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Centre of Excellence in Mathematics
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# YEAR 12 – MATHS EXT.2

# REVIEW TOPIC (PAPER 1): VOL BY CYLINDRICAL SHELLS

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## CSSA 2001 Q5

(b) (i) On the same diagram and without using calculus, sketch the graphs of $y=e^{-x}$, $y=-e^{-x}$ and $y=e^{-x}\cos x$, $0\le x\le 2\pi$. Shade the region bounded by $y=e^{-x}$, $y=e^{-x}\cos x$ and $x=\pi$ for $x\ge 0$. 3

(ii) The region shaded in (i) is rotated through one revolution about the line $x=\pi$. Use the method of cylindrical shells to show that the volume of the solid of revolution is given by 2

$$V=2\pi\int_0^{\pi}(\pi-x)\,e^{-x}(1-\cos x)\;dx.$$

Part (i); $I$ is given by

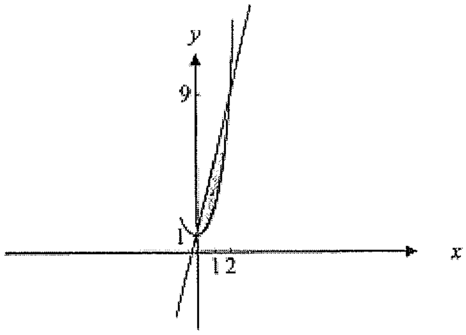
$$I=\int_0^{\pi} xe^{-x}\cos x\;dx$$

(iii) Use the substitution $u = \pi - x$ to show $V = 2\pi e^{-\pi}\left\{\int_0^{\pi} ue^{u}\,du + I\right\}$, where $I$ is as defined in (a). **2**

(iv) Hence find the volume of the solid. **3**

**HEFFERNAN 2002 Q3**

(c)




The shaded area shown in the diagram above is the area between the graph of $y = 4x + 1$ and the graph of $y = 2x^2 + 1$. This shaded area is rotated about the $y$ axis to form a solid. 4

Use the method of cylindrical shells to find the volume of the solid.

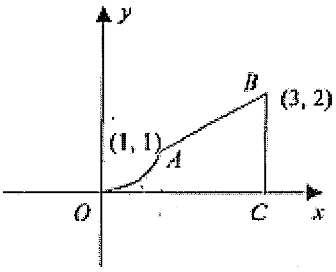
## RC 2002 Q5

a) A doughnut is formed by rotating the area of the circle $(x-3)^2 + y^2 = 4$ about the $y$-axis.

Calculate the volume of this doughnut using cylindrical shells. 5

**SBHS 2001 Q6**

(c)

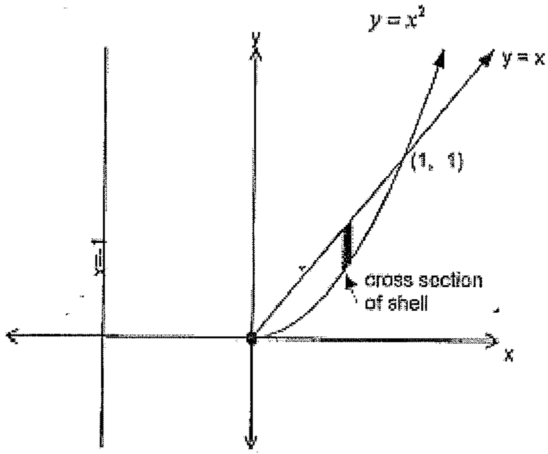



*OA* is an arc of the parabola $y = x^2$. The region *OABC* is rotated about the *y* axis forming a bowl. By using cylindrical shells determine the holding capacity of the bowl.

6

**SGHS 2002 Q6**

3. [6]

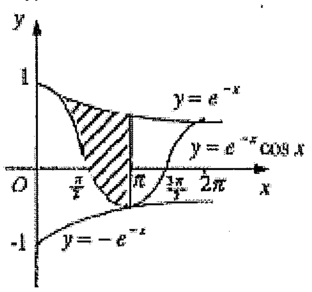



Use the method of cylindrical shells to calculate the volume of the solid formed when the area bounded by $y = x$ and $y = x^2$ is rotated about the line $x = -1$

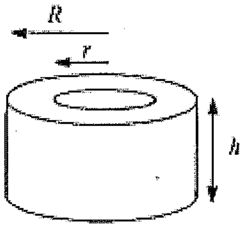
## SOLUTIONS

## CSSA 2001 Q5

(i)




(ii)




$R=\pi-x+\delta x$ , $r=\pi-x$

$h=e^{-x}-e^{-x}\cos x$

Cylindrical shell has volume

$\delta V=\pi\left(R^2-r^2\right)e^{-x}(1-\cos x)$

where

$R^2-r^2=(R+r)(R-r)$

$=\{2(\pi-x)+\delta x\}\delta x$

$=2(\pi-x)\delta x$

ignoring terms in $(\delta x)^2$ .

Hence volume of solid of revolution is given by

$$V=\lim_{\delta x\to 0}\sum_{x=0}^{x=\pi}\delta V = 2\pi\int_0^{\pi}(\pi-x)e^{-x}(1-\cos x)\ dx.$$

(iii)

$u=\pi-x$ $du=-dx$

$x=0\Rightarrow u=\pi$

$x=\pi\Rightarrow u=0$

$1-\cos x = 1-\cos(\pi-u)$

$=1+\cos u$

$$V=2\pi\int_{\pi}^{0}u\,e^{u-\pi}\{1+\cos u\}\ (-du)$$

$$=2\pi\ e^{-\pi}\int_0^{\pi}u\,e^{u}\{1+\cos u\}\ du$$

$$=2\pi\ e^{-\pi}\left\{\int_0^{\pi}u\,e^{u}\ du+\int_0^{\pi}u\,e^{u}\cos u\ du\right\}$$

$$=2\pi\ e^{-\pi}\left\{\int_0^{\pi}u\,e^{u}\ du+I\right\}$$

(iv)

$$\int_0^{\pi}u\,e^{u}\ du = \left[u\,e^{u}\right]_0^{\pi}-\int_0^{\pi}e^{u}\ du$$

$$=\pi\,e^{\pi}-\left[e^{u}\right]_0^{\pi}$$

$$=\pi\,e^{\pi}-\left(e^{\pi}-1\right)$$

$$V=2\pi\,e^{-\pi}\left\{\pi e^{\pi}-e^{\pi}+1+I\right\}$$

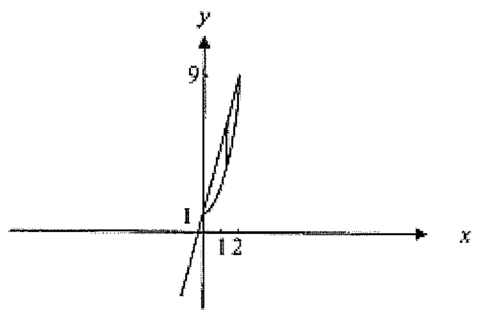
$$=2\pi\,e^{-\pi}\left\{\pi e^{\pi}-e^{\pi}+1-\tfrac{1}{2}\pi\,e^{\pi}\right\}$$

$$=\pi\,(\pi-2)+2\pi\,e^{-\pi}$$

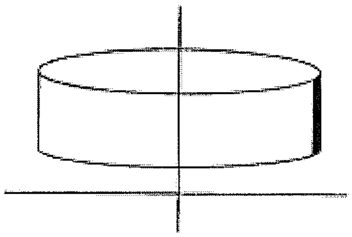
Hence volume is $\pi\,(\pi-2)+2\pi\,e^{-\pi}$ cu. units.

## HEFFERNAN 2002 Q3

(c)




Consider a small strip of the area to be rotated.



When this strip is rotated around the $y$ axis a cylindrical shell is formed with a base of perimeter $2\pi x$, a length of $4x+1-(2x^3+1)=-2x^2+4x$ **(1 mark)**

So we have $\delta V \approx 2\pi x \times (4x+1-2x^2-1) \times \delta x$ **(1 mark)**

So $V = 2\pi \int_0^2 (-2x^3+4x^2)dx$ **(1 mark)**

$$= 2\pi\left[\frac{-2x^4}{4}+\frac{4x^3}{3}\right]_0^2$$

$$= 2\pi\left\{\frac{-2\times 16}{4}+\frac{4\times 8}{3}\right\}$$

$$= 2\pi\left(-8+\frac{32}{3}\right)$$

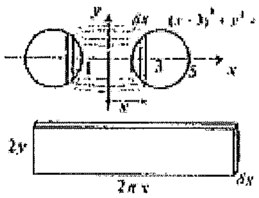
$$= 2\pi \times \frac{-24+32}{3}$$

$$= 2\pi \times \frac{8}{3}$$

$$= \frac{16\pi}{3} \text{ cubic units}$$

**(1 mark)**

**RC 2002 Q5**




$$\delta V \approx 4\pi xy\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=1}^{5} 4\pi xy\delta x$$ 1

$$= 4\pi \int_1^5 x\sqrt{4-(x-3)^2}\,dx$$

Put $x-3 = 2\sin\theta$

$$dx = 2\cos\theta\, d\theta$$

$$x = 1 \Rightarrow \theta = -\tfrac{\pi}{2}$$ 1

$$x = 5 \Rightarrow \theta = \tfrac{\pi}{2}$$

$$V = 4\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3+2\sin\theta)2\cos\theta 2\cos\theta d\theta$$ 1

$$= 8\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(12\cos^2\theta + 8\cos^2\theta\sin\theta\right)d\theta$$

$$= 48\pi \int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta - 64\pi \int_0^{\frac{\pi}{2}} \cos^2\theta(-\sin\theta)\,d\theta$$ 1

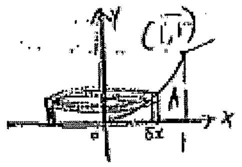
$$= 48\pi\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} - 64\pi\left[\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}}$$

$$= 48\pi\left[\tfrac{\pi}{2}+0\right] - 64\pi\left[0-\tfrac{1}{3}\right]$$ 1

$$= 24\pi^2 - \frac{64\pi}{3}$$

**SBHS 2001 Q6**

(c)




Height = y.

Vol. when region A is rotated about Y axis

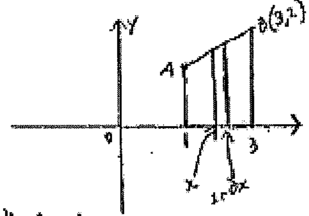
Vol. of shell $\delta V = \pi\left[(x+\delta x)^2 - x^2\right]y$

$= \pi\left[x^2 + 2x\delta x + \delta x^2 - x^2\right]y$

$= 2\pi x y \delta x$

$V = 2\pi\int_0^1 x \times x^2\,dx$ 1

(3) $= 2\pi\left[\frac{x^4}{4}\right]_0^1 = \frac{\pi}{2}$ units$^3$ 1




Eqⁿ of line AB is $x = 2y - 1$ ½

or $y = \frac{x+1}{2}$

$V = 2\pi\int_1^3 x\left(\frac{x+1}{2}\right)dx$ 1

$= \pi\int_1^3 (x^2 + x)\,dx$

(3) $= \pi\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_1^3$ 1

$= \pi\left[\left(9 + \frac{9}{2}\right) - \left(\frac{1}{3} + \frac{1}{2}\right)\right] = \frac{38\pi}{3}$

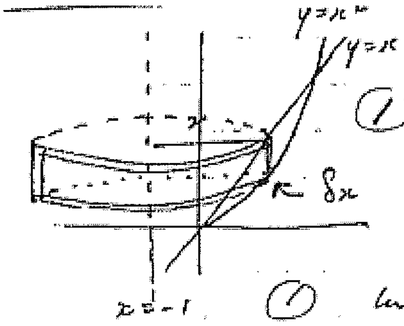
½

$\therefore$ Total volume $= \frac{\pi}{2} + \frac{38\pi}{3} = \frac{79\pi}{6}$

Vol. of bowl $= 18\pi - \frac{79\pi}{6} = \frac{29\pi}{6}$ u³

**SGHS 2002 Q6**

(3)




Volume of a Typical Shell

(1) $\Delta V = \pi(R^2 - r^2) \times \text{height}$

$R = r + \delta x$

$\therefore \Delta V = \pi((r+\delta x)^2 - r^2)h.$

$\Delta V = \pi(2r\,\delta x + (\delta x)^2)h.$

(1) lim as $\delta x \to 0$ $(\delta x)^2$ is negligible

$\therefore \Delta V \doteqdot 2\pi r h . \delta x .$

Now. $r = 1 + x$ and $h = x - x^2$. (1)

(1) $\therefore V \doteqdot \lim_{\delta x \to 0} \sum_0^1 2\pi r h . \delta x . =$

(1) $= \int_0^1 2\pi (1+x)(x - x^2).dx = \int_0^1 2\pi (x - x^3).dx$

$= 2\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 2\pi\left[\frac{1}{2} - \frac{1}{4}\right] - 0$

$= 2\pi\left[\frac{1}{4}\right]$

(1) $= \frac{\pi}{2}$ c.u.