frac{3}{x+3}\right)dx = -\log(x+1) + 3\log(x+3) + C$ #

(b) Let $u = x^3 + 1$, $\therefore du = 3x^2\,dx$, i.e. $x^2\,dx = du/3$

Given integral $= \int \frac{1}{3} du.e^u = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3+1} + C$ #

(c) Let $I = \int \frac{x+1}{\sqrt{4-9x^2}}dx = \int \left(\frac{x}{\sqrt{4-9x^2}} + \frac{1}{\sqrt{4-9x^2}}\right) dx = I_1 + I_2$

In I_1, put $u = \sqrt{4-9x^2}$, i.e. $u^2 = 4 - 9x^2$, i.e. $2u\,du = -18x\,dx$

$I_1 = \int \frac{2u\,du/-18}{u} = -\frac{1}{9}\int du = -\frac{1}{9}u + C_1 = -\frac{1}{9}\sqrt{4-9x^2} + C_1$

$I_2 = \int \frac{dx}{3\sqrt{4/9 - x^2}} = \frac{1}{3}\sin^{-1}\left(\frac{x}{2/3}\right) + C_2 = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C_2$

Thus $I = -\frac{1}{9}\sqrt{4-9x^2} + \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C$ #

(ii)(a) $\int_0^{\pi/2} \sin^2 x \cos^2 x\,dx = \int_0^{\pi/2} \frac{1}{4}\sin^2 2x\,dx = \frac{1}{4}.\frac{1}{2}\int_0^{\pi/2} (1 - \cos 4x)\,dx$

$= \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi/2} = \frac{1}{8}.\frac{\pi}{2} = \frac{\pi}{16}$ #

{We use the 2θ results: sin 2θ = 2 sin θ cos θ and cos 2θ = 1 - 2 sin²θ.}

(b) Let $t = \tan\frac{\theta}{2}$, $\therefore dt = \sec^2\frac{\theta}{2}.\frac{1}{2}d\theta = (1+t^2).\frac{1}{2}d\theta$, i.e. $d\theta = \frac{2dt}{1+t^2}$

When $\theta = 0$, $t = 0$ and when $\theta = \pi/2$, $t = 1$

$\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta} = \int_0^1 \frac{2dt/(1+t^2)}{2 + (1-t^2)/(1+t^2)} = \int_0^1 \frac{2dt}{2(1+t^2) + (1-t^2)}$

$= \int_0^1 \frac{2dt}{3+t^2} = 2.\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right]_0^1 = \frac{2}{\sqrt{3}}.\frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$ #

{In this type, we use the 't' results $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$*}*

(iii) $u_n = \int_0^{\pi/2} x^n \sin x\,dx = \int_0^{\pi/2} x^n.\frac{d}{dx}(-\cos x)\,dx$, *(integrating by parts)*

$= \left[x^n.-\cos x\right]_0^{\pi/2} - \int_0^{\pi/2} -\cos x.n\,x^{n-1}\,dx = 0 + n\int_0^{\pi/2} x^{n-1}\frac{d}{dx}(\sin x)\,dx$

$= n\{\left[x^{n-1}\sin x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x.(n-1)x^{n-2}\,dx\} = n\{(\pi/2)^{n-1} - (n-1)\,u_{n-2}\}$

i.e. $u_n + n(n-1)\,u_{n-2} = n(\pi/2)^{n-1}$ # --------------- (L)

Now $u_1 = \int_0^{\pi/2} x^1 \sin x\,dx = \int_0^{\pi/2} x\frac{d}{dx}(-\cos x)\,dx$

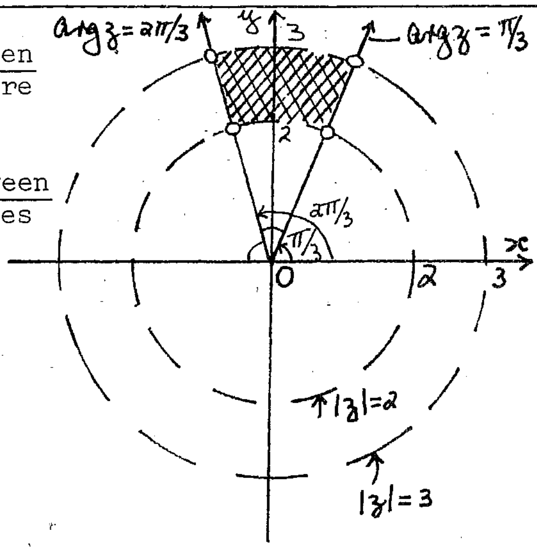
$= \left[x.-\cos x\right]_0^{\pi/2} - \int_0^{\pi/2} -\cos x.1.dx$

$= 0 + \left[\sin x\right]^{\pi/2} = 1$

Subst. $n = 3$ in (L) gives $u_3 + 3.2\,u_1 = 3(\pi/2)^2$

Thus $u_3 = 3(\pi/2)^2 - 6u_1 = 3(\pi/2)^2 - 6.1 = \frac{3}{4}\pi^2 - 6$ #

3.(i) $2 < |z| < 3$ represents the region between the circles $|z| = 2$, $|z| = 3$; which have centre O and radius 2, 3 units.



$2\pi/3 \leq \arg z \leq \pi/3$ represents the region between (and on) the rays directed from 0 making angles of $\pi/3$, $2\pi/3$ with the positive x-axis.

The reqd. region is shaded (points of intersection not incl.) #

(ii) De Moivre's Thm. states that if n is a positive or negative integer, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Proof Consider the positive integer n, and use the method of induction.

When n = 1, $(\cos\theta + i\sin\theta)^1 = \cos 1\theta + i\sin 1\theta$, which is true

Assume that $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ (k + ve integer)

Now $(\cos\theta + i\sin\theta)^{k+1}$

$= (\cos\theta + i\sin\theta)^k.(\cos\theta + i\sin\theta) = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= (\cos k\theta \cos\theta - \sin k\theta \sin\theta) + i(\sin k\theta \cos\theta + \cos k\theta \sin\theta)$

$= \cos(k\theta + \theta) + i\sin(k\theta + \theta) = \cos(k+1)\theta + i\sin(k+1)\theta$,

and the result is true for n = k + 1

Since the result is true for n = 1, it is true for n = 1 + 1 = 2 and hence for n = 2 + 1 = 3, and so on for all +ve integral n.

Consider the negative integer m = -n (where n > 0)

$(\cos\theta + i\sin\theta)^m$

$= (\cos\theta + i\sin\theta)^{-n} = \dfrac{1}{(\cos\theta + i\sin\theta)^n} = \dfrac{1}{\cos n\theta + i\sin n\theta}$

$= \dfrac{1}{\cos n\theta + i\sin n\theta} \times \dfrac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta} = \dfrac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta}$

$= \dfrac{\cos(-n\theta) + i\sin(-n\theta)}{1} = \cos m\theta + i\sin m\theta$,

and the result is true when m is a negative integer #

(a) Noting $\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$;

$\cos 2\theta - i\sin 2\theta = (\cos\theta + i\sin\theta)^{-2}$

given exp. $= (\cos\theta + i\sin\theta)^9.(\cos\theta + i\sin\theta)^{-15} \div (\cos\theta + i\sin\theta)^{-8}$

$$= (\cos\theta + i\sin\theta)^{9-15+8} = (\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta\#$$

OR exp. $= (\cos\theta + i\sin\theta)^9.(\cos 3\theta + i\sin 3\theta)^{-5}.\{\cos(-2\theta) + i\sin(-2\theta)\}^{-4}$

$= (\cos 9\theta + i\sin 9\theta).\{\cos(-15\theta) + i\sin(-15\theta)\}.\{\cos 8\theta + i\sin 8\theta\}$

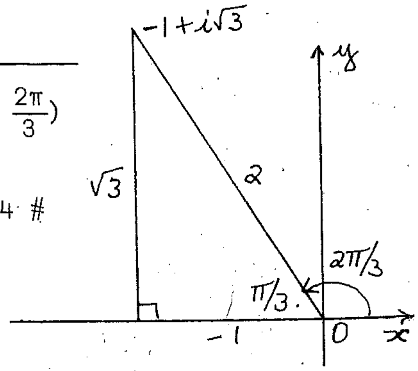
$= \cos(9\theta - 15\theta + 8\theta) + i\sin(9\theta - 15\theta + 8\theta) = \cos 2\theta + i\sin 2\theta\#$

<u>Note</u> $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha + \beta) + i\sin(\alpha + \beta)$ etc}

(b) In mod-arg form, $-1 + i\sqrt{3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$\therefore (-1 + i\sqrt{3})^{-6} = 2^{-6}\{\cos(-4\pi) + i\sin(-4\pi)\}$

$= 2^{-6}(\cos 4\pi - i\sin 4\pi) = 2^{-6}(1 - i.0) = 1/64\ \#$



(iii) Let $z = x + iy$ and $w = u + iv = 1/z$

Now $u + iv = \dfrac{1}{x + iy} = \dfrac{x - iy}{x^2 + y^2} = \dfrac{x}{x^2 + y^2} + \dfrac{i.-y}{x^2 + y^2}$

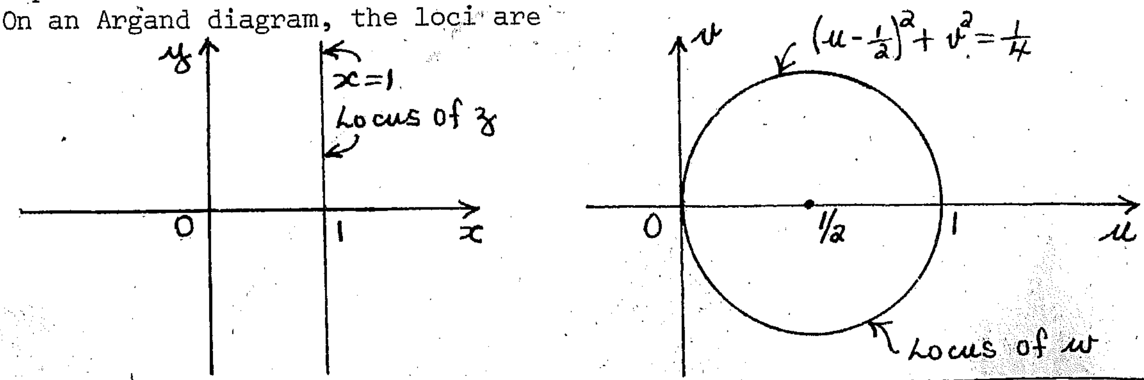
Thus $u = \dfrac{x}{x^2 + y^2} = \dfrac{1}{1 + y^2}$ and $v = \dfrac{-y}{x^2 + y^2} = \dfrac{-y}{1 + y^2}$, <u>when $x = 1$.</u>

To find the locus of Q ($w = 1/z$), we relate u, v and eliminate x,y.

Now $u^2 + v^2 = \left(\dfrac{1}{1 + y^2}\right)^2 + \left(\dfrac{-y}{1 + y^2}\right)^2 = \dfrac{1 + y^2}{(1 + y^2)^2} = \dfrac{1}{1 + y^2} = u$

Thus the locus of Q has eqn. $u^2 + v^2 = u$, i.e. $(u - \frac{1}{2})^2 + v^2 = \frac{1}{4}$, which represents a circle, centre $(\frac{1}{2}, 0)$ radius $\frac{1}{2}$ unit.#

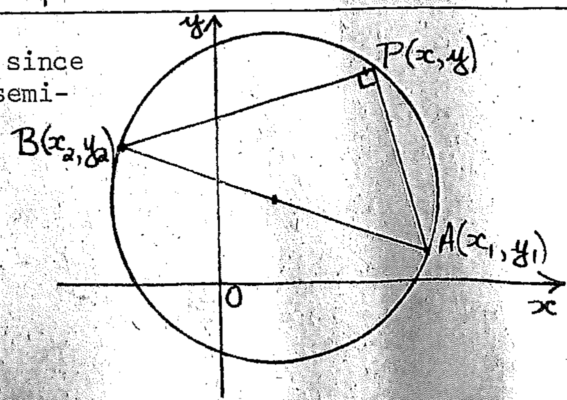
On an Argand diagram, the loci are



4.(i) Let P (x,y) be on the circle, and since AB is a diameter, then $A\hat{P}B = 90^\circ$, in a semi-circle..

Grad. PA $= \dfrac{y - y_1}{x - x_1}$; grad. PB $= \dfrac{y - y_2}{x - x_2}$

Thus $\dfrac{y - y_1}{x - x_1} . \dfrac{y - y_2}{x - x_2} = -1,$



i.e. $(y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$

and $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ is the req'd eqn. #

(ii) From $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\therefore \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$, i.e. $\frac{dy}{dx} = \frac{-b^2x}{a^2y}$

At P (a cos θ, b sin θ), $\frac{dy}{dx} = \frac{-b^2.a\cos\theta}{a^2.b\sin\theta} = \frac{-b\cos\theta}{a\sin\theta}$

{OR *if* $x = a\cos\theta$, $y = b\sin\theta$ *then* $\frac{dx}{d\theta} = -a\sin\theta$, $\frac{dy}{d\theta} = b\cos\theta$

and $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta}$ }

Eqn of tangent at P is $y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

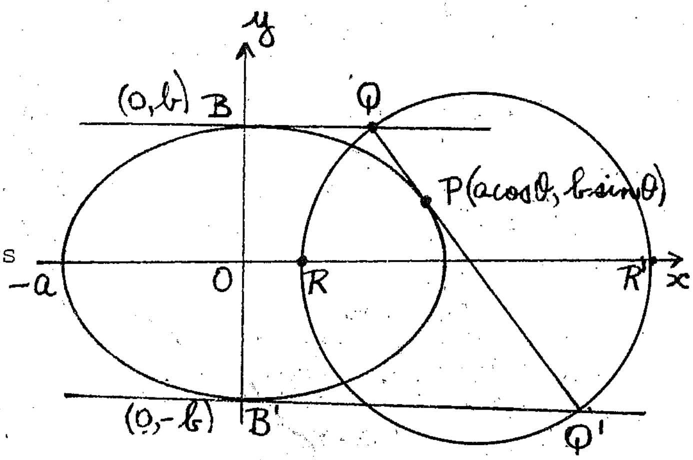
i.e. $a\sin\theta y - ab\sin^2\theta = -b\cos\theta x + ab\cos^2\theta$, i.e. $b\cos\theta x + a\sin\theta y = ab$

Dividing by ab gives eqn, as $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ #

(a) Tan at B has eqn. $y = b$. This meets tan at P in Q, where

$\frac{x\cos\theta}{a} + \frac{b\sin\theta}{b} = 1$,

i.e. $x = \frac{a}{\cos\theta}(1 - \sin\theta)$



Tan at B' has eqn. $y = -b$ and meets tan at P in Q', where

$\frac{x\cos\theta}{a} + \frac{-b\sin\theta}{b} = 1$

i.e. $x = \frac{a}{\cos\theta}(1 + \sin\theta)$

Thus $BQ = \frac{a}{\cos\theta}(1 - \sin\theta)$ and $B'Q' = \frac{a}{\cos\theta}(1 + \sin\theta)$,

Hence $BQ.\ B'Q' = \frac{a^2}{\cos^2\theta}(1 - \sin^2\theta) = a^2$ #

(b) Using result of (i), eqn. of circle on QQ' as diameter is

$$\{x - \frac{a}{\cos\theta}(1 - \sin\theta)\}\{x - \frac{a}{\cos\theta}(1 + \sin\theta)\} + (y - b)(y + b) = 0$$

i.e. $x^2 - \frac{a}{\cos\theta}\{(1 - \sin\theta) + (1 + \sin\theta)\}x + a^2 + (y^2 - b^2) = 0$

This circle meets the x - axis (y = 0) at pts R, R' whose abscissae OR, OR' are the roots of the eqn. $x^2 - \frac{2a}{\cos\theta}x + a^2 - b^2 = 0$.

Thus OR.OR' = product of roots of this eqn. $= a^2 - b^2$ #

5.(i) If $9y^2 = x(3-x)^2$ then $y = \pm\frac{1}{3}x^{\frac{1}{2}}(3-x) = \pm\frac{1}{3}(3x^{\frac{1}{2}} - x^{3/2})$, and given eqn. represents 2 sections, reflections of one another in x-axis.

Consider $y = \frac{1}{3}x^{\frac{1}{2}}(3-x) = \frac{1}{3}(3x^{\frac{1}{2}} - x^{3/2})$

Domain is $x \geq 0$; curve meets x-axis where $x = 0,3$ (for $0 < x < 3$, $y > 0$ and for $x > 3$, $y < 0$).

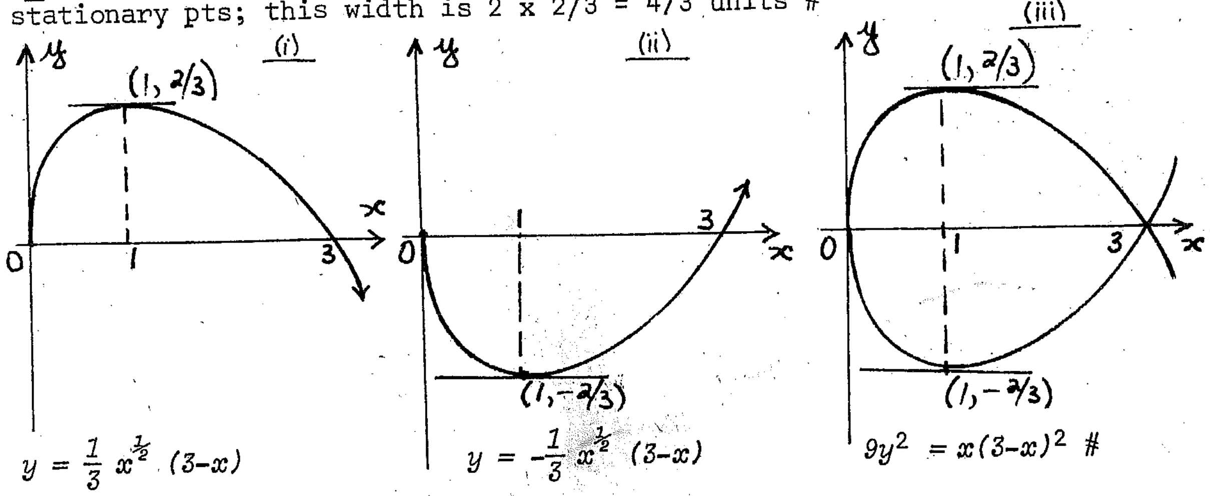
Now $y' = \frac{dy}{dx} = \frac{1}{2}(x^{-\frac{1}{2}} - x^{\frac{1}{2}}) = \frac{1}{2\sqrt{x}}(1-x)$

At (0,0), y' is undefined ($y' \to \infty$); the y-axis is a vertical tangent whilst at (3,0), $y' = -1/\sqrt{3}$ and the tang. is incl. at 150° to x-axis.

Also $y' = 0$ at (1,2/3); through $x = 1$, y' passes from + to - and there is a max. staty. pt. then.

This curve is sketched in fig (i), the reflected curve in fig (ii) and the given eqn. in fig (iii).

(a) Max. width of loop measured parallel to y-axis occurs at the stationary pts; this width is $2 \times 2/3 = 4/3$ units #

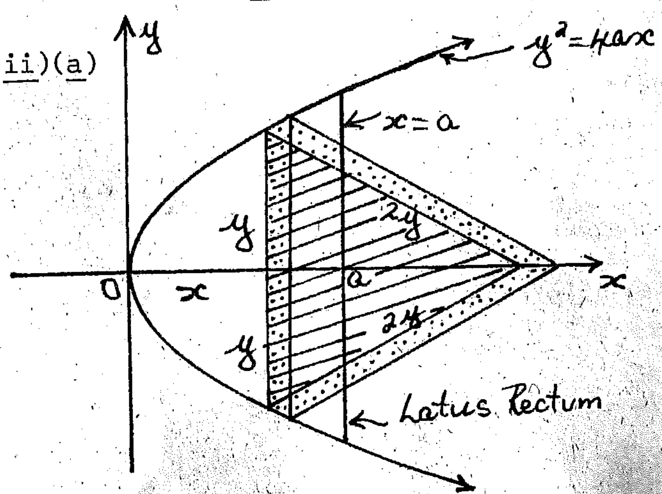


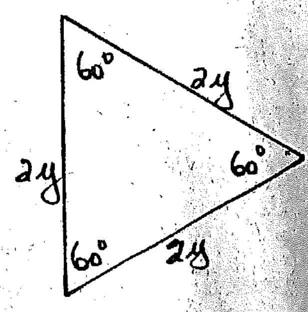
The curve $9y^2 = x(3-x)^2$ forms a loop as shown in the sketch #

(b) Area of loop = 2 × area 'above' x-axis $= 2\int_0^3 \frac{1}{3}(3x^{\frac{1}{2}} - x^{3/2})dx$

$= \frac{2}{3}\left[3.\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^3 = \frac{4}{3}\{3\sqrt{3} - \frac{1}{5}.9\sqrt{3}\} = \frac{8\sqrt{3}}{5}$ units^2 #

(ii)(a)





Area of face section

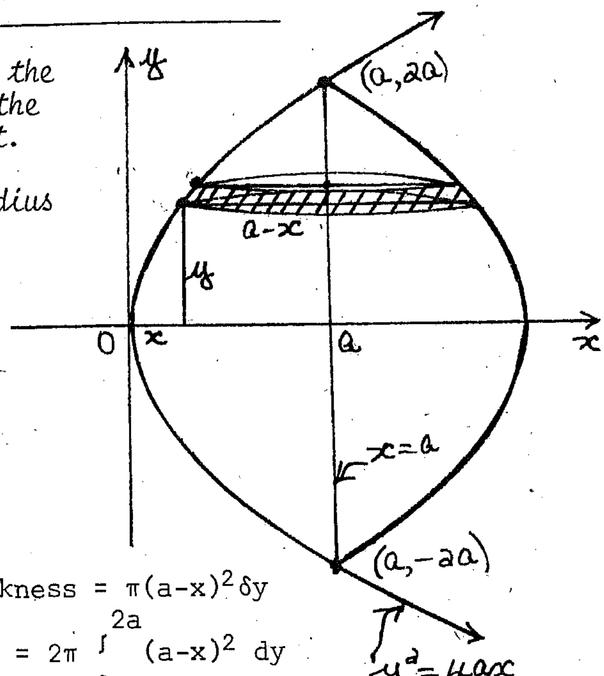
Consider an element of volume of the solid taken between 2 planes parallel to y-axis, distant x, x + δx from it. This element is of thickness δx and has equilateral triangles as faces; its volume ≑ area of face x thickness.

Area of face = area of equilat. triangle of side 2y = $\frac{1}{2}.2y.2y\sin 60^\circ$
$= \sqrt{3}y^2 = \sqrt{3}.4ax$ since $y^2 = 4ax$

Vol. of element ≑ $4\sqrt{3}ax.\delta x$

Vol. of solid $S_1 = \lim_{\delta x\to 0} \sum_{x=0}^{a} 4\sqrt{3}ax\delta x = \int_0^a 4\sqrt{3}ax\,dx = 2\sqrt{3}\,a^3$ units^3#

(b) Consider an element of volume of the solid taken between planes parl. to the x-axis and distant y, y + δy above it. This element is of thickness δy and approximates a cylinder with base radius (a - x).



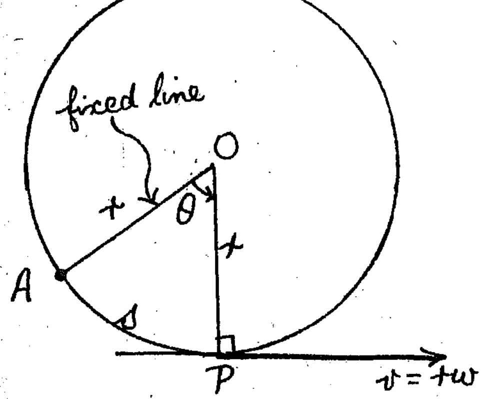
Vol. of element ≑ area of end x thickness = $\pi(a-x)^2\delta y$

Vol. of solid $S_2 = \lim_{\delta y\to 0} \sum_{y=-2a}^{y=2a} \pi(a-x)^2\delta y = 2\pi\int_0^{2a}(a-x)^2\,dy$

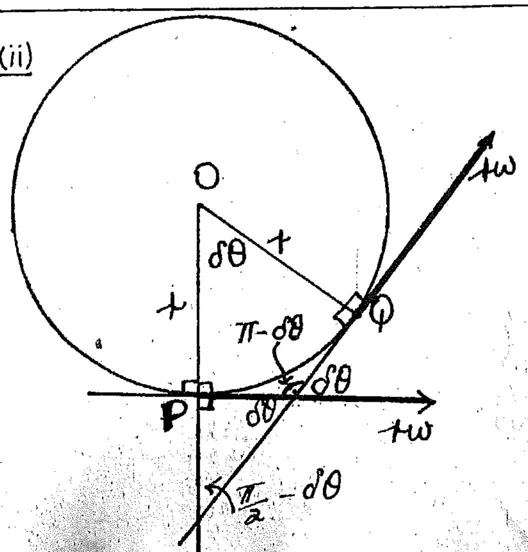
$= 2\pi\int_0^{2a}(a^2-2ax+x^2)dy = 2\int_0^{2a}(a^2-\frac{y^2}{2}+\frac{y^4}{16a^2})dy$

$= 2\pi\left[a^2y - \frac{y^3}{6} + \frac{y^5}{80a^2}\right]_0^{2a} = \frac{32\pi a^3}{15}$ units^3 #

6.(i)



(ii)



In fig (i), the point P moves on the circle centre O radius r, from A to P (distant s) in time t, where $A\hat{O}P = \theta$ and OA is a fixed line. The angular velocity ω of P is defined as $d\theta/dt$; the linear vel. v of P is defined as ds/dt, and is directed along the tangent at P.

Now $s = r\theta$, $\therefore \frac{ds}{dt} = r\frac{d\theta}{dt}$, i.e. $v = r\omega$, and since ω is constant, then v is constant for all points on the circle.

In fig (ii), the point traverses the arc PQ where $P\hat{O}Q = \delta\theta$ in the time interval δt. By geometry, we obtain the angles shown.

Resolving the velocity at Q parallel and perp. to the tangent at P, the vel. component parl. to this tangent is $r\omega\cos\delta\theta$ and the vel. component perp. to this tangent (i.e. along the radius PO towards O) is $r\omega\cos(\pi/2-\delta\theta) = r\omega\sin\delta\theta$.

The change in velocity δv of the point in going from P to Q

parallel to the tangent at P is $r\omega\cos\delta\theta - r\omega = 0$

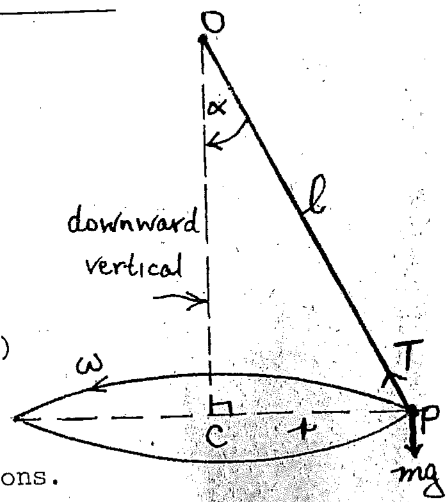
along the radius PO towards O is $r\omega\sin\delta\theta - r\omega\cos\pi/2 = r\omega\delta\theta$,

noting that to the first order of small quantities $\cos\delta\theta = 1$ and $\sin\delta\theta = \delta\theta$.

The only velocity change is directed towards O and is of magnitude $r\omega\delta\theta$, this occurs in time δt, and thus

$$\frac{dv}{dt} = \lim_{\delta t\to 0}\frac{\delta v}{\delta t} = \lim_{\delta t\to 0} r\omega\frac{\delta\theta}{\delta t} = r\omega.\frac{d\theta}{dt} = r\omega^2$$

Hence the acceln. of the point P is $r\omega^2$ towards the centre O #



(a) Let the particle P be of mass m; the only forces acting on the particle are its weight mg and the tension T in the string.

Since P moves with uniform ang. vel. ω about the vertical OC, the acceln. of P is $r\omega^2$ towards the centre C, where r is the radius of the circle.

There is no tangential acceleration in the plane of the circle, and also no upwards (or downwards) movement of P, i.e. f = 0 in ↓↑ directions.

The eqns. of motion of P can be obtained by resolving the forces on P in appropriate directions.

Resolving vertically, then $m.0 = mg - T\cos\alpha$, i.e. $T\cos\alpha = mg$ ---(1)

Resolving towards the centre C, then $mr\omega^2 = T\sin\alpha$ --- (2)

In ΔOCP, $\sin\alpha = r/\ell$, i.e. $r = \ell\sin\alpha$ and $\therefore T\sin\alpha = m\ell\sin\alpha\omega^2$

This gives $T = m\ell\omega^2$ and subst. in (1), $\therefore m\ell\omega^2\cos\alpha = mg$

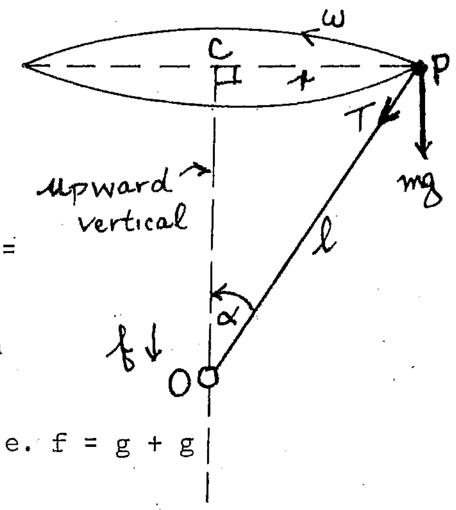
i.e. $\cos\alpha = g/(\ell\omega^2)$, i.e. $\alpha = \cos^{-1}(g/\ell\omega^2)$#

Now $0 \le \cos\alpha \le 1$, i.e. $0 \le \frac{g}{\ell\omega^2} \le 1$, i.e. $g \le \ell\omega^2$

Thus steady circular motion is impossible if $g > \ell\omega^2$, i.e. $\ell\omega^2 < g$ for then $\cos\alpha > 1$ which cannot be.#

If ω is increased, then $\cos\alpha = g/\ell\omega^2$ is decreased and thus α is increased #, noting as $\cos\alpha \to 0$, then $\alpha \to \pi/2$.

(b) Since the point O is descending with uniform acceleration f, then for the angle α to remain constant, the particle P must also descend with acceln. f.



The eqns. of motion of P now are

vertically, $mf = mg + T\cos\alpha$ --(3), where $\cos\alpha = g/\ell\omega^2$ from (a)

towards the centre C, $mr\omega^2 = T\sin\alpha$ --(4), which gives $T = m\ell\omega^2$

Subst. for T in (3), then $mf = mg + m\ell\omega^2 . \frac{g}{\ell\omega^2}$, i.e. $f = g + g$

Thus the reqd. value of f is 2g#

7.(i) $(1 + i)^2 = 1 + 2i + i^2 = 2i$; $(1 + i)^3 = 2i\,(1 + i) = 2i - 2$

$p(1 + i) = (1 + i)^3 + (1 + i)^2 - 4(1 + i) + 6 = 2i - 2 + 2i - 4 - 4i + 6 = 0$, and hence $1 + i$ is a root of p(x)#

Since $p(x) = x^3 + x^2 - 4x + 6$ has real coeffts, i.e. p(x) is a polynomial over the field of real numbers, R*, the complex roots occur in conjugate pairs, and thus $1 - i$ is also a root of p(x).

Now p(x) has roots $1 + i$, $1 - i$, γ (say) and the sum of these roots is $-$ coefft x^2/coefft x^3; i.e. $(1 + i) + (1 - i) + \gamma = -1$, i.e. $\gamma = -3$.

Hence the factors of p(x) over the field of

(a) *complex numbers*, C, are $\{x - (1+i)\}\{x - (1-i)\}(x+3)$#

(b) *real numbers*, R*, are $\{x^2 - (1+i + 1-i)x + (1+i)(1-i)\}(x+3)$, i.e. $(x^2 - 2x + 2)(x+3)$#

(ii) If α, β, γ are the roots of $x^3 + qx + r = 0$, then

$$x^3 + qx + r \equiv (x-\alpha)(x-\beta)(x-\gamma) = \{x^2 - (\alpha + \beta)x + \alpha\beta\}(x - \gamma)$$
$$= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$$

Equating coeffts of like powers, then

$\sum\alpha = \alpha + \beta + \gamma = 0$; $\sum\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = q$, $\sum\alpha\beta\gamma = \alpha\beta\gamma = -r$ #

Now $(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2 = 2(\alpha^2+\beta^2+\gamma^2) - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= 2\{(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)\} -2q = 2\{0^2-2q\} - 2q = -6q$ #

(iii) When $\theta = 18^\circ$, $\sin 2\theta = \sin 36^\circ$ and $\cos 3\theta = \cos 54^\circ$

Since $\sin 36^\circ = \cos 54^\circ$, then $\theta = 18^\circ$ satisfies eqn. $\sin 2\theta = \cos 3\theta$ #

{OR *$\sin 2\theta = \cos 3\theta = \sin(90^\circ-3\theta)$ i.e. one soln. of this eqn. is $2\theta = 90^\circ-3\theta$, i.e. $5\theta = 90^\circ$, i.e. $\theta = 18^\circ$*}

Solving $\sin 2\theta = \cos 3\theta$, gives $2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$, and thus $\cos\theta = 0$ or $2\sin\theta = 4\cos^2\theta - 3$, i.e. $2\sin\theta = 4(1-\sin^2\theta)-3$.

Since $\theta = 18^\circ$ is not a soln. of $\cos\theta = 0$, then it is a soln. of the eqn. $4\sin^2\theta + 2\sin\theta - 1 = 0$. If $\sin\theta = x$, then $\sin 18^\circ$ is a soln. of the eqn. $4x^2 + 2x - 1 = 0$.

By the quadratic formula, $x = \frac{-1\pm\sqrt{5}}{4}$. Since $\sin 18^\circ > 0$, then the value of $\sin 18^\circ = \frac{1}{4}(-1+\sqrt{5})$#

Now $\sin 2\theta = \cos 3\theta = \sin(90^\circ-3\theta) \rightarrow 5\theta = 90^\circ$, i.e. $\theta = 18^\circ$

$= \sin(450^\circ-3\theta) = \sin(810^\circ-3\theta) = \sin(1170^\circ-3\theta) = \sin(1530^\circ-3\theta)\ldots$

Solving these eqns successively gives $5\theta = 450^\circ, 810^\circ, 1170^\circ, 1530^\circ, \ldots$ whence $\theta \doteq 90^\circ, 162^\circ, 234^\circ, 306^\circ, \ldots$

Now $\sin 90^\circ = 1$, $\sin 162^\circ = \sin 18^\circ > 0$, $\sin 234^\circ = \sin 306^\circ = -\sin 54^\circ < 0$, and thus the other root of the eqn $4x^2+2x-1 = 0$ represents $\sin 234^\circ$#, i.e. $\sin 234^\circ = \frac{1}{4}(-1-\sqrt{5})$

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8. (i) The condition for the eqn. $(a^2+b^2)x^2 + 2(ac+bd)x + (c^2+d^2) = 0$ to have real roots is that $\Delta \geq 0$ (Δ is the discriminant)#

Here $\Delta = 4(ac+bd)^2 - 4(a^2+b^2)(c^2+d^2) = 4(-a^2d^2 + 2abcd - b^2c^2)$
$= -4(ad-bc)^2 \leq 0$ for all real a, b, c, d.

Thus for the roots to be real, then $\Delta = 0$, in which case they are equal# For this to occur, $ad - bc = 0$, i.e. $d = bc/a$.

If roots of given eqn. are α, α then

$$2\alpha = \frac{-2(ac+bd)}{a^2+b^2} = \frac{-2(ac + b.\frac{bc}{a})}{a^2+b^2} = \frac{-2c(a^2+b^2)}{a(a^2+b^2)} = \frac{-2c}{a}$$

and the equal roots equal $-c/a$ #

(ii) If n is a positive integer, then

$$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \dots + {}^nC_rx^r + \dots + {}^nC_nx^n \ \#$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots + x^n \ \#$$

(a) $(1 + \frac{1}{n})^k = 1 + k.\frac{1}{n} + \frac{k(k-1)}{2!}.\frac{1}{n^2} + \dots\dots\dots\dots + \frac{1}{n^k}$

Noting that k is fixed, and as $n\to\infty$, then $\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}, \dots \to 0$ hence $(1+\frac{1}{n})^k \to 1$ #

(b) $(1+\frac{1}{n})^n = 1 + n.\frac{1}{n} + \frac{n(n-1)}{2!}.\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}.\frac{1}{n^3} + \dots\dots + \frac{1}{n^n}$

$$= 1 + 1 + \frac{1}{2!}.\frac{n}{n}.\frac{(n-1)}{n} + \frac{1}{3!}.\frac{n}{n}.\frac{n-1}{n}.\frac{n-2}{n} + \dots\dots + \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}.1.(1-\frac{1}{n}) + \frac{1}{3!}.1.(1-\frac{1}{n})(1-\frac{2}{n}) + \dots\dots + \frac{1}{n^n}$$

As $n \to \infty$, $(1 + \frac{1}{n})^n \to 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$, without end.

Thus, as $n \to \infty$, $(1 + \frac{1}{n})^n$ approaches the sum of an infinite series; this sum is obviously greater than 1 + 1 = 2 #

(c) Now $n! = 1.2.3.4.5.6\dots\dots(n-1)n$, *there are n terms*
and $2^{n-1} = 2.2.2.2.2.2\dots\dots 2.\ 2$, *there are (n-1) terms*

Thus $n! > 2^{n-1}$ for all n, except n=1,2 when $n! = 2^{n-1}$, and
hence $\frac{1}{n!} < \frac{1}{2^{n-1}}$ for all positive integral $n \geq 3$.#

From this, $\frac{1}{3!} < \frac{1}{2^2}$, $\frac{1}{4!} < \frac{1}{2^3}$, $\frac{1}{5!} < \frac{1}{2^4}$, ...

Thus $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \dots$

$$< (1 + 1 + \frac{1}{2^1}) + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \dots$$

= 1 + (limiting sum of infinite geom. series with a = 1, r = ½)

$= 1 + \frac{1}{1-\frac{1}{2}} = 3$, i.e. sum of infinite series < 3.

Thus $\lim_{n\to\infty}(1 + \frac{1}{n})^n = \ell$, where $2 < \ell < 3$ #