2000 HIGHER SCHOOL CERTIFICATE EXAMINATION PAPER

2/3 UNIT (COMMON) MATHEMATICS

QUESTION 1

Marks

(a) Find the value of $\log_e 8$ correct to two decimal places. **2**

(b) Solve $x + 7 \geq 3$ and graph the solution on the number line. **2**

(c) What is the exact value of $\cos\frac{\pi}{6}$? **1**

(d) A bag contains red marbles and blue marbles in the ratio 2 : 3. A marble is selected at random.
What is the probability that the marble is blue? **1**

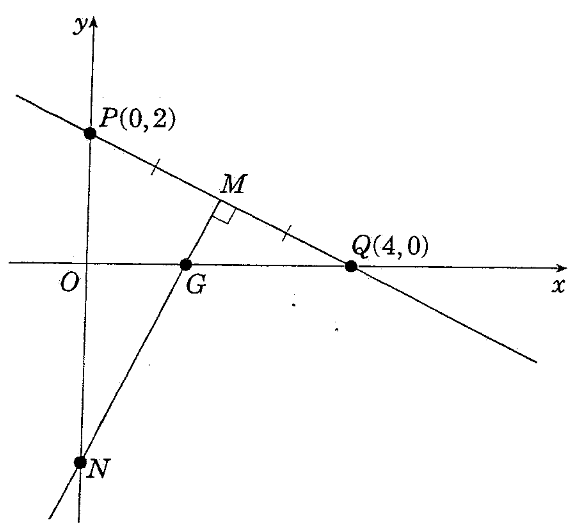
(e) Solve the pair of simultaneous equations: **2**

$$x - y = 2$$
$$3x + 2y = 1.$$

(f) Solve $|x - 5| = 3$. **2**

(g) Sketch the line $y = 2x + 3$ in the Cartesian plane. **2**

QUESTION 2



The diagram shows the points $P(0, 2)$ and $Q(4, 0)$. The point M is the midpoint of PQ. The line MN is perpendicular to PQ and meets the x axis at G and the y axis at N.

(a) Show that the gradient of PQ is $-\frac{1}{2}$. **1**

(b) Find the coordinates of M. **2**

(c) Find the equation of the line MN. **2**

(d) Show that N has coordinates $(0, -3)$. **1**

(e) Find the distance NQ. **1**

(f) Find the equation of the circle with centre N and radius NQ. **2**

(g) Hence show that the circle in part (f) passes through the point P. **1**

(h) The point R lies in the first quadrant, and $PNQR$ is a rhombus. Find the coordinates of R. **2**

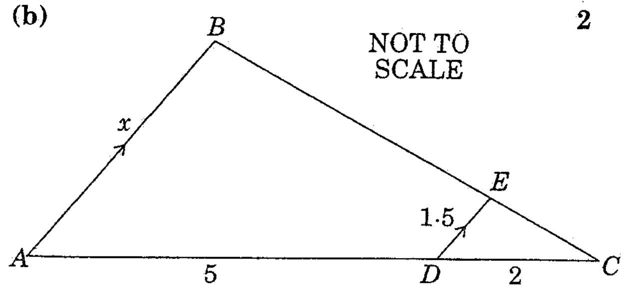
QUESTION 3

(a) Differentiate the following: **4**

(i) $3xe^x$

(ii) $\sin(x^2 + 1)$

(b) **2**



In the diagram, AB is parallel to DE, AD is 5 cm, DC is 2 cm and DE is 1·5 cm. Find the length of AB.

(c) Find: **3**

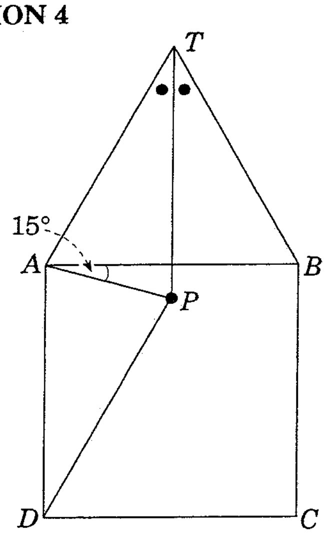
(i) $\int \sec^2 5x \, dx$

(ii) $\int_{-2}^{1} \frac{2}{x+3} \, dx$

(d) Find the equation of the tangent to to the curve $y = 2\log_e x$ at $(1, 0)$. **3**

QUESTION 4

(a) **6**



In the diagram, $ABCD$ is a square and ABT is an equilateral triangle. The line TP bisects $\angle ATB$, and $\angle PAB = 15°$.

(i) Copy the diagram and explain why $\angle PAT = 75°$.

(ii) Prove that $\Delta TAP \equiv \Delta DAP$.

(iii) Prove that triangle DAP is isosceles.

(b) In the construction of a 5 km expressway a truck delivers materials from a base. After depositing each load, the truck returns to the base to collect the next load. The first load is deposited 200 m from the base, the second 350 m from the base, the third 500 m from the base. Each subsequent load is deposited 150 m from the previous one. **6**

(i) How far is the fifteenth load deposited from the base?

(ii) How many loads are deposited along the total length of the 5 km expressway? (The last load is deposited at the end of the expressway.)

(iii) How many kilometres has the truck travelled in order to make all the deposits and then return to the base?

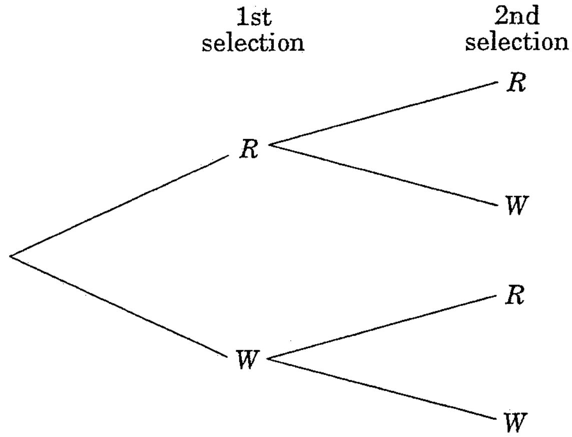
QUESTION 5

(a) Solve $\tan x = 2$ for $0 < x < 2\pi$. **2**

Express your answer in radian measure correct to two decimal places.

(b) Four white (W) balls and two red (R) balls are placed in a bag. One ball is selected at random, removed and replaced by a ball of the other colour. The bag is then shaken and another ball is randomly selected. **5**

(i) Copy the tree diagram. Complete the tree diagram, showing the probability on each branch.



(ii) Find the probability that both balls selected are white.

(iii) Find the probability that the second ball selected is white.

(c) The population of a certain insect is growing exponentially according to $N = 200e^{kt}$, where t is the time in weeks after the insects are first counted. **5**

At the end of three weeks the insect population has doubled.

(i) Calculate the value of the constant k.

(ii) How many insects will there be after 12 weeks?

(iii) At what rate is the population increasing after three weeks?

QUESTION 6

(a) Sketch the curve **3**

$$y = 1 - \sin 2x \text{ for } 0 \le x \le \pi.$$

(b) The number N of students logged onto a website at any time over a five-hour period is approximated by the formula **9**

$$N = 175 + 18t^2 - t^4, \quad 0 \le t \le 5.$$

(i) What was the initial number of students logged onto the website?

(ii) How many students were logged onto the website at the end of the five hours?

(iii) What was the maximum number of students logged onto the website?

(iv) When were the students logging onto the website most rapidly?

(v) Sketch the curve

$$N = 175 + 18t^2 - t^4, \quad 0 \le t \le 5.$$

QUESTION 7

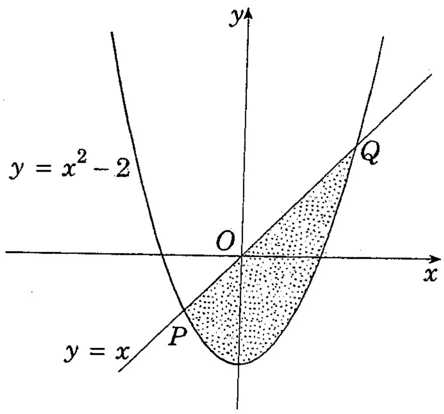
(a) The area under the curve **4**

$$y = \frac{1}{\sqrt{x}}, \text{ for } 1 \le x \le e^2,$$

is rotated about the x axis. Find the exact volume of the solid of revolution.

(b) Estimate $\int_0^1 \sin\left(1+x^2\right)\,dx$ by using Simpson's rule with three function values. **3**

(c) The diagram shows the graphs of $y = x^2 - 2$ and $y = x$. **5**



(i) Find the x values of the points of intersection, P and Q.

(ii) Calculate the area of the shaded region.

QUESTION 8

(a) A particle is moving in a straight line, starting from the origin. At time t seconds the particle has a displacement of x metres from the origin and a velocity v m s^{-1}. The displacement is given by $x = 2t - 3\log_e(t+1)$. **7**

(i) Find an expression for v.

(ii) Find the initial velocity.

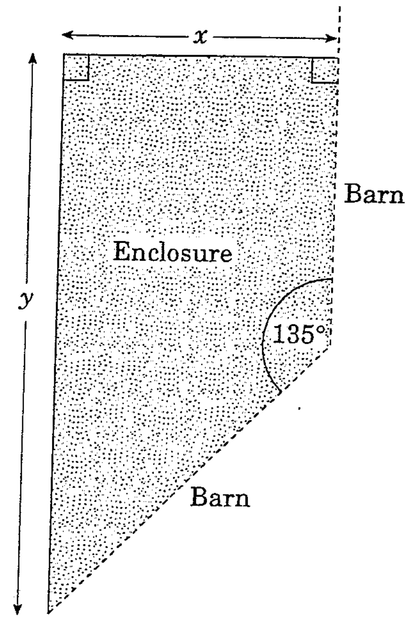
(iii) Find when the particle comes to rest.

(iv) Find the distance travelled by the particle in the first three seconds.

(b) An enclosure is to be built adjoining a barn, as in the diagram. The walls of the barn meet at 135°, and 117 metres of fencing is available for the enclosure, so that $x + y = 117$ where x and y are as shown in the diagram. **5**

(i) Show that the shaded area of the enclosure in square metres is given by

$$A = 117x - \frac{3}{2}x^2.$$



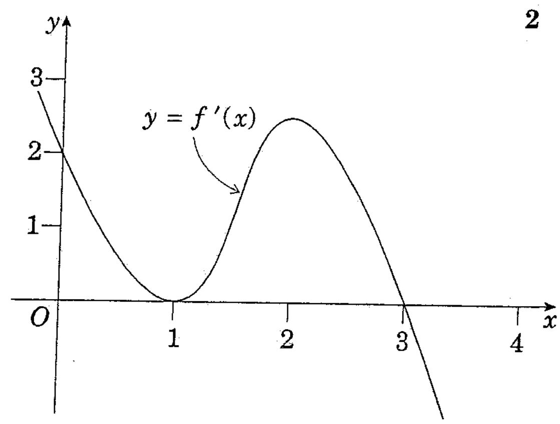
(ii) Show that the largest area of the enclosure occurs when $y = 2x$.

QUESTION 9

(a) (i) Without using calculus, sketch $y = \log_e x$. **3**

(ii) On the same sketch, find, graphically, the number of solutions of the equation $\log_e x - x = -2$.

(b) **2**

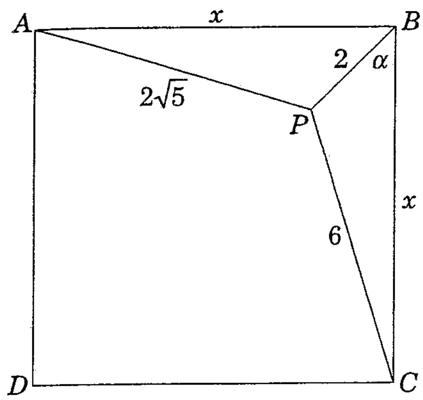


The above diagram shows a sketch of the gradient function of the curve $y = f(x)$.

Draw a sketch of the function $y = f(x)$ given that $f(0) = 0$.

(c) The diagram on page 4 shows a square $ABCD$ of side x cm, with a point P within the square, such that $PC = 6$ cm, $PB = 2$ cm and $AP = 2\sqrt{5}$ cm. **7**

Let $\angle PBC = \alpha$.



(i) Using the cosine rule in triangle PBC, show that
$$\cos\alpha = \frac{x^2-32}{4x}.$$

(ii) By considering triangle PBA, show that
$$\sin\alpha = \frac{x^2-16}{4x}.$$

(iii) Hence, or otherwise, show that the value of x is a solution of $x^4 - 56x^2 + 640 = 0$.

(iv) Find x. Give reasons for your answer.

QUESTION 10

(a) A store offers a loan of \$5000 on a computer for which it charges interest at the rate of 1% per month. As a special deal, the store does not charge interest for the first three months, however the first repayment is due at the end of the first month. **6**

A customer takes out the loan and agrees to repay the loan over three years by making 36 equal monthly repayments of \$$M$.

Let \$$A_n$ be the amount owing at the end of the nth repayment.

(i) Find an expression for A_3.

(ii) Show that
$$A_5 = (5000 - 3M)1{\cdot}01^2 - M(1 + 1{\cdot}01).$$

(iii) Find an expression for A_{36}.

(iv) Find the value of M.

(b) The first snow of the season begins to fall during the night. The depth of the snow, h, increases at a constant rate through the night and the following day. At 6 am a snow plough begins to clear the road of snow. The speed, v km/h, of the snow plough is inversely proportional to the depth of snow. **6**

(This means $v = \dfrac{A}{h}$ where A is a constant.)

Let x km be the distance the snow plough has cleared and let t be the time in hours from the beginning of the snowfall. Let $t = T$ correspond to 6 am.

(i) Explain carefully why, for $t \geq T$,
$$\frac{dx}{dt} = \frac{k}{t},\text{ where } k \text{ is a constant.}$$

(ii) In the period from 6 am to 8 am the snow plough clears 1 km of road, but it takes a further 3·5 hours to clear the next kilometre.

At what time did it begin snowing?

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2000 HIGHER SCHOOL CERTIFICATE SOLUTIONS

2/3 UNIT (COMMON) MATHEMATICS

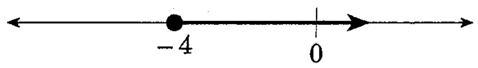
QUESTION 1

(a) $\log_e 8 = 2{\cdot}079\,441\,542\ldots$

$\doteqdot 2{\cdot}08$ (2 d.p.).

(b) $x + 7 \geq 3$

$x \geq -4.$



(c) $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$

(d) $P(\text{blue}) = \frac{3}{5}.$

(e) By substitution:

$$x - y = 2 \quad ①$$
$$3x + 2y = 1. \quad ②$$

From ①, $x = y + 2.$

Substitute for x in ②:

$$3(y+2) + 2y = 1$$
$$5y + 6 = 1$$
$$5y = -5$$
$$y = -1.$$

Substitute for y in ①:

$$x - (-1) = 2$$
$$x = 1.$$

Solution: $x = 1,\ y = -1.$

OR By elimination:

$$x - y = 2 \quad ①$$
$$3x + 2y = 1. \quad ②$$

Multiply ① by 2:

$$2x - 2y = 4 \quad ③$$
$$3x + 2y = 1. \quad ②$$

Add ③ + ②: $5x = 5$

$\therefore\ x = 1.$

Substitute into ①:

$$1 - y = 2$$

$\therefore\ y = -1.$

Solution: $x = 1,\ y = -1.$

(f) $|x - 5| = 3$

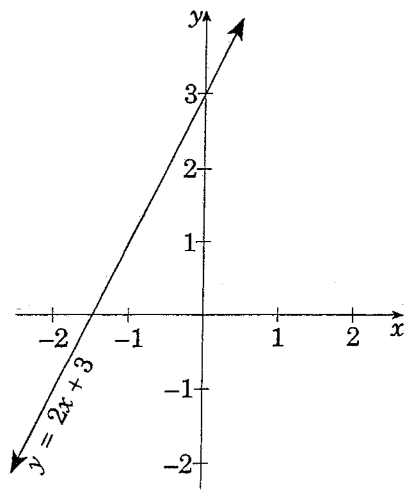
$x - 5 = 3$ or $-(x-5) = 3$

$\therefore\ x = 8.$ $\quad x - 5 = -3$

$\therefore\ x = 2.$

Solution: $x = 8$ or $x = 2$.

(g)



QUESTION 2

(a) Gradient of $PQ = \frac{0-2}{4-0} = -\frac{1}{2}.$

(b) M, midpoint of PQ is $M\left(\frac{0+4}{2}, \frac{2+0}{2}\right).$

$\therefore\ M(2, 1).$

(c) $MN \perp PQ$,

$\therefore$ gradient of $MN = 2 \quad (m_1 m_2 = -1).$

Equation of MN is

$$y - 1 = 2(x-2)$$
$$= 2x - 4$$
$$2x - y - 3 = 0.$$

(d) Substitute $x = 0$ in equation of MN.

$$2(0) - y - 3 = 0$$
$$y = -3.$$

$\therefore$ N has coordinates $(0, -3)$.

(e) $NQ = \sqrt{(4-0)^2 + (0+3)^2} = 5.$

$\therefore$ distance $NQ = 5$ units.

(f) $r = 5$. Centre is $(0, -3)$.

$\therefore$ Equation of circle is $(x-0)^2 + (y+3)^2 = 5^2$

or $x^2 + (y+3)^2 = 25$.

(g) $PN = 2 - (-3)$
$= 5$
$=$ radius of circle.

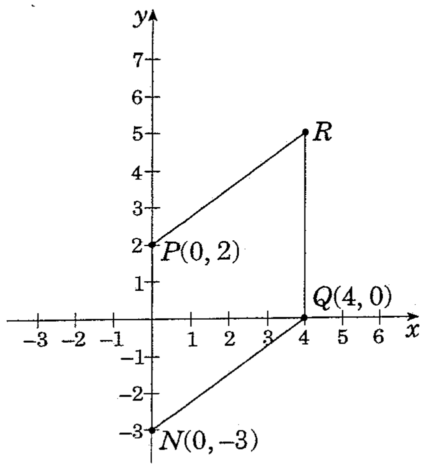
$\therefore$ Circle passes through P.

OR Substitute coordinates of P in the equation of the circle.

$\text{LHS} = 0^2 + (2+3)^2$
$= 25$
$= \text{RHS}$.

$\therefore$ Circle passes through P.

(h)



By observation: R is $(4, 5)$, since Q is 4 units to the right of PN and P is 5 units up the y axis from N.

QUESTION 3

(a) (i) $\dfrac{d}{dx}\left(3xe^x\right) = 3e^x + 3xe^x$
$= 3e^x(1+x)$.

(ii) $\dfrac{d}{dx}\left[\sin\left(x^2+1\right)\right] = 2x\cos\left(x^2+1\right)$.

(b) $\Delta ABC \parallel\!\!| \Delta DEC$.

$\therefore \quad \dfrac{AB}{DE} = \dfrac{AC}{DC}$

$\dfrac{x}{1{\cdot}5} = \dfrac{7}{2}$

$x = 5{\cdot}25$.

$\therefore\ AB = 5{\cdot}25$ cm.

(c) (i) $\displaystyle\int \sec^2 5x\, dx = \frac{1}{5}\tan 5x + C$.

(ii) $\displaystyle\int_{-2}^{1} \frac{2}{x+3}\, dx = 2\Big[\ln(x+3)\Big]_{-2}^{1}$
$= 2(\ln 4 - \ln 1)$
$= 2\ln 4$.

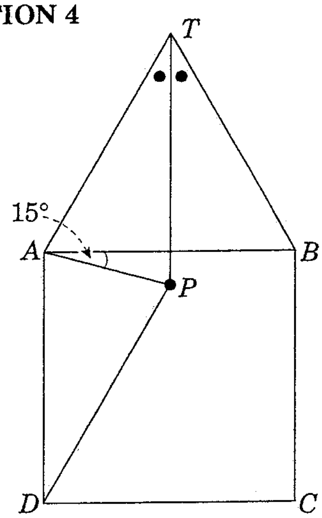
(d) $y = 2\log_e x$, $\dfrac{dy}{dx} = \dfrac{2}{x}$.

At $x = 1$, $\dfrac{dy}{dx} = 2$.

$\therefore$ Equation of the tangent is $y - 0 = 2(x-1)$
$y = 2x - 2$.

QUESTION 4

(a)



(i) $\angle TAB = 60°$ (ΔABT is equilateral.)

$\angle PAT = \angle TAB + \angle PAB$
$= 60° + 15°$
$= 75°$.

(ii) In ΔTAP and ΔDAP,

$AT = AD$ (ΔTAB equilateral; $ABCD$ square)

AP is common.

$\angle DAP = \angle DAB - \angle PAB$
$= 90° - 15°$
$= 75°$
$= \angle PAT$.

$\therefore\ \Delta TAP \equiv \Delta DAP$ (**SAS**).

(iii) $\angle ATB = 60°$ (ΔABT is equilateral.)

$\angle ATP = 30°$ (TP bisects $\angle APB$.)

$\angle ADP = \angle ATP$ (corresponding $\angle$s in congruent Δs)
$= 30°$.

$\angle APD = 180° - 75° - 30°$ ($\angle$ sum of a Δ)
$= 75°$
$= \angle DAP$.

$\therefore\ \Delta DAP$ is isosceles.

(b) (i) Base ... last load

0 200 350 500

|←------ 5000 m ------→|

This is an arithmetic series, with $a = 200$, $d = 150$.

$$T_n = a + (n-1)d$$
$$\therefore\ T_{15} = 200 + 14 \times 150 = 2300.$$

$\therefore$ The fifteenth load is deposited 2300 m from the base.

(ii) $T_n = 5000$,

$$5000 = 200 + (n-1) \times 150$$
$$150(n-1) = 4800$$
$$n - 1 = 32$$
$$n = 33.$$

$\therefore$ There are 33 loads.

(iii) Using $S_n = \frac{n}{2}(a + \ell)$,

total distance travelled

$$= 2 \times \left[\frac{33}{2}(200 + 5000)\right]$$
$$= 171600 \text{ m}$$
$$= 171{\cdot}600 \text{ km}.$$

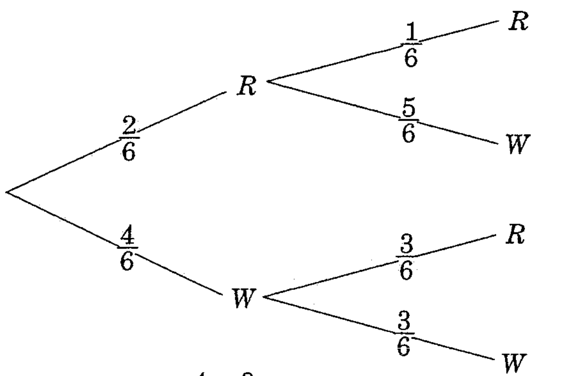
QUESTION 5

(a) $\tan x = 2$.

1st quadrant: $x = 1{\cdot}107\,14\ldots \doteqdot 1{\cdot}11$ (2 d.p.).

3rd quadrant: $x = 1{\cdot}107\,14 + \pi = 4{\cdot}248\,74\ldots \doteqdot 4{\cdot}25$ (2 d.p.).

(b) (i)



(ii) $P(WW) = \frac{4}{6} \times \frac{3}{6} = \frac{1}{3}$.

(iii) $P(RW) + P(WW) = \frac{2}{6} \times \frac{5}{6} + \frac{4}{6} \times \frac{3}{6} = \frac{11}{18}$.

(c) (i) At $t = 0$, $N = 200$.

At $t = 3$, $N = 400$.

$$N = 200e^{kt}$$
$$t = 3: \quad 400 = 200e^{3k}$$
$$e^{3k} = 2$$
$$3k = \ln 2.$$
$$\therefore\ k = \frac{1}{3}\ln 2 = 0{\cdot}210\,4906 \doteqdot 0{\cdot}2310 \text{ (4 d.p.)}.$$

(ii) At $t = 12$, $N = 200e^{k \times 12} = 3198{\cdot}116\,643\ldots \doteqdot 3200$ (2 sig. figs).

(iii) $\frac{dN}{dt} = kN$.

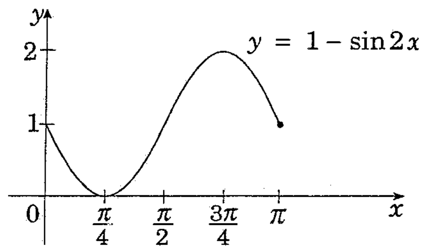
When $t = 3$, $N = 400$,

$$\therefore\ \frac{dN}{dt} = k \times 400 \doteqdot 92{\cdot}4 \text{ (3 sig. figs)}.$$

$\therefore$ The rate is 92·4 insects per week.

QUESTION 6

(a)



(b) $N = 175 + 18t^2 - t^4,\ 0 \le t \le 5$.

(i) When $t = 0$, $N = 175$.

$\therefore$ The initial number of students was 175.

(ii) When $t = 5$, $N = 175 + 18(5)^2 - (5)^4 = 0$.

$\therefore$ There were no students logged on at the end of five hours.

(iii) $\frac{dN}{dt} = 36t - 4t^3$

Let $\frac{dN}{dt} = 0$, $\quad 36t - 4t^3 = 0$

$$4t(9 - t^2) = 0.$$

$\therefore$ $t = 0$ or $t = 3$ or $t = -3$ (out of domain).

At $t = 0$, $N = 175$.

At $t = 3$, $N = 256$.

$$\frac{d^2N}{dt^2} = 36 - 12t^2 < 0 \text{ when } t = 3.$$

$\therefore$ Maximum value of N occurs at $t = 3$.

$\therefore$ The maximum value of N is 256.

(iv) Students are logging onto the website most rapidly when $\frac{dN}{dt}$ is a maximum.

That is, when $\frac{d^2N}{dt^2} = 0$

$$36t - 12t^2 = 0$$
$$t = +\sqrt{3} \text{ or } -\sqrt{3}$$

(not in domain).

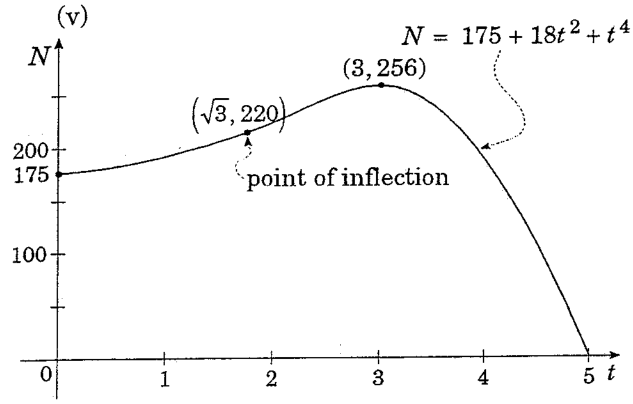
Test for maximum or minimum of $\frac{dN}{dt}$:

t	$\sqrt{3}^-$	$\sqrt{3}$	$\sqrt{3}^+$
$\frac{d^2N}{dt^2}$	+	0	−

N.B. This is checking the first derivative of $\frac{dN}{dt}$.

$\therefore$ Students are logging onto the website most rapidly when $t = \sqrt{3}$ hours.

(v)



QUESTION 7

(a) $V = \pi\int_1^{e^2} y^2\,dx$

$= \pi\int_1^{e^2}\left(\frac{1}{\sqrt{x}}\right)^2 dx$

$= \pi\int_1^{e^2}\frac{1}{x}\,dx$

$= \pi\Big[\ln x\Big]_1^{e^2}$

$= \pi\left(\ln e^2 - \ln 1\right)$

$= \pi(2\ln e - \ln 1).$

$\therefore$ Volume $= 2\pi$ unit^3.

(b)

x	0	0·5	1
y	sin 1	sin 1·25	sin 2

$\int_0^1 \sin\left(1+x^2\right)dx$

$\doteqdot \frac{1-0}{6}(\sin 1 + 4\sin 1{\cdot}25 + \sin 2)$

$\doteqdot 0{\cdot}924\,451\,148$

$\doteqdot 0{\cdot}92$ (2 d.p.).

(c) (i) $y = x^2 - 2$ ①

$y = x.$ ②

Substitute for y in ①:

$x = x^2 - 2$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = -1$ or $x = 2.$

$\therefore$ At P, $x = -1$ and at Q, $x = 2$.

(ii) Area $= \int_{-1}^{2} x - \left(x^2 - 2\right) dx$

$= \int_{-1}^{2} x - x^2 + 2\,dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2}$

$= \left(\frac{2^2}{2} - \frac{2^3}{3} + 2\times 2\right) - \left(\frac{1}{2} + \frac{1}{3} - 2\right)$

$= 4\frac{1}{2}.$

$\therefore$ Area is $4\frac{1}{2}$ square units.

QUESTION 8

(a) (i) $x = 2t - 3\log_e(t+1)$

$v = \frac{dx}{dt}$

$= 2 - \frac{3}{t+1}.$

(ii) Let $t = 0$, $v = 2 - 3 = -1.$

$\therefore$ Initial velocity is -1 m s^{-1}.

(iii) Let $v = 0$, $2 - \frac{3}{t+1} = 0$

$2t + 2 - 3 = 0$

$t = \frac{1}{2}.$

$\therefore$ Particle is at rest when $t = \frac{1}{2}$ s.

(iv) At $t = 0$, $x = 2(0) - 3\ln 1 = 0.$

At $t = \frac{1}{2}$, $x = 2(0{\cdot}5) - 3\ln 1{\cdot}5$

$= 1 - 3\ln 1{\cdot}5$

$= -0{\cdot}2164.$

At $t = 3$, $x = 2\times 3 - 3\ln 4$

$= 6 - 3\ln 4$

$= 1{\cdot}8411.$

A	O	B
$t = \frac{1}{2}$	$t = 0$	$x = 1{\cdot}8411$
$x = -0{\cdot}2164$	$x = 0$	$t = 3$

Distance OA = 0·2164 m.

Distance OB = 1·8411 m.

Distance travelled $= 2OA + OB$

$= 2\times 0{\cdot}2164 + 1{\cdot}8411$

$= 2{\cdot}2739\ldots$

$= 2{\cdot}27$ m (3 sig. figs).

OR At $t = 0$, $x = 0$.

At $t = \frac{1}{2}$, $x = 1 - 3\ln 1{\cdot}5$.

At $t = 3$, $x = 6 - 3\ln 4$.

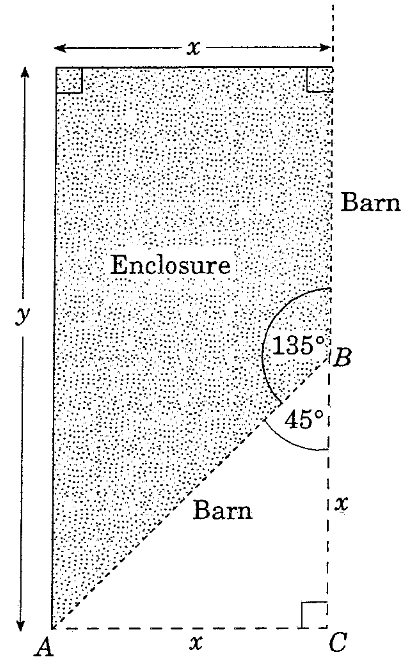
Distance travelled in the first 3 seconds

$= |1 - 3\ln 1{\cdot}5 - 0| + |6 - 3\ln 4 - (1 - 3\ln 1{\cdot}5)|$

$= 3\ln 1{\cdot}5 - 1 + 6 - 3\ln 4 - 1 + 3\ln 1{\cdot}5$

$= 2{\cdot}27$ m.

(b)



(i) Since $\angle ABC = 45°$, ABC is a right-angled isosceles triangle.

$\therefore\ AC = BC = x$.

$A = xy - \frac{1}{2}x^2$

$A = x(117 - x) - \frac{1}{2}x^2$ (since $x + y = 117$)

$A = 117x - \frac{3}{2}x^2$.

(ii) $A = 117x - \frac{3}{2}x^2$

$\frac{dA}{dx} = 117 - 3x$.

Let $\frac{dA}{dx} = 0$, $117 - 3x = 0$

$x = 39$.

Now $\frac{d^2A}{dx^2} = -3$.

$\therefore$ Area is a maximum when $x = 39$.

$x + y = 117$

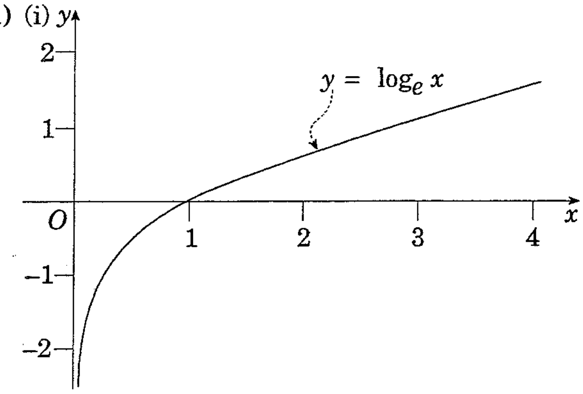
$y = 78$

$= 2 \times 39$.

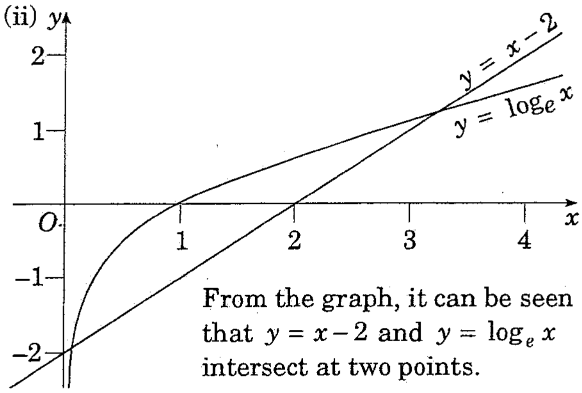
Hence area is a maximum when $y = 2x$.

QUESTION 9

(a) (i)



(ii)

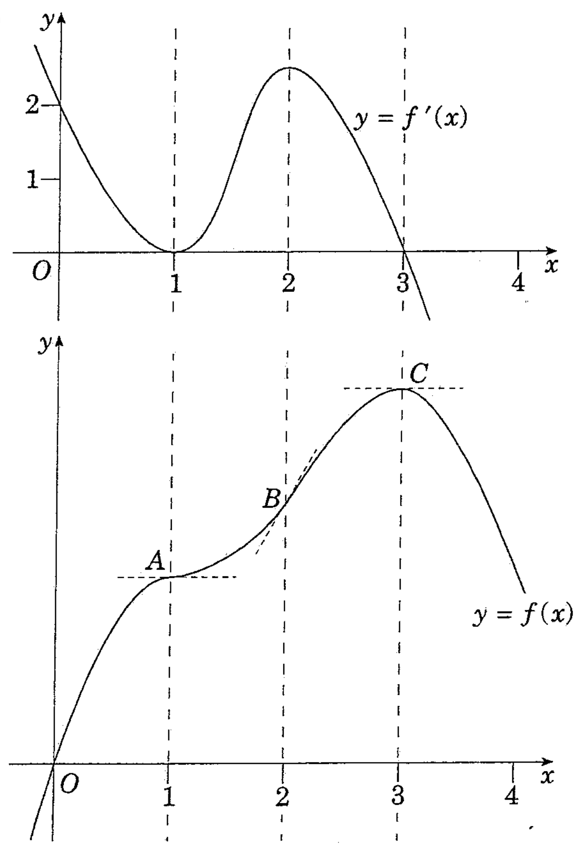


From the graph, it can be seen that $y = x - 2$ and $y = \log_e x$ intersect at two points.

$\therefore$ There are two solutions to $\log_e x = x - 2$. That is, there are two solutions to

$\log x - x = -2$.

(b)

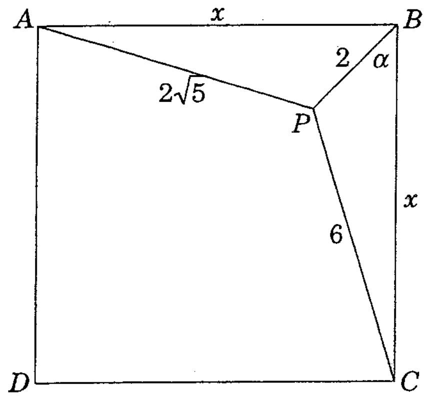


A: Horizontal point of inflection.

B: Point of inflection.

C: Maximum turning point.

(c)



(i) In ΔPBC, $\cos\alpha = \dfrac{PB^2 + CB^2 - PC^2}{2\times PB\times CB}$

$$= \frac{2^2 + x^2 - 6^2}{2\times 2\times x}$$

$$= \frac{x^2 - 32}{4x}.$$

(ii) Since $\angle ABC$ is an angle in a square and $\angle PBC = \alpha$, then $\angle ABP = 90° - \alpha$.

Using the cosine rule in ΔABP:

$$\cos\angle ABP = \frac{x^2 + 2^2 - \left(2\sqrt{5}\right)^2}{2\times x\times 2}$$

$$= \frac{x^2 - 16}{4x}.$$

But $\cos(90° - \alpha) = \sin\alpha$

$$\therefore \quad \sin\alpha = \frac{x^2 - 16}{4x}.$$

(iii) From (i), $\cos\alpha = \dfrac{x^2 - 32}{4x}$.

From (ii), $\sin\alpha = \dfrac{x^2 - 16}{4x}$.

But $\cos^2\alpha + \sin^2\alpha = 1$

$$\therefore \left(\frac{x^2 - 32}{4x}\right)^2 + \left(\frac{x^2 - 16}{4x}\right)^2 = 1.$$

$$\therefore x^4 - 64x^2 + 1024 + x^4 - 32x^2 + 256 = 16x^2$$

$$\therefore \quad 2x^4 - 112x^2 + 1280 = 0$$

$$\therefore \quad x^4 - 56x^2 + 640 = 0.$$

That is,
x is a solution of $x^4 - 56x^2 + 640 = 0$.

(iv) $x^4 - 56x^2 + 640 = 0$.

That is, $\left(x^2 - 16\right)\left(x^2 - 40\right) = 0$

$\therefore \; x = \pm 4$ or $x = \pm\sqrt{40}$.

But x cm is the length of the side of a square, so $x > 0$.

$\therefore \; x = 4$ or $\sqrt{40} \; \left(= 2\sqrt{10}\right)$.

If $x = 4$, $\cos\alpha = \dfrac{4^2 - 32}{4\times 4} < 0$.

But α is acute, so $\cos\alpha > 0$.

If $x = 2\sqrt{10}$, $\cos\alpha = \dfrac{\left(2\sqrt{10}\right)^2 - 32}{4\times 2\sqrt{10}} > 0$.

$\therefore$ Solution is $x = 2\sqrt{10}$.

QUESTION 10

(a) (i) $A_1 = 5000 - M$.

$A_2 = 5000 - 2M$.

$A_3 = 5000 - 3M$.

(ii) $A_4 = (5000 - 3M)\times 1{\cdot}01 - M$.

$A_5 = A_4 \times 1{\cdot}01 - M$.

$$\therefore \; A_5 = \left[(5000 - 3M)\times 1{\cdot}01 - M\right]\times 1{\cdot}01 - M$$

$$= (5000 - 3M)\times 1{\cdot}01^2 - M\times 1{\cdot}01 - M$$

$$= (5000 - 3M)\times 1{\cdot}01^2 - M(1 + 1{\cdot}01).$$

(iii) $A_6 = A_5 \times 1{\cdot}01 - M$

$$= (5000 - 3M)\times 1{\cdot}01^3 - M\left(1 + 1{\cdot}01 + 1{\cdot}01^2\right).$$

Continuing the pattern,

$$A_{36} = (5000 - 3M)\times 1{\cdot}01^{33} - M\left(1 + 1{\cdot}01 + \cdots + 1{\cdot}01^{32}\right)$$

$$= (5000 - 3M)\times 1{\cdot}01^{33} - M\times\frac{\left(1{\cdot}01^{33} - 1\right)}{(1{\cdot}01 - 1)}.$$

(iv) But $A_{36} = 0$.

$$\therefore (5000 - 3M)\times 1{\cdot}01^{33} - \frac{M\times\left(1{\cdot}01^{33} - 1\right)}{0{\cdot}01} = 0,$$

$$\therefore \; 5000\times 1{\cdot}01^{33} - 3M\times 1{\cdot}01^{33} - M\times\frac{1{\cdot}01^{33} - 1}{0{\cdot}01} = 0,$$

$$\therefore \; 5000\times 1{\cdot}01^{33} - M\left(3\times 1{\cdot}01^{33} + \frac{1{\cdot}01^{33} - 1}{0{\cdot}01}\right) = 0,$$

$$\therefore \; M = \frac{5000\times 1{\cdot}01^{33}}{\left(3\times 1{\cdot}01^{33} + \dfrac{1{\cdot}01^{33} - 1}{0{\cdot}01}\right)},$$

$\therefore \; M = 161{\cdot}343\,9692.$

$\therefore$ The monthly repayment is \$161·34 to the nearest cent.

(b) (i) The depth of the snow increases at a constant rate.

Let $\dfrac{dh}{dt} = c$, where c is a constant.

Then $h = ct + d$, where d is a constant.

But at $t = 0$, $h = 0$,

$\therefore \quad h = ct$.

It is given that $v = \dfrac{A}{h}$, where A is a constant.

Since $v = \dfrac{dx}{dt}$ and $h = ct$,

$$\frac{dx}{dt} = \frac{A}{ct}$$

$$= \frac{k}{t}, \text{ where } k = \frac{A}{c}.$$

(ii) *Without calculus:*

- 1st km of road takes 2 hours to clear (6 am to 8 am),
- 2nd km of road takes $3\frac{1}{2}$ hours.

This is a difference of $1\frac{1}{2}$ hours.

This means that it took an extra $1\frac{1}{2}$ hours to clear the snow which fell during the 2nd km clearing time; that is, it took an extra $1\frac{1}{2}$ hours to clear $3\frac{1}{2}$ hours of snow.

This is a rate of 1 hour to clear

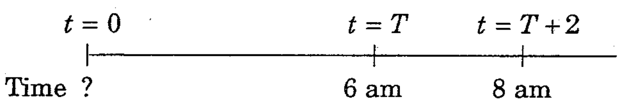
$$\frac{3\frac{1}{2}}{1\frac{1}{2}} = 2\frac{1}{3} \text{ hours of snow.}$$

$\therefore$ The first km taking 2 hours must have cleared $2 \times 2\frac{1}{3}$ hours, that is, $4\frac{2}{3}$ hours of snow. This time finished at 8 am.

$\therefore$ The snow started falling at $4\frac{2}{3}$ hours before 8 am. That is, it began to snow at 3:20 am.

OR x = distance in km that the snow plough has cleared.

t = time in hours that the snow has been falling.



$$\frac{dx}{dt} = \frac{k}{t}$$

$$\therefore \quad x = \int \frac{k}{t}\,dt.$$

From 6 am (time T) to 8 am (time $T + 2$), 1 km of snow was cleared.

Using $x = \int \frac{k}{t}\,dt$, $\quad 1 = \int_T^{T+2} \frac{k}{t}\,dt$.

Similarly in the next 3·5 hours, that is, from $t = T + 2$ to $t = T + 5{\cdot}5$, another 1 km of snow was cleared.

$$\therefore \quad 1 = \int_{T+2}^{T+5\cdot5} \frac{k}{t}\,dt.$$

$$\therefore \quad \int_T^{T+2} \frac{k}{t}\,dt = \int_{T+2}^{T+5\cdot5} \frac{k}{t}\,dt,$$

that is, $k\Big[\ln t\Big]_T^{T+2} = k\Big[\ln t\Big]_{T+2}^{T+5\cdot5}$

that is, $k\big[\ln(T+2) - \ln T\big]$

$$= k\big[\ln(T+5\cdot5) - \ln(T+2)\big].$$

$$\therefore \quad \ln\left(\frac{T+2}{T}\right) = \ln\left(\frac{T+5\cdot5}{T+2}\right)$$

$$\therefore \quad \frac{T+2}{T} = \frac{T+5\cdot5}{T+2}$$

$$\therefore \quad (T+2)^2 = T(T+5\cdot5)$$

$$\therefore \quad T = 2\tfrac{2}{3}.$$

$\therefore$ The snow started falling $2\frac{2}{3}$ hours before time T (6 am).

$\therefore$ The snow started at 3:20 am.

END OF 2/3 UNIT (COMMON) MATHEMATICS SOLUTIONS