SPECIMEN PAPER 2

Question 1

(i) Evaluate

(a) $\int_0^{\pi} x \cos x \, dx$ (b) $\int_3^6 \frac{dx}{x^2-6x+18}$ (c) $\int_0^{\ln 2} \frac{e^{2x}dx}{1+e^x}$

(ii) Find $\int_1^N \frac{dx}{x(1+x)}$ and hence find the area bounded by the curve $y = x^{-1}(1+x)^{-1}$ and the x-axis, to the right of the $x = 1$.

(iii) Differentiate $\cos^{n-1}\theta \sin\theta$ and express the result in terms of $\cos\theta$ only.

Deduce that $\int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2}\theta \, d\theta$, and hence evaluate $\int_0^{\pi/2} \cos^6 \theta \, d\theta$.

Question 2

(i) Sketch the graph of $y = x^2e^{-x}$. Find dy/dx and d^2y/dx^2, and find the values of x which correspond to the turning points and points of inflection on the curve. Mark these points on your graph.

(ii) Show that the curve $y = 3x^4 - 4x^3 + 6x^2 - 12x + k$ has only one stationary point and find its coordinates.

Hence prove that the equation $3x^4 - 4x^3 + 6x^2 - 12x + k = 0$ has no real root if $k > 7$ and state the number of real roots if $k \leq 7$.

Question 3

(i) The rectangular hyperbola H has cartesian (x, y) equation $x^2 - y^2 = 8$. Write down its eccentricity, the coordinates of its foci S and S', the equations of each directrix and of each asymptote.

Sketch the curve, indicating on your diagram, the foci, directrices and asymptotes.

(ii) If this curve is rotated through 45° in an anti-clockwise direction, the equation of the curve takes the form $xy = 4$.

Prove that the equation of the normal to the rectangular hyperbola $xy = 4$ at the point $(2p, 2/p)$ is $py - p^3x = 2(1 - p^4)$.

If this normal meets the hyperbola again at $(2q, 2/q)$ prove that $q = -1/p^3$.

Hence, or otherwise, show that there exists only one chord of the hyperbola which is a normal at both ends of the chord, and find its equation.

Question 4

(i) If z is a complex number so that $|z| = 3$ and $\arg z = \pi/6$, mark clearly (using geometry where possible) on the same Argand diagram the points representing the complex numbers:

z ; iz ; $\bar{z}$; $\frac{1}{z}$; $z\bar{z}$; z^2 ; $z^2 + z$; $z^2 - z$.

(ii) Find $\sqrt{6i - 8}$ and hence solve the equation $2z^2 - (3 + i)z + 2 = 0$, expressing values of z in the form $x + iy$.

(iii) Sketch on an Argand diagram, the locus of the point P which satisfies the equation $|z - 3| = 3$. From the figure, using Euclidean properties of the locus, show that P also satisfies the condition $\arg(z - 3) = \arg z^2$.

Determine the complete locus of points satisfying this condition.

Question 5

(i) Use the method of integration by parts to evaluate $\int_0^1 \sqrt{1 + t^2}\, dt$.

(ii) PQ, QR are two straight roads meeting at right angles; PQ = a km, QR = 3a km and S is a point on QR. A girl walks across country from P to S at 3 km/h and then along the road from S to R at 5 km/h. Find the distance of S from Q in order that the time for the whole journey should be a minimum.

(iii) The area bounded by the curve $y = x(2 - x)$ and the x-axis is rotated about the y-axis. By considering cylindrical shells with generators parallel to the y-axis, show that the volume V units³ of the solid so generated is given by $V = \int_0^2 2\pi xy\,dx$.

Hence determine the volume of this solid.

Question 6

A particle moves in a straight line. Prove that its acceleration at any instant is $v\frac{dv}{dx}$ where x denotes its position coordinate and v its velocity.

A particle of mass m is projected vertically upwards with initial speed U. If the air resistance at any instant is proportional to the velocity at that instant (say resistance = $-mkv$, where k is a constant), prove that the particle will reach its highest point in time T given by $kT = \log(1 + kU/g)$, where g is the acceleration due to gravity (assumed constant).

Also show that the particle will ascend to height H, where $kH = U - gT$.

Question 7

(i) Show that if the polynomial $P(x)$ has a root α of multiplicity m, then $P'(x)$ has the root α with multiplicity $(m - 1)$.

Given that the polynomial $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ has a 3-fold root, find all the roots of $P(x)$.

(ii) If $p^2 = 1 + p$, prove that $p^3 = 1 + 2p$. Express in the form $a + bp$, where a, b are integers (a) p^5 (b) $1/p^5$.

(iii) Assuming the results for $\sin(A \pm B)$, prove that

$$\sin C + \sin D = 2 \sin \tfrac{1}{2}(C + D) \cos \tfrac{1}{2}(C - D)$$

Find the solutions of $\sin 3x + \sin x = \cos x$ for $0 \le x \le 2\pi$.

Question 8

(i) Given a, b, c, d are real positive numbers, prove that

(a) $a^4 + b^4 \ge 2a^2b^2$

(b) $a^4 + b^4 + c^4 + d^4 \ge 4abcd$

(c) if $a^4 + b^4 + c^4 + d^4 \le 4$ then $a^{-4} + b^{-4} + c^{-4} + d^{-4} \ge 4$

If $a^4 + b^4 + c^4 + d^4 > 4$, what statement (if any) can be made about $a^{-4} + b^{-4} + c^{-4} + d^{-4}$?

(ii) If w is a complex root of the equation $x^3 = 1$, show that the other complex root is w^2.

(a) Show that $1 + w + w^2 = 0$.

(b) Find in its simplest form, the cubic equation whose roots are $a + b$, $aw + bw^2$, $aw^2 + bw$ where a, b are real number.

1.(i)(a) Integrating by parts, $I = \int_0^{\pi} x \cos x \, dx = \int_0^{\pi} x \frac{d}{dx}(\sin x)\, dx$

$\therefore I = \left[x \sin x\right]_0^{\pi} - \int_0^{\pi} \sin x . \frac{d}{dx}(x) dx = 0 - \int_0^{\pi} \sin x \, dx$

$= - \left[-\cos x\right]_0^{\pi} = -2$ #

(b) $\int_3^6 \frac{dx}{x^2-6x+18} = \int_3^6 \frac{dx}{(x-3)^2+9} = \frac{1}{3}\left[\tan^{-1}\left(\frac{x-3}{3}\right)\right]_3^6$, putting $u = x-3$

$= \frac{1}{3}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{12}$ #

(c) Let $u = 1 + e^x$, then $du = e^x \, dx$; note $e^{2x}dx = e^x . e^x dx = (u-1)du$

When $x = 0$, $u = 2$ and when $x = \ln 2$, $u = 1 + e^{\ln 2} = 1 + 2 = 3$

$\int_0^{\ln 2} \frac{e^{2x}dx}{1 + e^x} = \int_2^3 \frac{(u-1)du}{u} = \int_2^3 (1 - \frac{1}{u})du = \left[u - \ln u\right]_2^3$

$= 1 - \ln (3/2)$ #

{*OR let* $v = e^x$, *then* $dv = e^x dx$ *and* $e^{2x}dx = v\, dv$

$\int_0^{\ln 2} \frac{e^{2x}dx}{1+e^x} = \int_1^2 \frac{v dv}{1+v} = \int_1^2 \frac{(1+v)-1}{1+v} dv = \int_1^2 \left(1 - \frac{1}{1+v}\right) dv$ *etc* }

(ii) Using partial fractions, $\frac{1}{x(1+x)} = \frac{A}{x} + \frac{B}{1+x}$

i.e. $1 = A(1+x) + Bx$; $x = 0$ gives $A = 1$; $x = -1$ gives $B = -1$

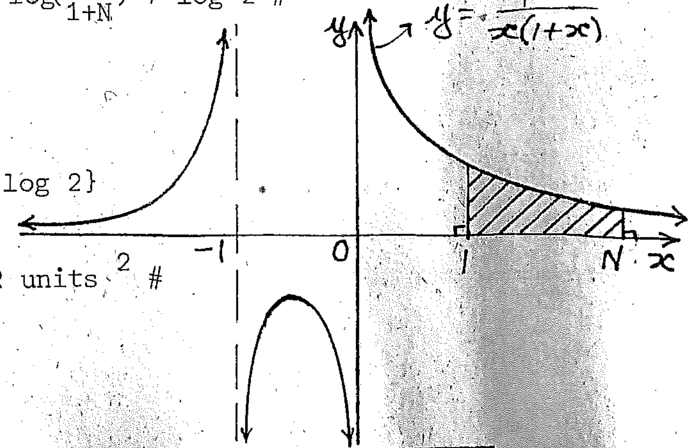
$\int_1^N \frac{dx}{x(1+x)} = \int_1^N \left(\frac{1}{x} - \frac{1}{1+x}\right)dx = \left[\ln x - \ln(1+x)\right]_1^N$

$= \ln N - \ln(1+N) + \ln 2 = \log\left(\frac{N}{1+N}\right) + \log 2$ #

Reqd area $= \int_1^N \frac{dx}{x(1+x)}$ as $N \to \infty$

$= \lim_{N\to\infty} \left\{\log\left(\frac{1}{1 + 1/N}\right) + \log 2\right\}$

$= \log 1 + \log 2 = \log 2$ units^2 #



(iii) $\frac{d}{d\theta}(\cos^{n-1}\theta \sin\theta) = \cos^{n-1}\theta \cos\theta + \sin\theta\{(n-1)\cos^{n-2}\theta.-\sin\theta\}$

$= \cos^n\theta - (n-1)\cos^{n-2}\theta \sin^2\theta = \cos^n\theta - (n-1)\cos^{n-2}\theta\ (1-\cos^2\theta)$

$= \cos^n\theta - (n-1)\cos^{n-2}\theta + (n-1)\cos^n\theta = n\cos^n\theta - (n-1)\cos^{n-2}\theta$ #

Integrating this result between the limits 0, $\pi/2$

$$\int_0^{\pi/2} \frac{d}{d\theta}(\cos^{n-1}\theta \sin\theta)d\theta = n\int_0^{\pi/2} \cos^n\theta\, d\theta - (n-1)\int_0^{\pi/2} \cos^{n-2}\theta\, d\theta$$

i.e. $\left[\cos^{n-1}\theta \sin\theta\right]_0^{\pi/2} = n\,u_n - (n-1)u_{n-2}$, where $u_n = \int_0^{\pi/2} \cos^n\theta\, d\theta$

Thus $nu_n - (n-1)u_{n-2} = 0$, i.e. $u_n = \frac{n-1}{n} u_{n-2}$

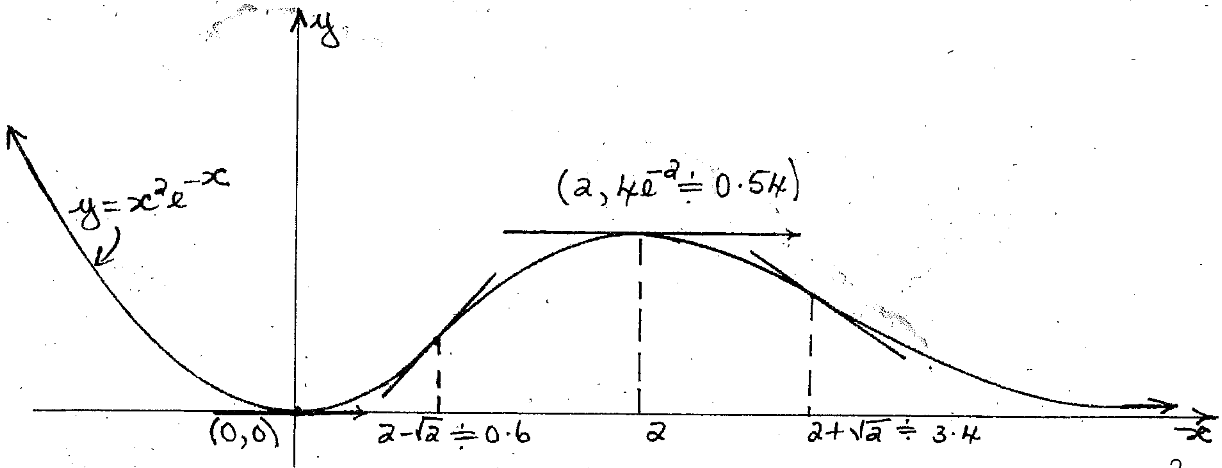
Now $u_6 = \frac{5}{6}u_4$; $u_4 = \frac{3}{4}u_2$; $u_2 = \frac{1}{2}u_0$ and $u_0 = \int_0^{\pi/2} 1\, d\theta = \frac{\pi}{2}$

Thus $\int_0^{\pi/2} \cos^6\theta\, d\theta = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot u_0 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}$ #

2.(i) For curve $y = x^2e^{-x}$; domain is all x, range is $y \geq 0$. Curve meets x - axis at origin; as $x \to +\infty$, $y \to 0$ and as $x \to -\infty$, $y \to +\infty$.

Now $dy/dx = x^2.-e^{-x} + e^{-x}.2x = xe^{-x}(2-x)$ #

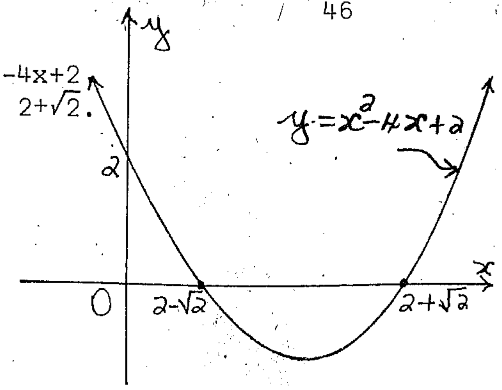
and $d^2y/dx^2 = \{x^2.e^{-x} - e^{-x}.2x\} + \{e^{-x}.2 + 2x.e^{-x}.-1\} = e^{-x}(x^2-4x+2)$#



Now $dy/dx = 0$ when $x = 0, 2$; these give stationary pts. (0,0) and $(2, 4e^{-2})$ resp. At $x = 0$, $d^2y/dx^2 > 0$ and at $x = 2$, $d^2y/dx^2 < 0$; the curve is concave up at $x = 0$ and concave down at $x = 2$. There is a min. turning pt. where $x = 0$ and a max. turning pt. where $x = 2$.#

The condition for inflections on the curve is $d^2y/dx^2 = 0$; this is so when $x^2 - 4x + 2 = 0$, i.e. at $x = 2\pm\sqrt{2}$.

From a sketch of $y = x^2-4x+2$, the value of x^2-4x+2 changes sign in passing through $x = 2-\sqrt{2}$, $x = 2+\sqrt{2}$. Hence, the sign of d^2y/dx^2 changes in passing through these values, and thus so does the concavity of the curve. That is, pts. of inflection occur where $x = 2\pm\sqrt{2}$.

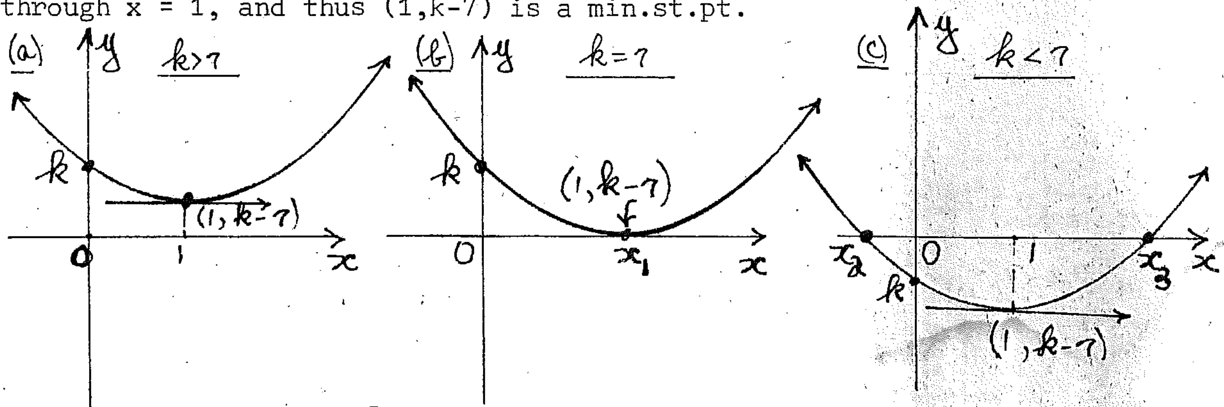


(ii) $y = 3x^4-4x^3+6x^2-12x+k$

$y' = 12x^3-12x^2+12x-12 = 12(x^3-x^2+x-1) = 12\{x^2(x-1) + 1(x-1)\}$

$= 12(x-1)(x^2+1) = 0$ at $x = 1$ only (for real x).

There is only one stat. pt. # When $x = 1$, $y = 3-4+6-12+k = -7+k$ and the st. pt. has coords. $(1,k-7)$ # Note y' changes sign from $-$ to $+$ in passing through $x = 1$, and thus $(1,k-7)$ is a min.st.pt.



Now graph of $y = 3x^4-4x^3+6x^2-12x+k$ cuts y-axis at $(0,k)$; as $x \to \pm\infty$, $y \to +\infty$. The graphs when $k>7$, $k=7$, $k<7$ are sketched in fig (a), (b), (c) resp. Note then the point $(1,k-7)$ is respectively above, on, below the x-axis.

When $k>7$, the curve does not meet the x-axis, and thus the eqn. $3x^4-4x^3+6x^2-12x+k = 0$ has no real roots #

When $k=7$, the curve touches the x-axis whilst when $k<7$, the curve crosses the x-axis in two distinct pts. Thus, the given eqn. has two real (and equal) roots when $k=7$ and two real (and distinct) roots when $k<7$ #

3.(i) From $x^2-y^2 = 8$, then $\frac{x^2}{8} - \frac{y^2}{8} = 1$, i.e. $a = b = 2\sqrt{2}$

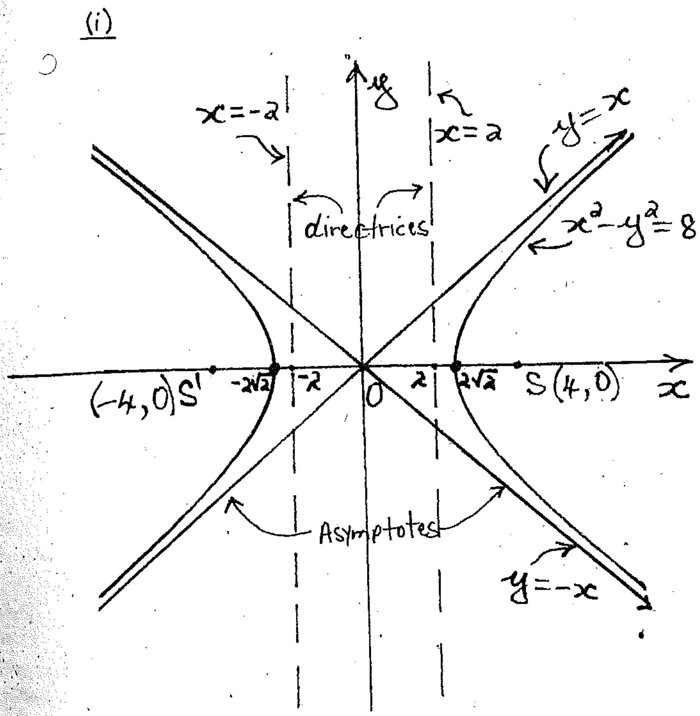
Using $b^2 = a^2(e^2-1)$ gives $8 = 8(e^2-1)$, i.e. $e = \sqrt{2}$ #

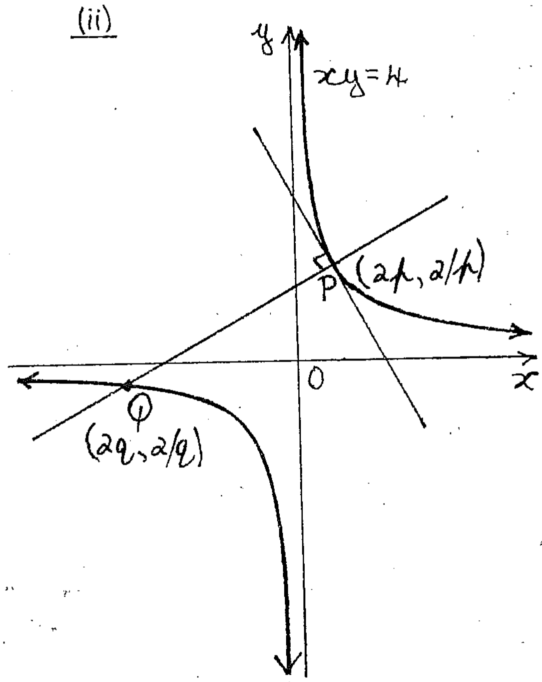
The foci S,S^1 are at $(\pm ae,o)$, i.e. at $(\pm 2\sqrt{2}.\sqrt{2}, 0)$, i.e. $(\pm 4,0)$ #

Eqns. of directrices are $x = \pm a/e = \pm 2\sqrt{2}/\sqrt{2} = \pm 2$ #

Eqns. of asymptotes are given by $\frac{x^2}{8} - \frac{y^2}{8} = 0$, i.e. $y = \pm x$ #

The hyperbola $x^2-y^2 = 8$ is sketched #





(ii) From xy = 4, $y = \frac{4}{x}$ and $\frac{dy}{dx} = \frac{-4}{x^2} = \frac{-1}{p^2}$ at P (2p,2/p)

{*OR from xy = 4, then* $(x\frac{dy}{dx} + y.1) = 0$, *i.e.* $\frac{dy}{dx} = \frac{-y}{x} = \frac{-1}{p^2}$

OR from $x = 2p$, $y = \frac{2}{p}$ *then* $\frac{dx}{dp} = 2$, $\frac{dy}{dp} = \frac{-2}{p^2}$ *and* $\frac{dy}{dx} = \frac{dy/dp}{dx/dp} = \frac{-1}{p^2}$}

The tangent to xy = 4 at P (2p,2/p) has slope $-1/p^2$ and thus the normal there has gradient $+p^2$.

Eqn. of normal at P is $y - \frac{2}{p} = p^2(x-2p)$, which gives

$py-2 = p^3x-2p^4$, i.e. $py-p^3x = 2(1-p^4)$#

If this normal passes through Q (2q,2/q) then $p.\frac{2}{q} - p^3.2q = 2(1-p^4)$

i.e. $p-p^3q^2 = q-p^4q$, i.e. $p-q = p^3q(q-p)$

Since $p \neq q$, then $1 = -p^3q$, and thus $q = -1/p^3$#

From above, if PQ is a normal at P, then $q = -1/p^3$, i.e. $p^3q = -1$ and a normal at Q, then $p = -1/q^3$, i.e. $pq^3 = -1$.

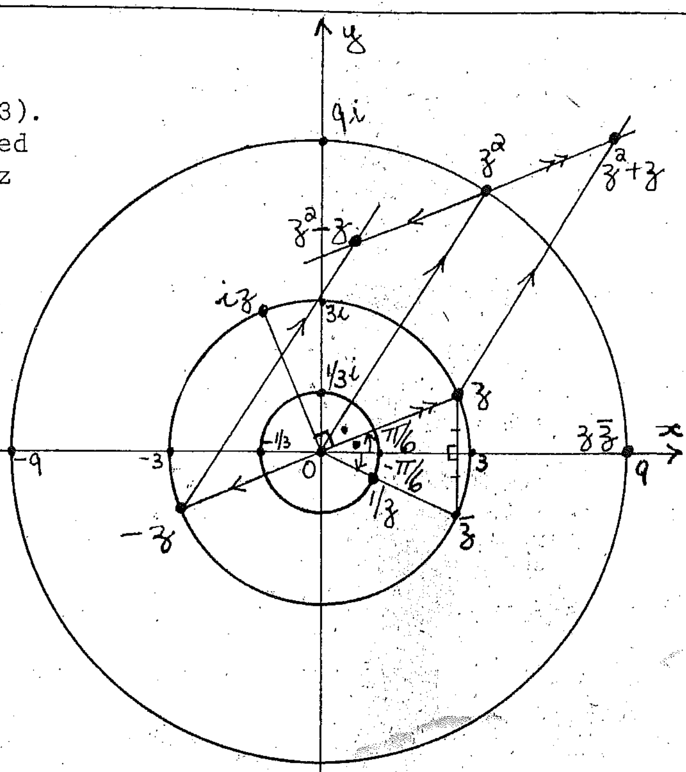
Thus, if PQ is a normal at both P, Q then $p^3q = pq^3$, i.e. $pq(p^2-q^2) = 0$, i.e. $pq(p-q)(p+q) = 0$ and since $p \neq 0$, $q \neq 0$, $p \neq q$ then $p = -q$, i.e. $q = -p$.

Since $p^3q = -1$, then $p^3.-p = -1$, $p^4 = 1$, i.e. $p = \pm 1$

When p = 1, eqn. of normal chord is $1y-1^3x = 2(1-1^4)$, i.e. $y = x$

When p = -1, eqn of normal chord is $-y+x = 2(1-1)$, i.e. $y = x$

Thus, there is only one chord of the hyperbola which is normal at both ends; its eqn. is $y = x$.#



4.(i) $z = 3(\cos \pi/6 + i\sin \pi/6)$

$iz = 3(\cos 2\pi/3 + i\sin 2\pi/3)$.

The pt. representing iz is obtained by rotating the pt. representing z anti-clockwise through $\pi/2$.

$\bar{z} = 3(\cos \pi/6 - i\sin \pi/6)$

The pt. representing $\bar{z}$ (the conjugate of z) is the reflection of the pt. rep. z in the x-axis.

$$\frac{1}{z} = \frac{1}{3(\cos \pi/6 + i\sin \pi/6)}$$

$$= \frac{1}{3}(\cos \pi/6 - i\sin \pi/6)$$

The pt. representing 1/z has mod. 1/3 and arg $-\pi/6$.

$$z\bar{z} = 3(\cos \frac{\pi}{6} + i\sin \frac{\pi}{6}).$$

$$3(\cos \frac{\pi}{6} - i\sin \frac{\pi}{6}) = 9$$

The pt. representing $z\bar{z}$ is on the real axis at (9,0).

$$z^2 = 3^2(\cos \frac{\pi}{6} + i\sin \frac{\pi}{6})^2 = 9(\cos \frac{\pi}{3} + i\sin \frac{\pi}{3}),$$ by de Moivre's Thm.

The points representing $z^2 + z$, $z^2 - z$ are the fourth vertices of parallelograms with vectors representing z^2, z and z^2, -z respectively as sides.

(ii) Let $\sqrt{6i-8} = a + ib$, where $a>0$ (by convention)

Squaring, $6i-8 = (a^2-b^2) + 2iab$ and thus $a^2-b^2 = -8$, $ab = 3$

Solving these equations at sight, $a = 1$, $b = 3$

{*OR from* $ab = 3$, $b = 3/a$ *and thus* $a^2-9/a^2 = -8$, *i.e.* $a^4+8a^2-9 = 0$, *i.e.* $(a^2+9)(a^2-1) = 0$, *i.e.* $a = 1$ *and then* $b = 3$}.

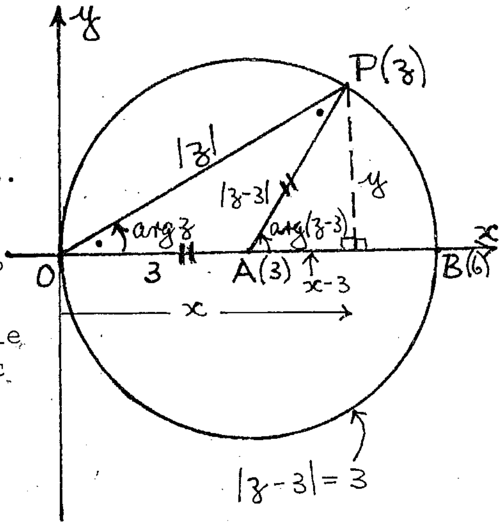
Thus $\sqrt{6i-8} = 1 + 3i$ #

Using the quadratic formula, the roots of $2z^2-(3+i)z + 2 = 0$

are $z = \dfrac{(3+i) \pm \sqrt{(3+i)^2-4.2.2}}{4} = \dfrac{(3+i) \pm \sqrt{6i-8}}{4}$

$= \dfrac{(3+i) \pm (1+3i)}{4} = 1 + i$ or $\frac{1}{2} - \frac{1}{2}i$ #

(iii) On an Argand diagram, $|z-3|$ represents the distance from the pt. P representing the complex no. z to the pt. A representing 3. If this distance is 3 units, then P lies on the circle centre A (3,0) with radius 3 units. The locus is sketched. #



Now arg $(z-3) = x\hat{A}P$. Arg $z^2 = 2$ arg $z = 2.x\hat{O}P$

By geometry, $x\hat{A}P$ at the centre A of the circle is twice $x\hat{O}P$ at the circumf., on the same arc PB, i.e. arg $(z-3) = 2$ arg $z =$ arg z^2#

{*OR use fact that arg (z-3) is the exterior angle of ΔOAP and is equal to the sum of the angles AOP, APO*}.

If $z = x + iy$, then $z-3 = (x-3) + iy$ and $z^2 = (x^2-y^2) + 2ixy$.

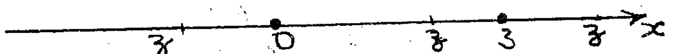
Since arg $(z-3) =$ arg (z^2), then tan arg $(z-3) =$ tan arg z^2

i.e. $\dfrac{y}{x-3} = \dfrac{2xy}{x^2-y^2}$ (*Note: if $Z = X + iY$, then tan arg $Z = Y/X$*)

i.e. $y = 0$ or $x^2-y^2 = 2x(x-3)$ i.e. $x^2+y^2-6x = 0$

Complete locus is $y = 0$ (i.e. the x-axis) or the circle $(x-3)^2+y^2 = 9$ #

{*Note: When z is on the x-axis,*



to the right of 3, *arg $(z-3) = 0$ and arg $z^2 = 0$, i.e. arg $(z-3) =$ arg z^2.*

between 0 and 3, *arg $(z-3) = \pi$ and arg $z^2 = 0$, i.e. arg $(z-3) \neq$ arg z^2.*

to the left of 0, *arg $(z-3) = \pi$ and arg $z^2 = 0$, i.e. arg $(z-3) \neq$ arg z^2.*

Note that $z \neq 0$, $z \neq 3$ since arg 0 is undefined.

The locus of P is the x-axis ($y=0$) for $x>3$ only, as well as the circle $(x-3)^2+y^2 = 9$, i.e. $|z-3| = 3$}

5.(i) By parts, $\int u \dfrac{dv}{dx} dx = uv - \int v \dfrac{du}{dx} dx$, where u, v are functions of x.

$$\int_0^1 \sqrt{1+t^2}\, dt$$

$$= \int_0^1 \sqrt{1+t^2} . \frac{d}{dt}(t)dt = \left[\sqrt{1+t^2} . t\right]_0^1 - \int_0^1 t . \frac{1}{2\sqrt{1+t^2}} . 2t\, dt$$

$$= \sqrt{2} - \int_0^1 \frac{(1+t^2) - 1}{\sqrt{1+t^2}} dt = \sqrt{2} - \int_0^1 \sqrt{1+t^2}\, dt + \int_0^1 \frac{dt}{\sqrt{1+t^2}}$$

$\therefore 2\int_0^1 \sqrt{1+t^2} dt = \sqrt{2} + \left[\log (t + \sqrt{t^2+1})\right]_0^1$, *from table of standard integrals*

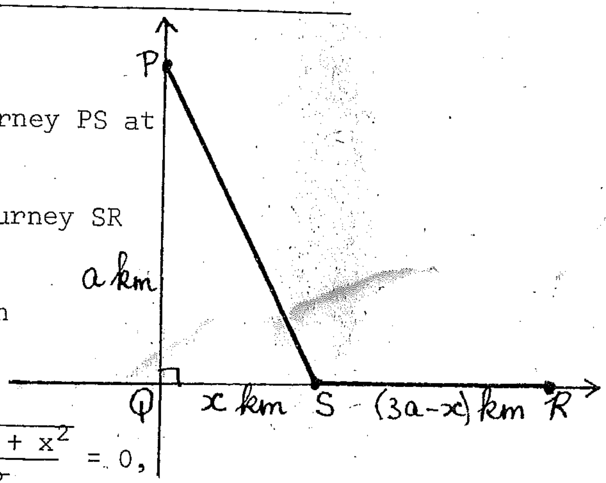
$= \sqrt{2} + \log (1+\sqrt{2})$, and thus value of given definite integral is $\frac{1}{2}\{\sqrt{2} + \log (1+\sqrt{2})\}$#

{OR $\int_0^1 \frac{dt}{\sqrt{1+t^2}} = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec\theta}$, *by letting* $t = \tan\theta$, *i.e.* $dt = \sec^2\theta\, d\theta$

$= \int_0^{\pi/4} \sec\theta\, d\theta = \int_0^{\pi/4} \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} d\theta = \left[\log (\sec\theta + \tan\theta)\right]_0^{\pi/4}$

$= \log (\sqrt{2}+1)$}

(ii) Let QS = x km (x is a variable)



Now PS = $\sqrt{a^2 + x^2}$ km and time for journey PS at 3 km/h is $\sqrt{a^2 + x^2}/3$ h.

Also SR = $(3a - x)$ km and time for journey SR at 5 km/h is $(3a - x)/5$ h.

If total time for journey is T h, then

$$T = \frac{\sqrt{a^2 + x^2}}{3} + \frac{(3a - x)}{5}$$

$$\frac{dT}{dx} = \frac{1}{2\sqrt{a^2 + x^2}} . \frac{2x}{3} - \frac{1}{5} = \frac{5x - 3\sqrt{a^2 + x^2}}{15\sqrt{a^2 + x^2}} = 0,$$

when $5x = 3\sqrt{a^2 + x^2}$, i.e. $25x^2 = 9(a^2 + x^2)$,

i.e. $x = 3a/4$.

Consider $x = 0$, $x = a$ (values through $x = 3a/4$), then dT/dx changes sign from - to +, and thus a minimum value of T occurs when $x = 3a/4$. Thus least time for whole journey occurs when distance of S from Q is $3a/4$ km #

(iii) Consider the area between the lines distant x, $x + \delta x$ from y-axis and parl. to it, rotated about the y-axis. Element of volume of solid is a cylindrical shell of radius x, height y and thickness δx.

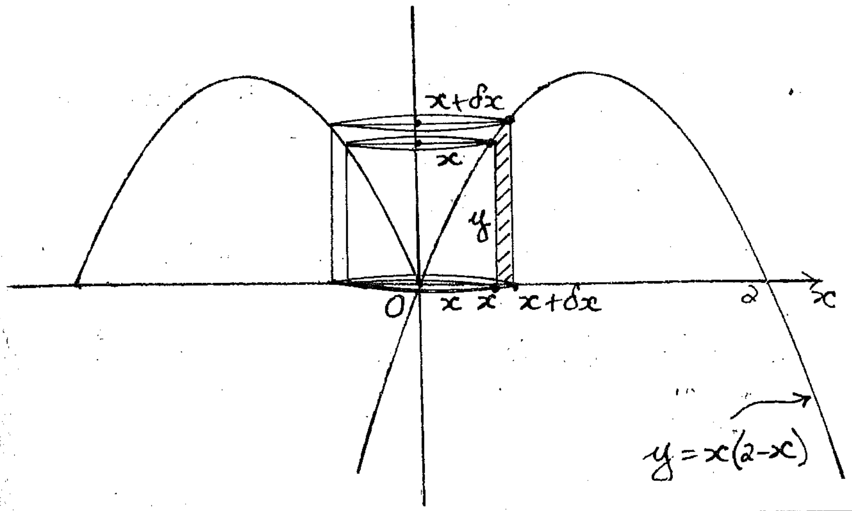
Vol. of element $\doteqdot$ area of cross-section x length $= \pi\{(x+\delta x)^2 - x^2\} \times y$

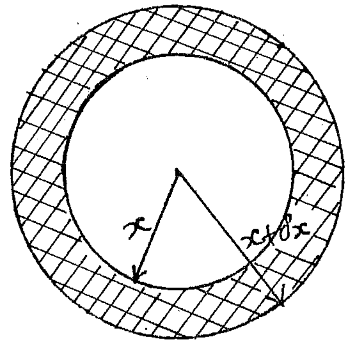
$= 2\pi x y \delta x$, *ignoring term in* $(\delta x)^2$

Total vol. of solid $= \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{2} 2\pi xy\delta x = \int_0^2 2\pi xy\,dx$ #

$= 2\pi\int_0^2 x.x(2 - x)dx$, *since* $y = x(2 - x)$

$= 2\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \frac{8\pi}{3}$ units^3 #





Cross Section

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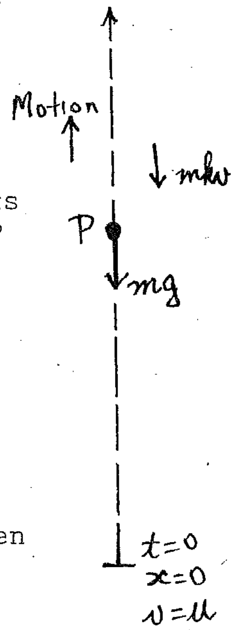
6. With the usual notation, the acceleration $f = \ddot{x}$ is given by

$\ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = \frac{dv}{dx}\cdot v = v\frac{dv}{dx}$ #



The forces acting on the particle P in its upwards path are its weight mg and the air resistance mkv. The eqn. of motion of P is mf = -mg-mkv, i.e. f = -(g+kv) ...(B)

To find the time of ascent, we use $f = \frac{dv}{dt}$

i.e. $\frac{dv}{dt} = -(g+kv)$, i.e. $\frac{dt}{dv} = \frac{-1}{g+kv}$ ... (L)

If T is the time for P to reach its highest point, i.e. in proceeding from v = U (initially) to v = 0 (at highest pt) then from (L)

$T = -\int_U^0 \frac{1}{g+kv}dv = \int_0^U \frac{1}{g+kv}dv = \frac{1}{k}\left[\log(g+kv)\right]_0^U$

$\therefore kT = \log(g+kU) - \log g = \log\left(\frac{g+kU}{g}\right) = \log(1+kU/g)$#

{OR from (L), $t = -\int \frac{1}{g+kv}dv = -\frac{1}{k}\log(g+kv) + C_1$

By data, when $t = 0$, $v = U$, then $0 = -\frac{1}{k}\log(g+kU) + C_1$

Thus $t = -\frac{1}{k}\log(g+kv) + \frac{1}{k}\log(g+kU) = \frac{1}{k}\log\left(\frac{g+kU}{g+kv}\right)$

At the highest pt, $v = 0$ *and time* $t = T$

$\therefore T = \frac{1}{k}\log\left(\frac{g+kU}{g}\right)$, *i.e.* $kT = \log(1 + kU/g)$ #}

From (B) above, f = -(g+kv), using $f = v\frac{dv}{dx}$ then

$$v\frac{dv}{dx} = -(g+kv), \text{ i.e. } \frac{dv}{dx} = \frac{-(g+kv)}{v}, \text{ i.e. } \frac{dx}{dv} = \frac{-v}{g+kv} \quad \ldots(M)$$

If H is the greatest height reached, in proceeding from v = U to v = 0, then from (M), $H = -\int_U^0 \frac{v}{g+kv}\,dv = \frac{1}{k}\int_0^U \frac{(g+kv)-g}{g+kv}\,dv$

$$\text{i.e. } H = \frac{1}{k}\int_0^U \left(1-\frac{g}{g+kv}\right) dv = \frac{1}{k}\left[v - \frac{g}{k}\log(g+kv)\right]_0^U$$

$$\text{i.e. } kH = U - \frac{g}{k}\{\log(g+kU) - \log g\} = U - \frac{g}{k}\log(1 + kU/g)$$

$$= U - \frac{g}{k}\,.\,kT, \text{ from above, i.e. } kH = U - gT \quad \#$$

{OR from (M), $x = \int \frac{-v\,dv}{g+kv} = -\frac{1}{k}\int \frac{(g+kv)-g}{g+kv}\,dv$

$$\text{i.e. } x = -\frac{1}{k}\int\left(1 - \frac{g}{g+kv}\right) dv = -\frac{1}{k}\{v - \frac{g}{k}\log(g+kv)\} + C_2$$

By data, when x = 0, v = U, ∴ $0 = -\frac{1}{k}\{U - \frac{g}{k}\log(g+kU)\} + C2$

$$\text{Thus } x = -\frac{1}{k}\{v - \frac{g}{k}\log(g+kv)\} + \frac{1}{k}\{U - \frac{g}{k}\log(g+kU)\}$$

At the highest point v = 0 and x = H,

$$\therefore H = \frac{g}{k^2}\log g + \frac{U}{k} - \frac{g}{k^2}\log(g+kU)$$

$$\text{i.e. } kH = U - \frac{g}{k}\log\left(\frac{g+kU}{g}\right) = U - \frac{g}{k}\,.\,kT = U - gT \quad \#\}$$

7.(i) Let $P(x) = (x-\alpha)^m Q(x)$ where $Q(\alpha) \neq 0$

$$\therefore P'(x) = (x-\alpha)^m Q'(x) + Q(x).m(x-\alpha)^{m-1}.1$$

$= (x-\alpha)^{m-1}\{(x-\alpha)Q'(x) + mQ(x)\}$, and hence P'(x) has the root α with multiplicity (m-1) #

Since $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ has a 3-fold root, then P'(x) has this root as a 2-fold root, and hence P'' (x) has this root as a 1-fold root.

Now $P'(x) = 4x^3 + 3x^2 - 6x - 5$ and $P''(x) = 12x^2 + 6x - 6 = 6(2x-1)(x+1)$

$P''(x) = 0$ when $x = \frac{1}{2}, -1$

Now $P'(\frac{1}{2}) = 4(1/8) + 3(1/4) - 6(\frac{1}{2}) - 5 \neq 0$,

but $P'(-1) = -4 + 3 + 6 - 5 = 0$ and since -1 is a 1-fold root of P''(x) and also a root of P'(x) = 0, then -1 is a 2-fold root of P'(x).

Also $P(-1) = 1 - 1 - 3 + 5 - 2 = 0$ and thus -1 is a 3-fold root of $P(x)$, i.e. $P(x) = (x+1)^3 Q(x)$ where $Q(x)$ is a polyn. of degree 1

$= (x+1)^3(x-2)$ at sight or by long division of $P(x)$ by $(x+1)^3$

Thus the roots of $P(x)$ are $-1, -1, -1, 2$ #

{*OR the sum of the 4 roots* $-1, -1, -1, \beta$ *of* $P(x)$ *is - coefft. of* x^3*/coefft. of* x^4, *i.e.* $-3 + \beta = -1$ *and then* $\beta = 2$}

(ii) If $p^2 = 1 + p$

then $p^3 = p.p^2 = p(1 + p) = p + p^2 = p + (1 + p) = 1 + 2p$ #

(a) Now $p^5 = p^3.p^2 = (1 + 2p)(1 + p) = 1 + 3p + 2p^2 = 1 + 3p + 2(1 + p)$

$= 3 + 5p$ #

{*OR* $p^4 = p.\ p^3 = p(1 + 2p) = p + 2p^2 = p + 2(1 + p) = 2 + 3p$

and $p^5 = p.\ p^4 = p(2 + 3p) = 2p + 3p^2 = 2p + 3(1 + p) = 3 + 5p$ #}

(b) From $p^2 = 1 + p$, ÷by p then $p = \frac{1}{p} + 1$, i.e. $\frac{1}{p} = p - 1$

Now $\frac{1}{p^2} = \frac{1}{p} \cdot \frac{1}{p} = \frac{1}{p}(p - 1) = 1 - \frac{1}{p} = 1 - (p - 1) = 2 - p$

and $\frac{1}{p^3} = \frac{1}{p} \cdot \frac{1}{p^2} = \frac{1}{p}(2 - p) = \frac{2}{p} - 1 = 2(p - 1) - 1 = 2p - 3$

Thus $\frac{1}{p^5} = \frac{1}{p^2} \cdot \frac{1}{p^3} = (2 - p)(2p - 3) = -2p^2 + 7p - 6 = -2(1 + p) + 7p - 6$

$= -8 + 5p$ #

There are other ways of obtaining these results

e.g. $\frac{1}{p^2} = \frac{1}{p} \cdot \frac{1}{p} = (p - 1)^2 = p^2 - 2p + 1 = (1 + p) - 2p + 1 = 2 - p$

$\frac{1}{p^3} = \frac{1}{p} \cdot \frac{1}{p^2} = (p - 1)(2 - p) = -p^2 + 3p - 2 = -(1 + p) + 3p - 2$ *etc*}

(iii) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\therefore \sin(A+B) + \sin(A-B) = 2\sin A \cos B$

i.e. $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$ #

Let $A + B = C$

and $A - B = D$

$\therefore\ 2A = C + D$

and $2B = C - D$

Now $\sin 3x + \sin x = 2 \sin \frac{3x + x}{2} \cos \frac{3x - x}{2} = 2 \sin 2x \cos x$

If $\sin 3x + \sin x = \cos x$, then $2 \sin 2x \cos x = \cos x$

i.e. $\cos x\,(2 \sin 2x - 1) = 0$, i.e. $\cos x = 0$ or $\sin 2x = \frac{1}{2}$

From $\cos x = 0$, $x = \pi/2,\ 3\pi/2$ #

From $\sin 2x = \frac{1}{2}$, $2x = \pi/6,\ 5\pi/6,\ 13\pi/6,\ 17\pi/6$

i.e. $x = \pi/12,\ 5\pi/12,\ 13\pi/12,\ 17\pi/12$ #

8.(i)(a) Now $a^4 + b^4 - 2a^2b^2 = (a^2 - b^2)^2 \geq 0$ for all real a, b

and thus $a^4 + b^4 \geq 2a^2b^2$ #

(b) Now $a^4 + b^4 \geq 2a^2b^2$ and similarly $c^4 + d^4 \geq 2c^2d^2$

By addition, $(a^4 + b^4) + (c^4 + d^4) \geq 2a^2b^2 + 2c^2d^2 = 2(a^2b^2 + c^2d^2)$

From (a), if X, Y are real then $X^4 + Y^4 \geq 2X^2Y^2$. If $X^2 = P$, $Y^2 = Q$ then we have $P^2 + Q^2 \geq 2PQ$ and thus $(ab)^2 + (cd)^2 \geq 2\,(ab)(cd)$

Hence $a^4 + b^4 + c^4 + d^4 \geq 2\,(a^2b^2 + c^2d^2) \geq 2.(2abcd) = 4abcd$ #

(c) From (b), $4abcd \leq a^4 + b^4 + c^4 + d^4$, and if $a^4 + b^4 + c^4 + d^4 \leq 4$, then $4abcd \leq a^4 + b^4 + c^4 + d^4 \leq 4$, i.e. $4abcd \leq 4$, i.e. $abcd \leq 1$.

Now $a^{-4} + b^{-4} + c^{-4} + d^{-4}$

$$= \frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} = \frac{b^4c^4d^4 + a^4c^4d^4 + a^4b^4d^4 + a^4b^4c^4}{a^4b^4c^4d^4}$$

$$\geq \frac{4(bcd)(acd)(abd)(abc)}{a^4b^4c^4d^4}$$, *using (b), noting the numerator is the sum of four terms, each to the power 4*

$$= \frac{4a^3b^3c^3d^3}{a^4b^4c^4d^4} = \frac{4}{abcd} = 4.\frac{1}{abcd} \geq 4,$$ *if $abcd \leq 1$, $\frac{1}{abcd} \geq 1$* #

From (b), $a^4 + b^4 + c^4 + d^4 \geq 4abcd$, and if $a^4 + b^4 + c^4 + d^4 > 4$ then we cannot decide on the relative sizes of 4abcd and 4. As in the working above, we can show

$a^{-4} + b^{-4} + c^{-4} + d^{-4} \geq 4.\frac{1}{abcd}$, but can proceed no further, since we do not know any further information about the size of abcd. #

(ii) The cubic eqn. $x^3 = 1$ has one real root $x = 1$, and two complex roots. If ω is one of these complex roots, then $\omega^3 = 1$.

Consider $x = \omega^2$, then $x^2 = (\omega^2)^3 = (\omega^3)^2 = 1^2 = 1$ and thus ω^2 is the other complex root of $x^3 = 1$ #

(a) The eqn. $x^3-1 = 0$ has 3 roots $1, \omega, \omega^2$. The sum of these roots is $-$ coefft x^2/coefft $x^3 = 0$, i.e. $1 + \omega + \omega^2 = 0$ #

{OR $x^3-1 = (x-1)(x^2+x+1) = 0$ when $x = 1$ or $x^2+x+1 = 0$. The root ω must satisfy $x^2+x+1 = 0$, i.e. $\omega^2+\omega+1 = 0$ #}

(b) If the reqd cubic eqn. has roots $a + b$, $a\omega + b\omega^2$, $a\omega^2 + b\omega$ then the form of this eqn. is $x^3 - S_1x^2 + S_2x - S_3 = 0$ where

$S_1 = (a + b) + (a\omega + b\omega^2) + (a\omega^2 + b\omega) = a(1 + \omega + \omega^2) + b(1 + \omega + \omega^2) = 0$

$S_2 = (a + b)(a\omega + b\omega^2) + (a\omega + b\omega^2)(a\omega^2 + b\omega) + (a\omega^2 + b\omega)(a + b)$

$= a^2(\omega + \omega^3 + \omega^2) + ab(\omega^2 + \omega + \omega^2 + \omega^4 + \omega^2 + \omega) + b^2(\omega^2 + \omega^3 + \omega)$

$= ab\ (3\omega + 3\omega^2)$, noting $\omega^3 = 1$, $\omega^4 = \omega^3.\omega = \omega$, $1 + \omega + \omega^2 = 0$

$= -3ab$, since $3(\omega + \omega^2) = 3(-1)$, using $1 + \omega + \omega^2 = 0$

$S_3 = (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega) = (a + b)\{a^2\omega^3 + ab(\omega^2 + \omega^4) + b^2\omega^3\}$

$= (a + b)(a^2 - ab + b^2) \qquad = a^3 + b^3$

Thus reqd. eqn. is $x^3 - 3abx - (a^3 + b^3) = 0$ #