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S201 / 414 GARDENERS RD. ROSEBERY 2018
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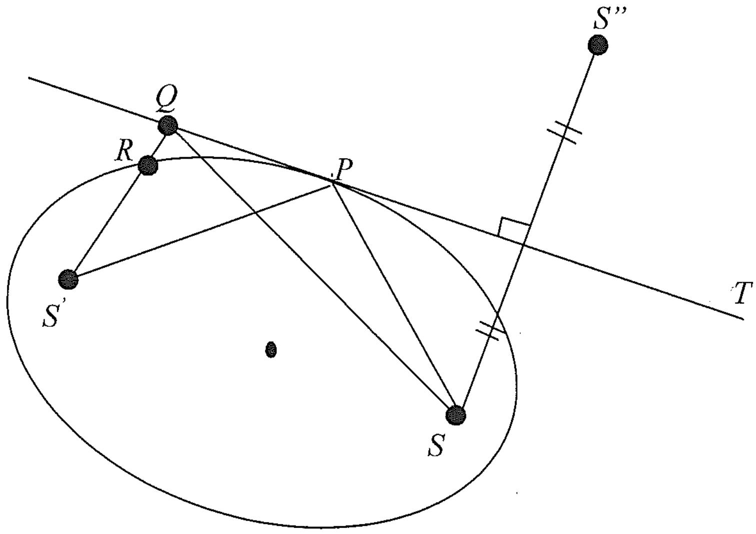
YEAR 12 – EXT.2 MATHS

REVIEW TOPIC (SP2)
THE ELLIPSE

HSC 2000 **Marks**

(7) (b)



6

In the diagram, P is an arbitrary point on the ellipse, and QPT is a tangent to the ellipse at P. The points S' and S are the foci of the ellipse, and S'' is the reflection of S across the tangent, as shown. Let the line $S'Q$ intersect the ellipse at R.

(i) Assuming $Q \neq P$, prove that

$$S'Q + QS > S'R + RS.$$

(ii) Deduce that the shortest path from S' to S passing through a point on the tangent is that through P, having length $S'P + PS$.

(iii) By considering the point S'', deduce that $\angle QPS' = \angle TPS$

HSC '99

(3)(a) Consider the ellipse E with equation $\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$. **9**

and let $P=(x_0, y_0)$ be an arbitrary point on E.

(i) Calculate the eccentricity of E.

$e=\frac{4}{5}$

(ii) Find the coordinates of the foci of E and the equations of the directrices of E.

$S(4,0), S'(-4,0); x=\pm\frac{25}{4}$

(iii) Show that the equation of the tangent at P is

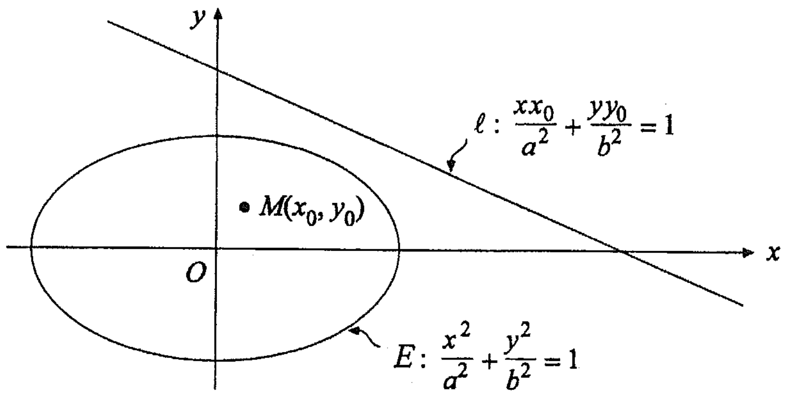
$$\frac{x_0 x}{5^2}+\frac{y_0 y}{3^2}=1.$$

(iv) Let the tangent at P meet a directrix at a point L. Show that $\angle PFL$ is a right angle where F is the corresponding focus.

HSC '95

(8) (a) Suppose that p and q are real numbers. Show that $pq \le \frac{p^2+q^2}{2}$. **1**

(b) **6**



The ellipse E is given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The point $M(x_0, y_0)$ lies inside E, so that $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} < 1$.

The line l is given by the equation $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$.

(i) Using the result of part (a), or otherwise, show that the line l lies entirely outside E. That is, show that if $P(x_1, y_1)$ is any point on l, then

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} > 1.$$

(ii) The chord of contact to E from any point $Q(x_2, y_2)$ outside E has equation

$$\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1.$$

Show that M lies on the chord of contact to E from any point on l.

HSC '89

(3) (a) The ellipse E: $\left(\frac{x}{5}\right)^2+\left(\frac{y}{3}\right)^2=1$ has foci $S(4,0)$ and $S'(-4,0)$.

(i) Sketch the ellipse E indicating its foci S, S' and its directrices.

(ii) Show that the tangent at $P(x_1, y_1)$ on the ellipse E has the equation

$$9x_1x+25y_1y=225$$

(iii) The line joining $P(x_1, y_1)$ to $Q(x_2, y_2)$ passes through S.

Show that $4(y_2 - y_1) = x_1 y_2 - x_2 y_1$.

(iv) It is known that $Q(x_2, y_2)$ lies on E. Show that the tangents at P and Q on the ellipse intersect on the directrix corresponding to S.

(v) Find the equation of the normal to E at P and decide under what circumstances, if any, it passes through S and S'.

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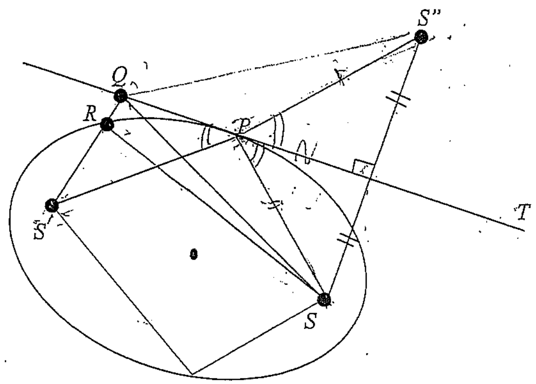


6

In the diagram, P is an arbitrary point on the ellipse, and QPT is a tangent to the ellipse at P. The points S' and S are the foci of the ellipse, and S'' is the reflection of S across the tangent, as shown. Let the line $S'Q$ intersect the ellipse at R.

(i) Assuming $Q \neq P$, prove that

$$S'Q + QS > S'R + RS.$$

In $\triangle QRS$

$QS + QR > RS$ ($\triangle$ inequalities)

adding $S'R$ to both sides

$S'R + QS + QR > RS + S'R$

$S'R + QR = S'Q$

$\therefore S'Q + QS > RS + S'R$

(ii) Deduce that the shortest path from S' to S passing through a point on the tangent is that through P, having length $S'P + PS$.

$S'P + PS = S'R + RS < S'Q + QS$ ✓

$\therefore$ As $R \to Q \to P$ ✓

$\therefore$ P is the shortest path

(iii) By considering the point S'', deduce that $\angle QPS' = \angle TPS$

In $\triangle SPZ$ & $\triangle S''PZ$:

$SZ = S''Z$ (given)

$\angle PZS'' = \angle SZP = 90^\circ$ (given) ✓

$PZ = ZP$ (common)

$\therefore \triangle SPZ \equiv S''PZ$ ✓ (SAS)

$\therefore PS = PS''$ (corresponding sides of a congruent $\triangle$) ✓

$\therefore \angle S''PZ = \angle SPZ$

Since $S'P$ is the shortest distance, $S'PS''$ is collinear ✓

$\therefore \angle QPS' = \angle TPS$ (vert. opp) ✓

HSC '99

(3)(a) Consider the ellipse E with equation $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$. **9**

and let $P = (x_0, y_0)$ be an arbitrary point on E.

(i) Calculate the eccentricity of E.

$9 = 25(1 - e^2)$

$\frac{9}{25} = 1 - e^2$ ✓

$e^2 = 1 - \frac{9}{25}$

$e^2 = \frac{16}{25}$ ✓

$e = \frac{4}{5}$

$e = \frac{4}{5}$

(ii) Find the coordinates of the foci of E and the equations of the directrices of E.

$(ae, 0)$

$\therefore (5 \times \frac{4}{5}, 0)$ ✓

$\therefore (\pm 4, 0)$

directrices $= \frac{a}{e}$

$= 5 \times \frac{5}{4}$ ✓

$x = \pm\frac{25}{4}$

$S(4,0), S'(-4,0); x = \pm\frac{25}{4}$

(iii) Show that the equation of the tangent at P is

$$\frac{x_0 x}{5^2}+\frac{y_0 y}{3^2}=1.$$

$$\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$$

$$\frac{2x}{5^2}=-\frac{2y}{3^2}\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{2x}{5^2}\times-\frac{3^2}{2y}$$

$$\frac{dy}{dx}=-\frac{3^2x_0}{5^2y_0}$$

$$y-y_0=\frac{-3^2x_0}{5^2y_0}(x-x_0)$$

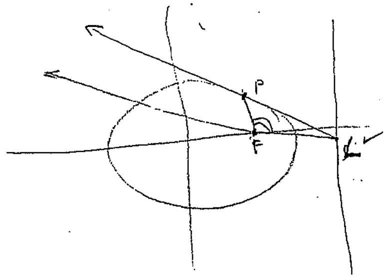
$$5^2yy_0-5^2y_0^2=-3^2x_0x+3^2x_0^2$$

$$\frac{5^2yy_0+3^2x_0x}{5^2.3^2}=\frac{5^2y_0^2+3^2x_0^2}{5^2.3^2}$$

$$\frac{yy_0}{3^2}+\frac{x_0x}{5^2}=\frac{y_0^2}{3}+\frac{x_0^2}{5^2}$$

$$\frac{yy_0}{3^2}+\frac{x_0x}{5^2}=1$$

(iv) Let the tangent at P meet a directrix at a point L. Show that $\angle PFL$ is a right angle where F is the corresponding focus.



$L\left(\frac{25}{4}, \frac{9(4-x_0)}{y_0 4}\right)$ ← finding

$F(ae, 0)$

$P(x_0, y_0)$

$$\frac{yy_0}{3^2}+\frac{x_0 25}{4}\div 5^2=1$$

$$\frac{yy_0}{3^2}+\frac{x_0 25}{4}\times\frac{1}{25}=1$$

$$\frac{yy_0}{3^2}=1-\frac{x_0}{4}$$

$$yy_0=3^2-\frac{3^2x_0}{4}$$

$$y=\frac{9-\frac{9x_0}{4}}{y_0}$$

gradient $PF=\dfrac{y_0}{x_0-ae}$

gradient $FL=\dfrac{\frac{9(4-x_0)}{y_0 4}}{\frac{25}{4}-ae}\times\dfrac{y_0-0}{x_0-ae}$

times $PF\times FL=\dfrac{\frac{9(4-x_0)}{y_0 4}}{\frac{25-4ae}{4}}\times\dfrac{y_0}{x_0-ae}$

$$=\frac{9(4-x_0)}{y_0(25-4ae)}\times\frac{y_0}{x_0-ae}$$

$$=\frac{9(4-x_0)}{x_0-ae\,(25-4ae)}$$

$$=\frac{9(4-x_0)}{x_0-4\times\frac{5}{5}(25-4\times\frac{5}{5}\times\frac{4}{5})}$$

$$=\frac{9(4-x_0)}{(x_0-4)(9)}$$

$$=\frac{-(x_0-4)}{x_0-4}$$

$\therefore=-1$

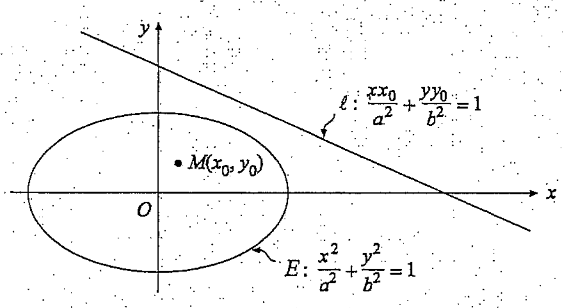
HSC '95

(8) (a) Suppose that p and q are real numbers. Show that $pq \le \frac{p^2+q^2}{2}$. **1**

$p^2+q^2 \geqslant 2pq$

$pq \le \frac{2pq}{2}$

(b) **6**



The ellipse E is given by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The point $M(x_0, y_0)$ lies inside E, so that $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}<1$.

The line l is given by the equation $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$.

(i) Using the result of part (a), or otherwise, show that the line l lies entirely outside E. That is, show that if $P(x_1, y_1)$ is any point on l, then

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}>1.$$

since $P(x_1, y_1)$ is any point on ℓ

$\frac{x_1x_0}{a^2}+\frac{y_1y_0}{b^2}=1$

$1 = \frac{x_1}{a}\cdot\frac{x_0}{a}+\frac{y_1}{b}\cdot\frac{y_0}{b} \le \left(\frac{x_1}{a}\right)^2+\left(\frac{x_0}{b}\right)^2+\left(\frac{y_1}{b}\right)^2+\left(\frac{y_0}{b}\right)^2$ (using a)

$1 \le \frac{\frac{x_1^2}{a^2}+\frac{x_0^2}{a^2}}{2}+\frac{\frac{y_1^2}{b^2}+\frac{y_0^2}{b^2}}{2}$

$1 < \frac{1}{2}\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}\right)+\frac{1}{2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right) < \frac{1}{2}+\frac{1}{2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)$

$1 < \frac{1}{2}+\frac{1}{2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)$

$\therefore\ 1 < \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$

(ii) The chord of contact to E from any point $Q(x_2, y_2)$ outside E has equation

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1.$$

Show that M lies on the chord of contact to E from any point on l.

$Q(x_2, y_2)$ is outside E

$\frac{x_2x_0}{a^2}+\frac{y_2y_0}{b^2}=1$

which means that $M(x_0, y_0)$ is on the line

$\frac{x_2x}{a^2}+\frac{y_2y}{b^2}=1$

HSC '89

(3) (a). The ellipse E: $\left(\frac{x}{5}\right)^2+\left(\frac{y}{3}\right)^2=1$ has foci $S(4,0)$ and $S'(-4,0)$.

(i) Sketch the ellipse E indicating its foci S, S' and its directrices.

$\frac{x^2}{25}+\frac{y^2}{9}=1$

$9=25(1-e^2)$

$\frac{9}{25}=1-e^2$

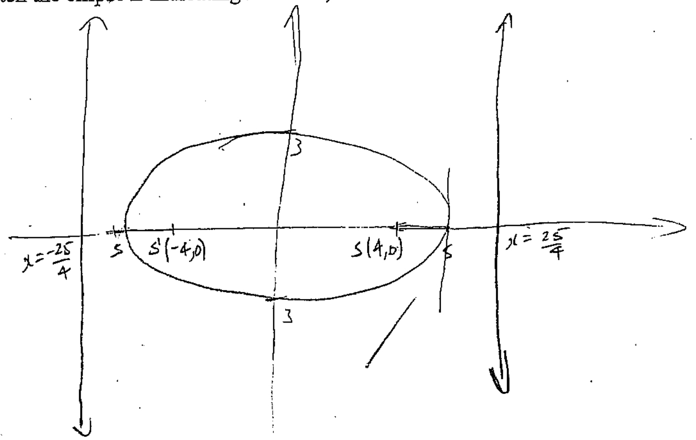
$e^2=1-\frac{9}{25}$

$e=\frac{4}{5}$ ✓

directrix $=\frac{a}{e}$

$=5\times\frac{5}{4}$

$x=\frac{25}{4}$



(ii) Show that the tangent at $P(x_1,y_1)$ on the ellipse E has the equation

$9x_1x+25y_1y=225$

by implicit differentiation

$\frac{2x}{25}+\frac{2y}{9}\frac{dy}{dx}=0$

$\frac{2x}{25}=-\frac{2y}{9}\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{2x}{25}\times\frac{9}{-2y}$ ✓

$=\frac{9x_1}{-25y_1}$ @ point (x_1,y_1)

$y-y_1=\frac{9x_1}{-25y_1}(x-x_1)$ ✓

$25y_1y-25y_1^2=-9x_1x+9x_1^2$

$9x_1x+25y_1y=25y_1^2+9x_1^2$ ✓

$\therefore 9x_1x+25y_1y=225$

@ $\frac{x_1^2}{25}+\frac{y_1^2}{9}=1$ ✓

$\frac{9x_1^2+25y_1^2}{225}=1$ ✓

$\therefore 9x_1^2+25y_1^2=225$

(iii) The line joining $P(x_1,y_1)$ to $Q(x_2,y_2)$ passes through S.
Show that $4(y_2-y_1)=x_1y_2-x_2y_1$.

$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

$yx_2-yx_1-y_1x_2+y_1x_1=(y_2-y_1)x-(y_2-y_1)x_1$

$yx_2-yx_1-y_1x_2+y_1x_1=xy_2-y_1x-y_2x_1+x_1y_1$

$x_1y_2-x_2y_1=xy_2-y_1x-yx_2+yx_1$

@ $(4,0)$

$x_1y_2-x_2y_1=4y_2-y_14-0+0$ ✓

$\therefore 4(y_2-y_1)=x_1y_2-x_2y_1$

(iv) It is known that $Q(x_2,y_2)$ lies on E. Show that the tangents at P and Q on the ellipse intersect on the directrix corresponding to S.

$9xx_1+25yy_1=225\ \times y_2$

$9xx_2+25yy_2=225\ \times y_1$

$x=\frac{a}{e}=\frac{5}{4}\times 5=\frac{25}{4}$

$9xx_1y_2-9xx_2y_1=225(y_2-y_1)$ ✓

$9x(x_1y_2-x_2y_1)=225(y_2-y_1)$

$9x(4(y_2-y_1))=225(y_2-y_1)$

$9x=\frac{225(y_2-y_1)}{4(y_2-y_1)}$ ✓

$x=\frac{25}{4}$

(v) Find the equation of the normal to E at P and decide under what circumstances, if any, it passes through S and S'.

by implicit

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

$$\frac{2x}{25}+\frac{2y}{9}\frac{dy}{dx}=0$$

$$-\frac{2x}{25}\times\frac{9}{2y}=\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{-9x}{25y} \quad \text{tangent}$$

$$\text{normal}=\frac{25y_1}{9x_1}$$

$$y-y_1=\frac{25y_1}{9x_1}(x-x_1)$$

@ (4, 0): $-y_1 9x_1=25y_1(4-x_1)$

$$-y_1 9x_1=100y_1-25x_1y_1$$

$$16x_1y_1=100y_1 \qquad y_1=0$$

$$16x_1y_1-100y_1=0$$

$$y_1(16x_1-100)=0$$

$$x_1\neq\frac{100}{16}=\frac{25}{4}$$