BOARD OF STUDIES
NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

1998
MATHEMATICS
2/3 UNIT (COMMON)

Time allowed—Three hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Answer each question in a SEPARATE Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

QUESTION 1. Use a SEPARATE Writing Booklet. **Marks**

(a) Express $\frac{3}{11}$ as a recurring decimal. **1**

(b) Simplify $|-5|-|8|$. **1**

(c) A coin is tossed three times. What is the probability that 'heads' appears every time? **2**

(d) Find a primitive of $x^2 + 7$. **2**

(e) Find the exact value of $\sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{2\pi}{3}\right)$. **2**

(f) By rationalising the denominators, express $\frac{1}{3-\sqrt{2}}+\frac{1}{3+\sqrt{2}}$ in simplest form. **2**

(g) A merchant buys tea from a wholesaler and then sells it at a profit of 37·5%. If the merchant sells a packet of tea for \$3·08, what price does he pay to the wholesaler per packet of tea? **2**

QUESTION 2. Use a SEPARATE Writing Booklet. **Marks**

(a) Differentiate the following functions: **6**

(i) $\left(3x^2+4\right)^5$

(ii) $x\,\sin(x+1)$

(iii) $\dfrac{\tan x}{x}$.

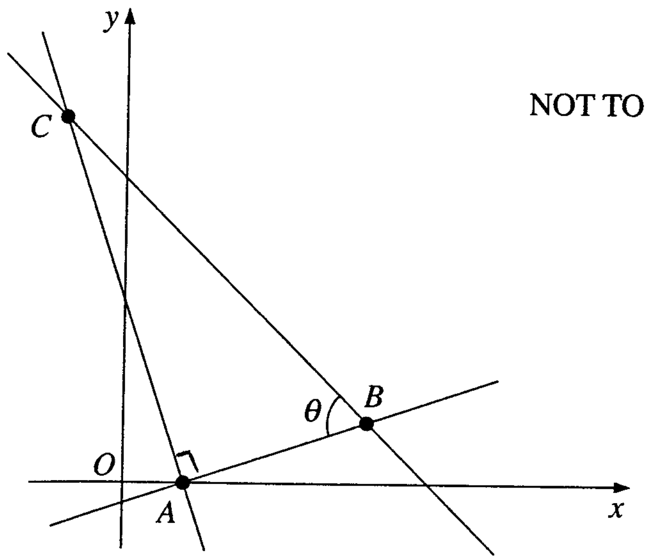
(b) Evaluate the following integrals: **4**

(i) $\displaystyle\int_1^2 \frac{1}{x^2}\,dx$

(ii) $\displaystyle\int_0^3 e^{4x}\,dx$.

(c) Find $\displaystyle\int \frac{x}{x^2+3}\,dx$. **2**

QUESTION 3. Use a SEPARATE Writing Booklet. **Marks**



The diagram shows points $A(1, 0)$, $B(4, 1)$ and $C(-1, 6)$ in the Cartesian plane. Angle ABC is θ.

Copy or trace this diagram into your Writing Booklet.

(a) Show that A and C lie on the line $3x + y = 3$. **2**

(b) Show that the gradient of AB is $\frac{1}{3}$. **1**

(c) Show that the length of AB is $\sqrt{10}$ units. **1**

(d) Show that AB and AC are perpendicular. **1**

(e) Find $\tan\theta$. **2**

(f) Find the equation of the circle with centre A that passes through B. **2**

(g) The point D is not shown on the diagram. The point D lies on the line $3x + y = 3$ between A and C, and $AD = AB$. Find the coordinates of D. **2**

(h) On your diagram, shade the region satisfying the inequality $3x + y \leq 3$. **1**

QUESTION 4. Use a SEPARATE Writing Booklet. **Marks**

(a) The following table lists the values of a function for three values of x. 4

x	1·0	2·0	3·0
$f(x)$	1·7	9·0	4·3

Use these three function values to estimate $\int_1^3 f(x)\,dx$ by:

(i) Simpson's rule

(ii) the trapezoidal rule.

(b) The third term of an arithmetic series is 32 and the sixth term is 17. 3

(i) Find the common difference.

(ii) Find the sum of the first ten terms.

(c) The first term of a geometric series is 16 and the fourth term is $\frac{1}{4}$. 2

(i) Find the common ratio.

(ii) Find the limiting sum of the series.

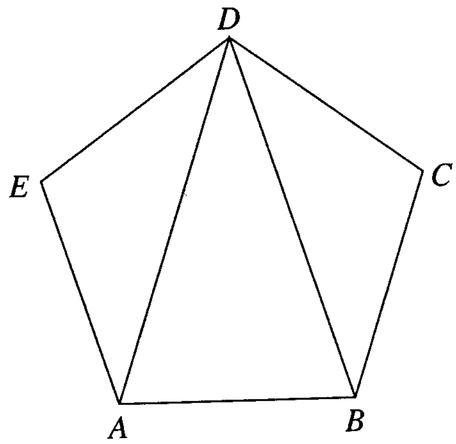
(d) The equation of a parabola is $x^2 = 8(y+3)$. 3

(i) Find the coordinates of the vertex of the parabola.

(ii) Find the equation of the directrix of the parabola.

QUESTION 5. Use a SEPARATE Writing Booklet. **Marks**

(a) **8**



The diagram shows a regular pentagon $ABCDE$. Each of the sides AB, BC, CD, DE and EA is of length x metres. Each of the angles $\angle ABC$, $\angle BCD$, $\angle CDE$, $\angle DEA$ and $\angle EAB$ is $108°$. Two diagonals, AD and BD, have been drawn.

Copy or trace the diagram into your Writing Booklet.

(i) State why triangle BCD is isosceles, and hence find $\angle CBD$.

(ii) Show that triangles BCD and DEA are congruent.

(iii) Find the size of $\angle ADB$.

(iv) Find an expression for the area of the pentagon in terms of x and trigonometric ratios.

(b) The population P of a city is growing at a rate that is proportional to the current population. The population at time t years is given by 4

$$P = Ae^{kt},$$

where A and k are constants.

The population at time $t = 0$ was 1 000 000 and at time $t = 2$ was 1 072 500.

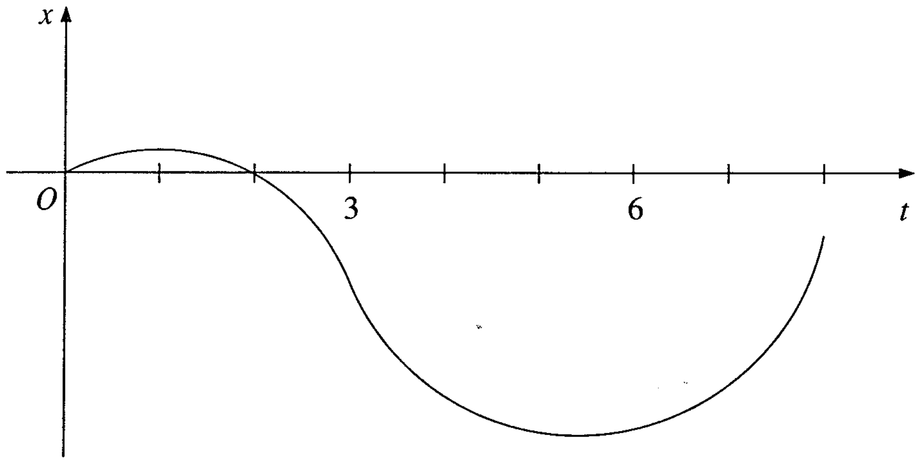
(i) Find the value of A.

(ii) Find the value of k.

(iii) At what time will the population reach 2 000 000?

QUESTION **6**. Use a SEPARATE Writing Booklet. **Marks**

(a) A particle P moves along a straight line for 8 seconds, starting at the fixed point S at time $t = 0$. At time t seconds, P is $x(t)$ metres to the right of S. The graph of $x(t)$ is shown in the diagram. **5**



(i) At approximately what times is the velocity of the particle equal to 0?

(ii) At approximately what time is the acceleration of the particle equal to 0?

(iii) At approximately what time is the distance from S greatest?

(iv) At approximately what time is the particle moving with the greatest velocity?

(b) The function $f(x) = xe^{-2x} + 1$ has first derivative $f'(x) = e^{-2x} - 2xe^{-2x}$ and second derivative $f''(x) = 4xe^{-2x} - 4e^{-2x}$. **7**

(i) Find the value of x for which $y = f(x)$ has a stationary point.

(ii) Find the values of x for which $f(x)$ is increasing.

(iii) Find the value of x for which $y = f(x)$ has a point of inflection and determine where the graph of $y = f(x)$ is concave upwards.

(iv) Sketch the curve $y = f(x)$ for $-\frac{1}{2} \leq x \leq 4$.

(v) Describe the behaviour of the graph for very large positive values of x.

QUESTION 7. Use a SEPARATE Writing Booklet. **Marks**

(a) (i) Write down the discriminant of $3x^2 + 2x + k$. **2**

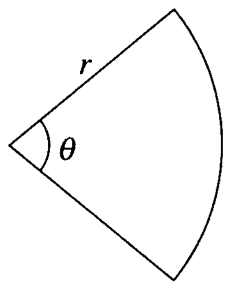
(ii) For what values of k does $3x^2 + 2x + k = 0$ have real roots?

(b) Consider the function $y = 1 + \sqrt{3}\sin x + \cos x$. **5**

(i) Find the equation of the tangent to the graph of the function at $x = \dfrac{5\pi}{6}$.

(ii) Find the maximum and minimum values of $1 + \sqrt{3}\sin x + \cos x$ in the interval $0 \le x \le 2\pi$.

(c) **5**



The diagram shows a sector of a circle with radius r cm. The angle at the centre is θ radians and the perimeter of the sector is 8 cm.

(i) Find an expression for r in terms of θ.

(ii) Show that A, the area of the sector in cm^2, is given by

$$A = \frac{32\theta}{(\theta + 2)^2}.$$

(iii) If $0 \le \theta \le \dfrac{\pi}{2}$, find the maximum area and the value of θ for which this occurs.

QUESTION 8. Use a SEPARATE Writing Booklet. **Marks**

(a) Sand is tipped from a truck onto a pile. The rate, R kg/s, at which the sand is flowing is given by the expression $R = 100t - t^3$, for $0 \le t \le T$, where t is the time in seconds after the sand begins to flow. **7**

(i) Find the rate of flow at time $t = 8$.

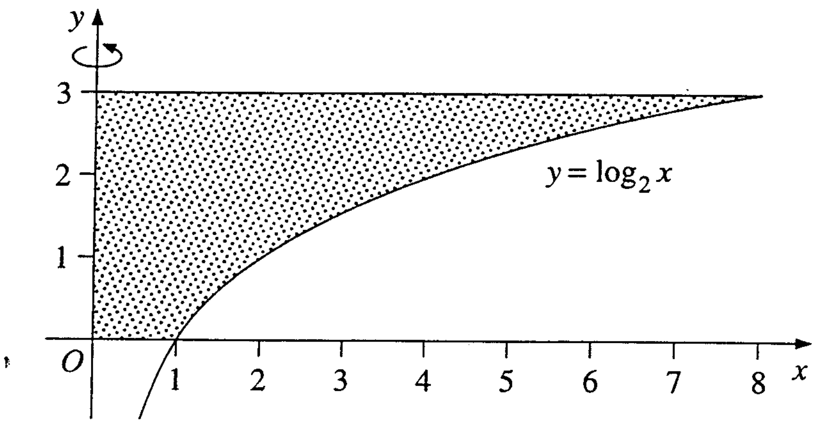
(ii) What is the largest value of T for which the expression for R is physically reasonable?

(iii) Find the maximum rate of flow of sand.

(iv) When the sand starts to flow, the pile already contains 300 kg of sand. Find an expression for the amount of sand in the pile at time t.

(v) Calculate the total weight of sand that was tipped from the truck in the first 8 seconds.

(b) **5**



The diagram shows the graph of $y = \log_2 x$ between $x = 1$ and $x = 8$. The shaded region, bounded by $y = \log_2 x$, the line $y = 3$, and the x and y axes, is rotated about the y axis to form a solid.

(i) Show that the volume of the solid is given by

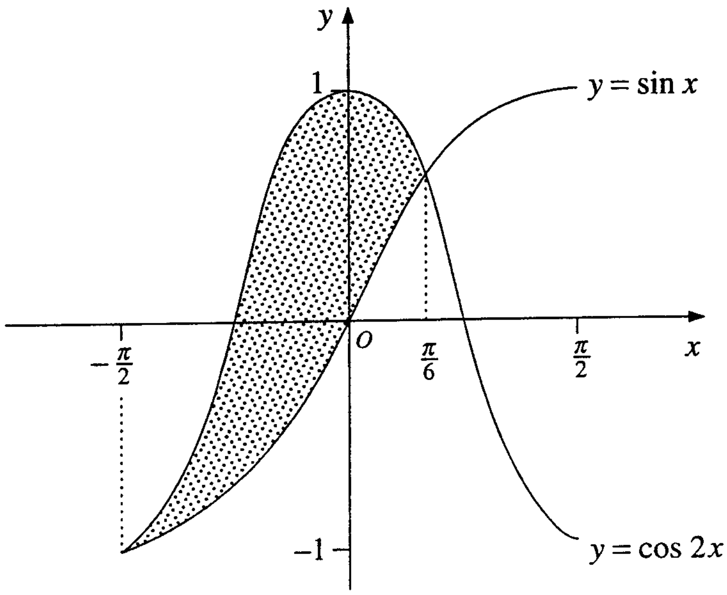
$$V = \pi \int_0^3 e^{y \ln 4}\, dy.$$

(ii) Hence find the volume of the solid.

QUESTION 9. Use a SEPARATE Writing Booklet. **Marks**

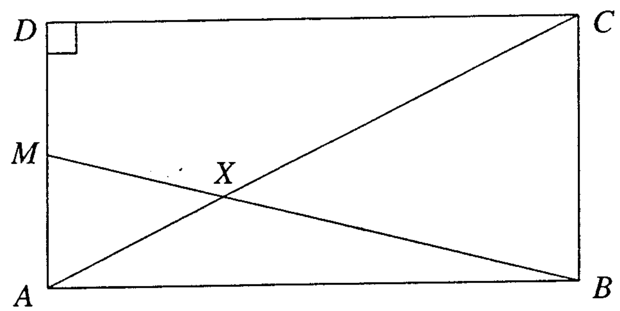
(a) Solve $\ln(7x - 12) = 2\ln x$. **2**

(b) **4**



The diagram shows the graphs of the functions $y = \cos 2x$ and $y = \sin x$ between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$. The two graphs intersect at $x = \frac{\pi}{6}$ and $x = -\frac{\pi}{2}$. Calculate the area of the shaded region.

(c) **6**



In the diagram, $ABCD$ is a rectangle and $AB = 2AD$. The point M is the midpoint of AD. The line BM meets AC at X.

(i) Show that the triangles AXM and BXC are similar.

(ii) Show that $3CX = 2AC$.

(iii) Show that $9(CX)^2 = 5(AB)^2$.

QUESTION 10. Use a SEPARATE Writing Booklet. **Marks**

(a) A game is played in which two coloured dice are thrown once. The six faces of the blue die are numbered 4, 6, 8, 9, 10 and 12. The six faces of the pink die are numbered 2, 3, 5, 7, 11 and 13. The player wins if the number on the pink die is larger than the number on the blue die. **5**

(i) By drawing up a table of possible outcomes, or otherwise, calculate the probability of the player winning a game.

(ii) Calculate the probability that the player wins at least once in two successive games.

(b) A fish farmer began business on 1 January 1998 with a stock of 100 000 fish. He had a contract to supply 15 400 fish at a price of \$10 per fish to a retailer in December each year. In the period between January and the harvest in December each year, the number of fish increases by 10%. **7**

(i) Find the number of fish just after the second harvest in December 1999.

(ii) Show that F_n, the number of fish just after the nth harvest, is given by

$$F_n = 154\,000 - 54\,000\,(1{\cdot}1)^n.$$

(iii) When will the farmer have sold all his fish, and what will his total income be?

(iv) Each December the retailer offers to buy the farmer's business by paying \$15 per fish for his entire stock. When should the farmer sell to maximise his total income?

End of paper

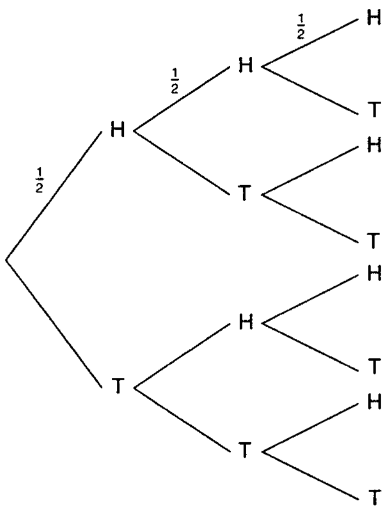
1998 Higher School Certificate Worked Answers

QUESTION 1

(a) $\frac{3}{11} = 0.2727\ldots$ (by calc.)
$= 0.\dot{2}\dot{7}$.

(b) $|-5| - |8| = 5 - 8$
$= -3$.

(c)



$$P(\text{HHH}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}.$$

(d) $\int (x^2 + 7)\,dx = \frac{x^3}{3} + 7x + c$.

(e)
$$\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{2\pi}{3}\right) = \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} = \frac{\sqrt{2} + \sqrt{3}}{2}.$$

(f)
$$\frac{1}{3 - \sqrt{2}} + \frac{1}{3 + \sqrt{2}}$$
$$= \frac{1}{3 - \sqrt{2}} \times \frac{3 + \sqrt{2}}{3 + \sqrt{2}} + \frac{1}{3 + \sqrt{2}} \times \frac{3 - \sqrt{2}}{3 - \sqrt{2}}$$
$$= \frac{3 + \sqrt{2}}{9 - 2} + \frac{3 - \sqrt{2}}{9 - 2}$$
$$= \frac{3 + \sqrt{2}}{7} + \frac{3 - \sqrt{2}}{7}$$
$$= \frac{3 + \sqrt{2} + 3 - \sqrt{2}}{7}$$
$$= \frac{6}{7}.$$

(g) old price + 37.5% of old price = new price

$$x + \frac{37.5}{100}x = \$3.08$$
$$x + 0.375x = \$3.08$$
$$1.375x = \$3.08$$
$$x = \frac{\$3.08}{1.375}$$
$$= \$2.24. \quad \text{(by calc.)}$$

QUESTION 2

(a) (i)
$$\frac{d}{dx}[(3x^2 + 4)^5]$$
$$= 5(3x^2 + 4)^4 . \frac{d}{dx}(3x^2 + 4)$$
$$= 5(3x^2 + 4)^4(6x)$$
$$= 30x(3x^2 + 4)^4.$$

(ii)
$$\frac{d}{dx}[x \sin(x + 1)]$$
$$= x . \frac{d}{dx}\sin(x + 1) + \sin(x + 1) . \frac{d}{dx}(x)$$
$$= x\cos(x + 1) . 1 + \sin(x + 1) . 1$$
$$= x\cos(x + 1) + \sin(x + 1).$$

(iii)
$$\frac{d}{dx}\left(\frac{\tan x}{x}\right) = \frac{x . \frac{d}{dx}(\tan x) - \tan x . \frac{d}{dx}(x)}{x^2}$$
$$= \frac{x\sec^2 x - \tan x . 1}{x^2}$$
$$= \frac{x\sec^2 x - \tan x}{x^2}.$$

(b) (i)
$$\int_1^2 \frac{1}{x^2}\,dx = \int_1^2 x^{-2}\,dx$$
$$= \left[\frac{x^{-1}}{-1}\right]_1^2$$
$$= \left[-\frac{1}{x}\right]_1^2$$
$$= \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right)$$
$$= -\frac{1}{2} + 1$$
$$= \frac{1}{2}.$$

(ii) $\int_0^3 e^{4x}\,dx = \left[\frac{e^{4x}}{4}\right]_0^3$

$= \left(\frac{e^{12}}{4}\right) - \left(\frac{e^0}{4}\right)$

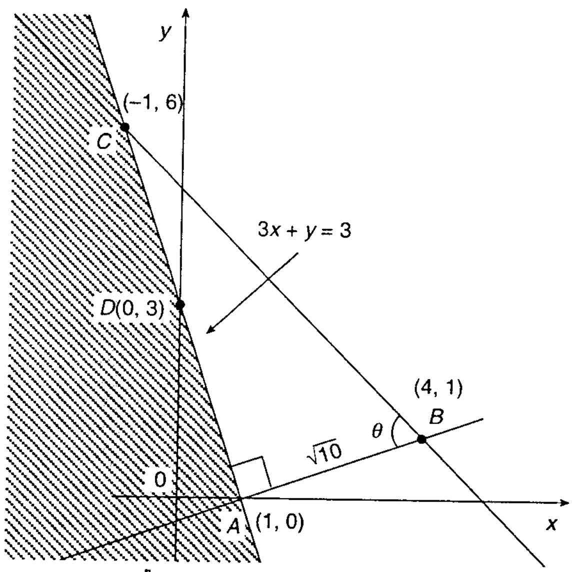
$= \frac{e^{12}}{4} - \frac{1}{4}$

$= \frac{e^{12}-1}{4}.$

(c) $\int \frac{x}{x^2+3}\,dx = \frac{1}{2}\int \frac{2x}{x^2+3}\,dx$

$= \frac{1}{2}\ln(x^2+3) + c.$

QUESTION 3



(a) Substituting (1, 0) and (−1, 6) into equation:

RHS = 3

LHS = 3(1) + 0

= 3

= RHS

∴ A lies on line

and RHS = 3

LHS = 3(−1) + 6

= −3 + 6

= 3

= RHS

∴ C lies on line.

(b) Gradient of AB: $m_1 = \frac{1-0}{4-1}$

$= \frac{1}{3}.$

(c) $AB = \sqrt{(4-1)^2 + (1-0)^2}$

$= \sqrt{9+1}$

$= \sqrt{10}$ units.

(d) Gradient of AC: $m_2 = \frac{6-0}{-1-1}$

$= \frac{6}{-2}$

$= -3$

Gradient of AB: $m_1 = \frac{1}{3}$ (from (b))

Now $m_1 \times m_2 = \frac{1}{3} \times -3$

$= -1$

∴ $AB \perp AC$.

(e) $AC = \sqrt{(-1-1)^2 + (6-0)^2}$

$= \sqrt{4+36}$

$= \sqrt{40}$

$= 2\sqrt{10}.$

Now $\tan\theta = \frac{AC}{AB}$

$= \frac{2\sqrt{10}}{\sqrt{10}}$

$= 2.$

(f) Equation of circle with centre (1, 0) and radius $AB = \sqrt{10}$ units is given by:

$(x-1)^2 + (y-0)^2 = (\sqrt{10})^2$

$x^2 - 2x + 1 + y^2 = 10$

$x^2 + y^2 - 2x - 9 = 0.$

(g) Now length of $AB = \sqrt{10}$ (from (c))

∴ length $AD = \sqrt{10}$ (since $AD = AB$)

and length $AC = 2\sqrt{10}$ (from (e))

∴ $DC = \sqrt{10}$

∴ $AD = DC$

∴ D is the mid point of AC.

∴ Coordinates of D are

$\left(\frac{-1+1}{2}, \frac{0+6}{2}\right)$

i.e. $\left(\frac{0}{2}, \frac{6}{2}\right)$

i.e. (0, 3)

(h) See diagram.

QUESTION 4

(a)

x	1.0	2.0	3.0
$f(x)$	1.7	9.0	4.3
	y_0	y_1	y_2

(i) $\int_1^3 f(x)\,dx \doteqdot \frac{h}{3}[y_0 + y_2 + 4y_1]$

$= \frac{1}{3}[1.7 + 4.3 + 4(9.0)]$

$= \frac{1}{3}[42]$

$= 14.$

(ii) $\int_1^3 f(x)\,dx \doteqdot \frac{h}{2}[y_0 + y_2 + 2(y_1)]$

$= \frac{1}{2}[1.7 + 4.3 + 2(9.0)]$

$= \frac{1}{2}[24]$

$= 12.$

(b) (i) $u_n = a + (n-1)d$

$u_3 = a + 2d = 32 \quad \ldots (1)$

$u_6 = a + 5d = 17 \quad \ldots (2)$

(1) – (2): $-3d = 15$

$d = -5$

(Substituting $d = -5$ into (1):

$a + 2(-5) = 32$

$a - 10 = 32$

$\therefore a = 42$.)

(ii) $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{10} = \frac{10}{2}[2(42) + 9(-5)]$

$= 5[84 - 45]$

$= 5[39]$

$= 195.$

(c) (i) $U_n = ar^{n-1}$

$U_4 = ar^3$

$\frac{1}{4} = 16r^3$

$r^3 = \frac{1}{64}$

$\therefore r = \frac{1}{4}.$

(ii) $S = \frac{a}{1-r}$

$= \frac{16}{1 - \frac{1}{4}}$

$= \frac{16}{\frac{3}{4}}$

$= \frac{64}{3}.$

(d) (i) $x^2 = 8(y + 3)$

This parabola is concave up and may be expressed in the form

$(x - 0)^2 = 4.2(y + 3)$

$\therefore$ Vertex = (0, –3) and focal length = 2

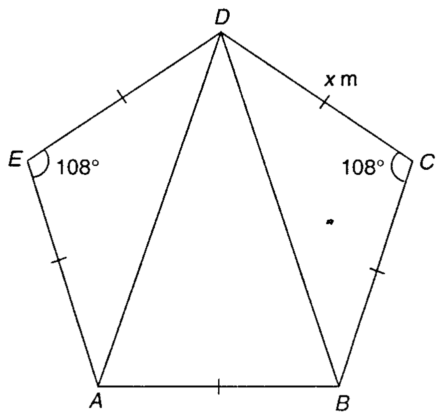
(ii) Equation of directrix is

i.e. $y = -3 - 2$

i.e. $y = -5$

QUESTION 5

(a) (i)



$CD = BC$ (equal sides of regular pentagon)

$\therefore \triangle BCD$ is isosceles

Now $\angle BCD = 108°$ (given)

and $\angle CDB = \angle CBD$

(equal angles of isosceles $\triangle$)

$\therefore \angle CDB + \angle CBD = 180° - 108°$

(angle sum of triangle)

$= 72°$

$\therefore \angle CBD = 36°$

(and $\angle CDB = 36°$)

(ii) In the $\triangle$'s BCD and DEA,

$CD = DE$ (given)

$BC = EA$ (given)

$\angle BCD = \angle DEA$ (given)

$\therefore \triangle BCD \equiv \triangle DEA$ (SAS).

(iii) $\angle CDE = \angle EDA + \angle ADB + \angle CDB$

Now $\angle EDA = 36°$

(corr. angle of congruent triangle)

and $\angle CDE = 108°$ (given)

$\therefore 108° = 36° + \angle ADB + 36°$

$108° = 72° + \angle ADB$

$\therefore \angle ADB = 36°$

(iv) Area $ABCDE$

$= \text{Area } \triangle BCD + \text{Area } \triangle AED + \text{Area } \triangle BDA$

Now

$$\text{Area } \triangle BCD = \frac{1}{2} \times DC \times BC \times \sin \angle DCB$$

$$= \frac{1}{2}(x)(x)(\sin 108°)$$

$$= \frac{1}{2}x^2 \sin 108°$$

and

$$\text{Area } \triangle AED = \text{Area } \triangle BCD \quad \text{(congruent triangles)}$$

$$= \frac{1}{2}x^2 \sin 108°.$$

and

$$\text{Area } \triangle BDA = \frac{1}{2} \times AD \times BD \times \sin \angle ADB$$

$$BD^2 = CD^2 + BC^2 - 2(CD)(BC) \cos 108°$$

$$= x^2 + x^2 - 2(x)(x) \cos 108°$$

$$= 2x^2 - 2x^2 \cos 108°$$

$$\therefore \text{Area } \triangle BDA = \frac{1}{2} \times BD^2 \times \sin 36° \quad \text{(since } AD = BD\text{)}$$

$$= \frac{1}{2}(2x^2 - 2x^2 \cos 108°) \sin 36°$$

$$= (x^2 - x^2 \cos 108°) \sin 36°$$

$\therefore$ Area $ABCDE$

$$= \frac{1}{2}x^2 \sin 108° + \frac{1}{2}x^2 \sin 108° + (x^2 - x^2 \cos 108°) \sin 36°$$

$$= x^2 \sin 108° + (x^2 - x^2 \cos 108°) \sin 36°$$

$$= x^2 \sin 108° + x^2 \sin 36° - x^2 \cos 108° \sin 36°$$

$$= x^2 (\sin 108° + \sin 36° - \cos 108° \sin 36°)$$

(Another method yields an equivalent answer of $x^2\left(\sin 108° + \frac{1}{4} \tan 72°\right)$

(b) (i) $P = Ae^{kt}$

At $t = 0$, $P = 1\,000\,000$,

$$\therefore 1\,000\,000 = Ae^0$$

$$1\,000\,000 = A(1)$$

$$\therefore A = 1\,000\,000$$

(ii) $P = 1\,000\,000e^{kt}$

At $t = 2$, $P = 1\,072\,500$

$$\therefore 1\,072\,500 = 1\,000\,000e^{2k}$$

$$\frac{1\,072\,500}{1\,000\,000} = e^{2k}$$

$$\ln \frac{1\,072\,500}{1\,000\,000} = \ln e^{2k}$$

$$\ln 1.072\,500 = 2k \ln e$$

$$\ln 1.072\,500 = 2k$$

$$k = \frac{\ln 1.072\,500}{2}$$

$$= 0.03499\ldots \quad \text{(by calc.)}$$

$$= 0.035 \text{ to 3 decimal places}$$

(iii) $2\,000\,000 = 1\,000\,000e^{kt}$ (where $k = 0.03499\ldots$)

$$\frac{2\,000\,000}{1\,000\,000} = e^{kt}$$

$$2 = e^{kt}$$

$$\ln 2 = \ln e^{kt}$$

$$\ln 2 = kt \ln e$$

$$= kt$$

$$\therefore t = \frac{\ln 2}{k}$$

$$= \frac{\ln 2}{(0.03499\ldots)} \quad \text{(using calc. memory for } k\text{)}$$

$$= 19.806\ldots \quad \text{(by calc.)}$$

$$\doteqdot 19.8 \text{ years}$$

QUESTION 6

(a) (i) 1 and $5\frac{1}{2}$ seconds.

(ii) 3 seconds

(iii) $5\frac{1}{2}$ seconds

(iv) 8 seconds

(b) (i) Stationary points occur when

$$f'(x) = 0$$

$$\therefore e^{-2x} - 2xe^{-2x} = 0$$

$$e^{-2x}(1 - 2x) = 0$$

$$\therefore 1 - 2x = 0 \text{ (since } e^{-2x} > 0)$$

$$2x = 1$$

$$x = \tfrac{1}{2}.$$

(ii) $f(x)$ is increasing when

$$f'(x) > 0$$

$$\therefore e^{-2x}(1 - 2x) > 0$$

Now $e^{-2x} > 0$ for all x

but $1 - 2x > 0$

when $1 > 2x$

$$\therefore 2x < 1$$

$$\therefore x < \tfrac{1}{2}.$$

$\therefore f(x)$ is increasing when $x < \frac{1}{2}$.

(iii) Possible points of inflection occur when

$$f''(x) = 0$$

$$4xe^{-2x} - 4e^{-2x} = 0$$

$$4e^{-2x}(x - 1) = 0$$

$$e^{-2x} > 0 \quad \therefore x - 1 = 0$$

$$x = 1$$

Checking for change of concavity:

x	$f''(x)$
0.9	$(+)(-) < 0$
1.0	0
1.1	$(+)(+) > 0$

$\therefore$ There is a change of concavity and a point of inflection at $x = 1$.

The graph is concave up
when $f''(x) > 0$
$4e^{-2x}(x - 1) > 0$
Now $e^{-2x} > 0$ and $x - 1 > 0$
when $x > 1$

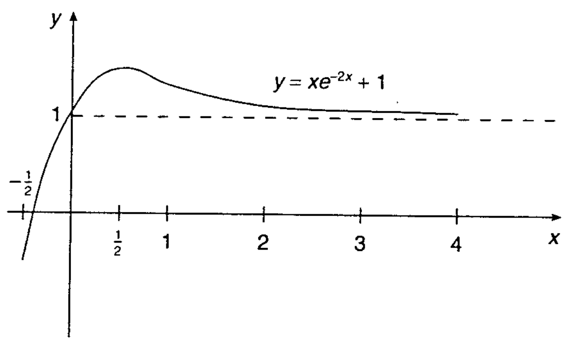
$\therefore$ Graph is concave upwards when $x > 1$.

(iv) Now $f(x) = xe^{-2x} + 1$

$f\left(-\frac{1}{2}\right) = -0.4$

$f(0) = 1.0$

$f(4) = 1.0013$



(v) $f(x) = xe^{-2x} + 1$

$= \dfrac{x}{e^{2x}} + 1$

As x gets very large, e^{2x} gets infinitesimally larger and $\dfrac{x}{e^{2x}}$ becomes very small and approaches zero.

$\therefore f(x)$ will approach 1.

QUESTION 7

(a) (i) $\Delta = b^2 - 4ac$

$= (2)^2 - 4(3)(k)$

$= 4 - 12k$

(ii) Real roots occur when

$\Delta \geqslant 0$

$\therefore 4 - 12k \geqslant 0$

$4 \geqslant 12k$

$k \leqslant \dfrac{1}{3}$.

(b) (i) $y = 1 + \sqrt{3}\sin x + \cos x$

$\dfrac{dy}{dx} = \sqrt{3}\cos x - \sin x$

At $x = \frac{5\pi}{6}$, $\dfrac{dy}{dx} = \sqrt{3}\cos\left(\frac{5\pi}{6}\right) - \sin\left(\frac{5\pi}{6}\right)$

$= \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)$

$= -\frac{3}{2} - \frac{1}{2}$

$= -\frac{4}{2}$

$= -2$

and $y = 1 + \sqrt{3}\sin\left(\frac{5\pi}{6}\right) + \cos\left(\frac{5\pi}{6}\right)$

$= 1 + \sqrt{3}\left(\frac{1}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)$

$= 1 + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$

$= 1$

Equation of tangent through $\left(\frac{5\pi}{6}, 1\right)$ with gradient $= -2$ is:

$y - 1 = -2\left(x - \frac{5\pi}{6}\right)$

$y - 1 = -2x + \frac{5\pi}{3}$

$y = -2x + \frac{5\pi}{3} + 1$

(ii) $y = 1 + \sqrt{3}\sin x + \cos x$

$\dfrac{dy}{dx} = \sqrt{3}\cos x - \sin x$

$\dfrac{d^2y}{dx^2} = -\sqrt{3}\sin x - \cos x$

Possible maximum and minimum values occur when

$\dfrac{dy}{dx} = 0$

$\sqrt{3}\cos x - \sin x = 0$

$\sqrt{3}\cos x = \sin x$

$\dfrac{\sqrt{3}\cos x}{\cos x} = \dfrac{\sin x}{\cos x}$

$\sqrt{3} = \tan x$

$\therefore x = \dfrac{\pi}{3}$ and $\dfrac{4\pi}{3}$, $0 \leqslant x \leqslant 2\pi$

At $x = \frac{\pi}{3}$, $y = 1 + \sqrt{3}\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)$

$= 1 + \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}$

$= 1 + \frac{3}{2} + \frac{1}{2}$

$= 3$

and $\frac{d^2y}{dx^2} = -\sqrt{3}\sin\frac{\pi}{3} - \cos\frac{\pi}{3}$

$= -\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}$

$= -\frac{3}{2} - \frac{1}{2}$

$= -\frac{4}{2}$

$= -2$

$\therefore$ A maximum turning point at $\left(\frac{\pi}{3}, 3\right)$

At $x = \frac{4\pi}{3}$, $y = 1 + \sqrt{3}\sin\left(\frac{4\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right)$

$= 1 + \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2}\right)$

$= 1 - \frac{3}{2} - \frac{1}{2}$

$= 1 - \frac{4}{2}$

$= -1$

and $\frac{d^2y}{dx^2} = -\sqrt{3}\sin\frac{4\pi}{3} - \cos\frac{4\pi}{3}$

$= -\sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2}\right)$

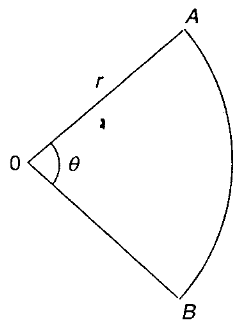
$= \frac{3}{2} + \frac{1}{2}$

$= 2$

$\therefore$ A minimum turning point at $\left(\frac{4\pi}{3}, -1\right)$

$\therefore$ Maximum and minimum values are 3 and −1 respectively.

(c) (i)



Perimeter = $OA + OB + \text{arc } AB$

$8 = r + r + r\theta$

$8 = 2r + r\theta$

$8 = r(2 + \theta)$

$r = \frac{8}{2 + \theta}$

(ii) Area of sector

$= \frac{1}{2}r^2\theta$

$= \frac{1}{2}\left(\frac{8}{2 + \theta}\right)^2 . \theta$

$= \frac{1}{2} . \frac{64}{(2 + \theta)^2} . \theta$

$= \frac{32\theta}{(2 + \theta)^2}$

(iii) $A = \frac{32\theta}{(2 + \theta)^2}$ for $0 < \theta < \frac{\pi}{2}$

$$\frac{dA}{d\theta} = \frac{(2 + \theta)^2 . \frac{d}{d\theta}(32\theta) - 32\theta . \frac{d}{d\theta}(2 + \theta)^2}{[(2 + \theta)^2]^2}$$

$$= \frac{(2 + \theta)^2 . 32 - 32\theta . 2(2 + \theta)}{(2 + \theta)^4}$$

$$= \frac{32(2 + \theta)^2 - 64\theta(2 + \theta)}{(2 + \theta)^4}$$

$$= \frac{32(2 + \theta)[(2 + \theta) - 2\theta]}{(2 + \theta)^4}$$

$$= \frac{32(2 + \theta)(2 - \theta)}{(2 + \theta)^4}$$

$$= \frac{32(2 - \theta)}{(2 + \theta)^3}$$

Stationery points occur when

$\frac{dA}{d\theta} = 0$

$\frac{32(2 - \theta)}{(2 + \theta)^3} = 0$

$\therefore \theta = 2$

Since $0 \leqslant \theta \leqslant \frac{\pi}{2}$, this solution lies outside of the range of possible values of θ.

$\therefore$ the maximum will lie on one of the endpoints of the interval.

Consider points on the graph of

$A = \frac{32\theta}{(\theta + 2)^2}$ for $0 < \theta < \frac{\pi}{2}$:

At $\theta = 0$, $A = 0$

At $\theta = \frac{\pi}{2}$, $A = \frac{32\left(\frac{\pi}{2}\right)}{\left(\frac{\pi}{2} + 2\right)^2}$

$= \frac{16\pi}{\left(\frac{\pi}{2} + 2\right)^2}$

$\doteqdot 3.94$

There are no turning points in this interval and the curve must be rising in this interval, therefore the maximum area will occur when $\theta = \frac{\pi}{2}$ and will be approximately

$\frac{16\pi}{\left(\frac{\pi}{2} + 2\right)^2} \doteqdot 3.94$ cm.

QUESTION 8

(a) (i) $R = 100t - t^3$

At $t = 8$, $R = 100(8) - (8)^3$

$= 800 - 512$

$= 288$

$\therefore$ Flow rate is 288 kg/s.

(ii) The expression is reasonable until $R = 0$

$\therefore\ 0 = 100t - t^3$

$= t(100 - t^2)$

$\therefore\ t = 0, 10, \text{ or } -10$ (not possible)

$\therefore$ The largest value of T is 10.

(iii) $R = 100t - t^3$

$\frac{dR}{dt} = 100 - 3t^2 \quad \frac{d^2R}{dt^2} = -6t$

Stationary points occur when $\frac{dR}{dt} = 0$.

$$100 - 3t^2 = 0$$
$$3t^2 = 100$$
$$t^2 = \frac{100}{3}$$
$$t = \pm\sqrt{\frac{100}{3}}$$
$$= \frac{10}{\sqrt{3}} \quad (\text{since } t > 0)$$

When $t = \frac{10}{\sqrt{3}}$, $\frac{d^2R}{dt^2} = -6\left(\frac{10}{\sqrt{3}}\right)$

$= -\frac{60}{\sqrt{3}} < 0$

$\therefore$ A maximum value occurs at $t = \frac{10}{\sqrt{3}}$.

When $t = \frac{10}{\sqrt{3}}$, $R = 100\left(\frac{10}{\sqrt{3}}\right) - \left(\frac{10}{\sqrt{3}}\right)^3$

$$= \frac{1000}{\sqrt{3}} - \frac{1000}{3\sqrt{3}}$$
$$= \frac{3000 - 1000}{3\sqrt{3}}$$
$$= \frac{2000}{3\sqrt{3}}$$
$$= \frac{2000\sqrt{3}}{9}$$
$$\doteqdot 384.9 \quad (\text{by calc.})$$

$\therefore$ The maximum flow rate is approximately 384.9 kg/s.

(iv) $R = 100t - t^3$

If M is the amount in the pile at time t:

$\frac{dM}{dt} = 100t - t^3$

$\therefore\ M = 50t^2 - \frac{t^4}{4} + C$

When $t = 0$, $M = 300$ $\therefore\ C = 300$

$\therefore\ M = 50t - \frac{t^4}{4} + 300.$

(v) $\int_0^8 (100t - t^3)\, dt$

$$= \left[\frac{100t^2}{2} - \frac{t^4}{4}\right]_0^8$$
$$= \left[50t^2 - \frac{t^4}{4}\right]_0^8$$
$$= \left(50(64) - \frac{4096}{4}\right) - 0$$
$$= 3200 - 1024$$
$$= 2176$$

$\therefore$ Total weight of sand tipped from truck in first 8 seconds was 2176 kg.

(b) (i) $y = \log_2 x$

$\therefore\ y = \frac{\log_e x}{\log_e 2}$ (change of base)

$$\log_e x = y \log_e 2$$
$$x = e^{y \log_e 2}$$
$$= e^{y \ln 2}$$

$$V = \pi \int_0^3 x^2\, dy$$
$$= \pi \int_0^3 (e^{y \ln 2})^2\, dy$$
$$= \pi \int_0^3 e^{2y \ln 2}\, dy$$
$$= \pi \int_0^3 e^{y \ln 2^2}\, dy$$
$$= \pi \int_0^3 e^{y \ln 4}\, dy$$

(ii) $V = \pi \int_0^3 e^{y \ln 4}\, dy$

$$= \pi \left[\frac{e^{y \ln 4}}{\ln 4}\right]_0^3$$
$$= \pi \left(\frac{e^{3 \ln 4}}{\ln 4} - \frac{e^0}{\ln 4}\right)$$
$$= \pi \left(\frac{e^{\ln 64}}{\ln 4} - \frac{1}{\ln 4}\right)$$
$$= \pi \left(\frac{64}{\ln 4} - \frac{1}{\ln 4}\right)$$
$$= \pi \times \frac{63}{\ln 4}$$
$$\doteqdot 142.8$$

$\therefore$ Volume of the solid is approximately 142.8 units^3.

QUESTION 9

(a) $\ln(7x - 12) = 2\ln x$

$\ln(7x - 12) = \ln x^2$

$\therefore 7x - 12 = x^2$

$x^2 - 7x + 12 = 0$

$(x - 3)(x - 4) = 0$

$\therefore x = 3$ or 4.

(b) The area of the shaded region is given by the integral of the upper function less the lower function as the curves are continuous and do not intersect in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{6}\right]$

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (\cos 2x - \sin x)\,dx$$

$$= \left[\frac{\sin 2x}{2} + \cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$$

$$= \left[\left(\frac{\sin\frac{\pi}{3}}{2} + \cos\frac{\pi}{6}\right) - \left(\frac{\sin(-\pi)}{2} + \cos\left(-\frac{\pi}{2}\right)\right)\right]$$

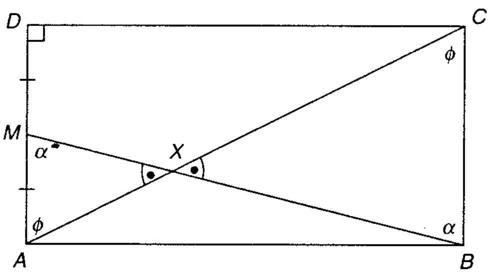
$$= \left(\frac{\sqrt{3}/2}{2} + \frac{\sqrt{3}}{2}\right) - \left(\frac{0}{2} + 0\right)$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3} + 2\sqrt{3}}{4}$$

$$= \frac{3\sqrt{3}}{4} \text{ units}^2$$

(c) (i)



In the Δ's AXM and BXC,

$\angle MXA = \angle CXB$ (vertically opposite)

$\angle MAX = \angle BCX$ (alternate, $BC \parallel AD$)

$\therefore \Delta AXM \,|||\, \Delta BXC$

(ii) (To show the relationship between CX and AC, it is necessary to find the relationship between CX and AX.)

Now $\frac{AX}{CX} = \frac{XM}{XB} = \frac{MA}{BC}$

(prop. sides of similar Δ's)

$\therefore \frac{AX}{CX} = \frac{MA}{BC}$

$\therefore \frac{AX}{CX} = \frac{MA}{DA}$ (since $BC = DA$)

$\therefore \frac{AX}{CX} = \frac{1}{2}$

i.e. $\frac{CX}{AX} = 2$

$\therefore \frac{CX}{AC} = \frac{2}{3}$

i.e. $3CX = 2AC$

(iii) $AC^2 = AB^2 + BC^2$

(Pythagoras Theorem)

and $AC = \frac{3CX}{2}$ (from (ii))

and $BC = \frac{AB}{2} = \frac{AD}{2}$ (given)

$\therefore \left(\frac{3CX}{2}\right)^2 = AB^2 + \left(\frac{AB}{2}\right)^2$

$\frac{9(CX)^2}{4} = AB^2 + \frac{AB^2}{4}$

$\therefore 9(CX)^2 = 4(AB)^2 + AB^2$

$\therefore 9(CX)^2 = 5(AB)^2$

QUESTION 10

(a) (i)

PINK DIE

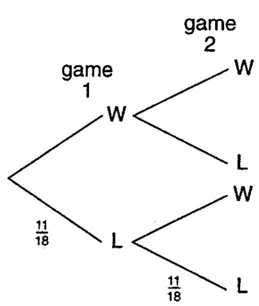
BLUE	2	3	5	7	11	13
4			✓	✓	✓	✓
6				✓	✓	✓
8					✓	✓
9					✓	✓
10					✓	✓
12						✓

$$P(\text{Win}) = \frac{14}{36}$$

$$= \frac{7}{18}$$

(ii) $P(\text{Loss}) = 1 - \frac{7}{18}$

$$= \frac{11}{18}$$



P(At least one win)
$= 1 - P(LL)$
$= 1 - \left(\frac{11}{18} \times \frac{11}{18}\right)$
$= 1 - \frac{121}{324}$
$= \frac{203}{324}$.

(b) (i) No. of fish after 1st harvest
$= 100\,000(1.1) - 15\,400$

No. of fish after 2nd harvest
$= [100\,000(1.1) - 15\,400]1.1 - 15\,400$
$= 100\,000(1.1)^2 - 15\,400(1.1) - 15\,400$
$= 88\,660$ fish

(ii) No. of fish after nth harvest, F_n
$= 100\,000(1.1)^n - 15\,400(1.1)^{n-1}$
$- 15\,400(1.1)^{n-2} \ldots - 15\,400(1.1) - 15\,400$
$= 100\,000(1.1)^n - [15\,400 + 15\,400(1.1)$
$+ \ldots + 15\,400(1.1)^{n-1}]$
$= 100\,000(1.1)^n - \left[\frac{a(r^n - 1)}{r - 1}\right]$
where $a = 15\,400$, $r = 1.1$ and $n = n$
$= 100\,000(1.1)^n - \left[\frac{15\,400[(1.1)^n - 1]}{1.1 - 1}\right]$
$= 100\,000(1.1)^n - \frac{15\,400[(1.1)^n - 1]}{0.1}$
$= 100\,000(1.1)^n - 154\,000[(1.1)^n - 1]$
$= 100\,000(1.1)^n - 154\,000(1.1)^n + 154\,000$
$= 154\,000 - 54\,000(1.1)^n$

(iii) When all fish are sold, $F_n = 0$
i.e. $154\,000 - 54\,000(1.1)^n = 0$
$54\,000(1.1)^n = 154\,000$
$(1.1)^n = \frac{154\,000}{54\,000}$
$(1.1)^n = 2.851\ldots$ (by calc.)
$\ln (1.1)^n = \ln 2.851\ldots$
$n \ln (1.1) = \ln 2.851\ldots$
$n = \frac{\ln 2.851\ldots}{\ln 1.1}$
$= 10.995\ldots$ (by calc.)
$\doteqdot 11$
$\therefore$ The farmer will have sold all his fish after the 11th harvest.

His total income will be
$11 \times 15\,400 \times \$10 = \$1\,694\,000$

(iv) If I is the total income received after n harvests:

$I = 15 \times F_n + 15\,400 \times 10 \times n$
$= 15[154\,000 - 54\,000(1.1)^n] + 154\,000n$
$= 2\,310\,000 - 810\,000(1.1)^n + 154\,000n$

By trial and error:

When $n = 6$, $I = \$1\,799\,036$.
When $n = 7$, $I = \$1\,809\,539$.
When $n = 8$, $I = \$1\,805\,693$.

$\therefore$ The farmer should sell after the 7th harvest.

(A more advanced approach:

$\frac{dI}{dn} = -810\,000 \ln (1.1).(1.1)^n + 154\,000$

$\left(\text{NB } \frac{d}{dn}(1.1)^n = \ln (1.1).(1.1)^n\right)$

Stationary points occur when
$\frac{dI}{dn} = 0$
$\therefore 0 = -810\,000 \ln (1.1).(1.1)^n + 154\,000$
$154\,000 = 810\,000 \ln (1.1).(1.1)^n$
$\frac{154\,000}{810\,000 \ln (1.1)} = (1.1)^n$
$1.9948 = (1.1)^n$
$\ln 1.9948 = \ln (1.1)^n$
$\ln 1.9948 = n \ln (1.1)$
$n = \frac{\ln 1.9948}{\ln 1.1}$
$\doteqdot 7.25$

Now when $n = 7$, $I = \$1\,809\,539$.
and when $n = 8$, $I = \$1\,805\,693$.

$\therefore$ The farmer should sell after the 7th harvest.