BOARD OF STUDIES
NEW SOUTH WALES

2005

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Total marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{1}{x^2+49}\,dx$. **1**

(b) Sketch the region in the plane defined by $y \le |2x+3|$. **2**

(c) State the domain and range of $y = \cos^{-1}\left(\frac{x}{4}\right)$. **2**

(d) Using the substitution $u = 2x^2+1$, or otherwise, find $\int x\left(2x^2+1\right)^{\frac{5}{4}}\,dx$. **3**

(e) The point $P(1, 4)$ divides the line segment joining $A(-1, 8)$ and $B(x, y)$ internally in the ratio $2:3$. Find the coordinates of the point B. **2**

(f) The acute angle between the lines $y = 3x+5$ and $y = mx+4$ is 45°. Find the two possible values of m. **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\frac{d}{dx}\left(2\sin^{-1}5x\right)$. **2**

(b) Use the binomial theorem to find the term independent of x in the expansion of $\left(2x - \frac{1}{x^2}\right)^{12}$. **3**

(c) (i) Differentiate $e^{3x}(\cos x - 3\sin x)$. **2**

(ii) Hence, or otherwise, find $\int e^{3x}\sin x\,dx$. **1**

(d) A salad, which is initially at a temperature of 25°C, is placed in a refrigerator that has a constant temperature of 3°C. The cooling rate of the salad is proportional to the difference between the temperature of the refrigerator and the temperature, T, of the salad. That is, T satisfies the equation

$$\frac{dT}{dt} = -k(T-3),$$

where t is the number of minutes after the salad is placed in the refrigerator.

(i) Show that $T = 3 + Ae^{-kt}$ satisfies this equation. **1**

(ii) The temperature of the salad is 11°C after 10 minutes. Find the temperature of the salad after 15 minutes. **3**

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that the function $g(x)=x^2-\log_e(x+1)$ has a zero between 0.7 and 0.9. **1**

(ii) Use the method of halving the interval to find an approximation to this zero of $g(x)$, correct to one decimal place. **2**

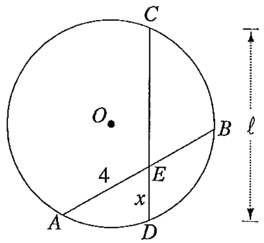
(b) (i) By expanding the left-hand side, show that **1**

$$\sin(5x+4x)+\sin(5x-4x)=2\sin 5x\cos 4x.$$

(ii) Hence find $\int \sin 5x \cos 4x\,dx$. **2**

(c) Use the definition of the derivative, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, to find $f'(x)$ when $f(x)=x^2+5x$. **2**

(d)



NOT TO SCALE

In the circle centred at O the chord AB has length 7. The point E lies on AB and AE has length 4. The chord CD passes through E.

Let the length of CD be ℓ and the length of DE be x.

(i) Show that $x^2-\ell x+12=0$. **2**

(ii) Find the length of the shortest chord that passes through E. **2**

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\int_0^{\frac{\pi}{4}} \cos x \sin^2 x\,dx$. **2**

(b) By making the substitution $t=\tan\frac{\theta}{2}$, or otherwise, show that **2**

$$\operatorname{cosec}\theta+\cot\theta=\cot\frac{\theta}{2}.$$

(c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2=4ay$. The equation of the normal to the parabola at P is $x+py=2ap+ap^3$ and the equation of the normal at Q is similarly given by $x+qy=2aq+aq^3$.

(i) Show that the normals at P and Q intersect at the point R whose coordinates are **2**

$$\left(-apq[p+q],\ a\left[p^2+pq+q^2+2\right]\right).$$

(ii) The equation of the chord PQ is $y=\frac{1}{2}(p+q)x-apq$. (Do NOT show this.) **1**

If the chord PQ passes through $(0,a)$, show that $pq=-1$.

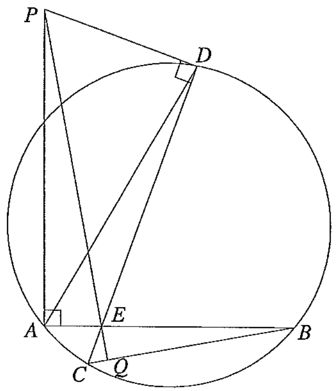
(iii) Find the equation of the locus of R if the chord PQ passes through $(0,a)$. **2**

(d) Use the principle of mathematical induction to show that $4^n-1-7n>0$ for all integers $n\geq 2$. **3**

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the exact value of the volume of the solid of revolution formed when the region bounded by the curve $y = \sin 2x$, the x-axis and the line $x = \frac{\pi}{8}$ is rotated about the x-axis. **3**

(b) Two chords of a circle, AB and CD, intersect at E. The perpendiculars to AB at A and CD at D intersect at P. The line PE meets BC at Q, as shown in the diagram.



(i) Explain why $DPAE$ is a cyclic quadrilateral. **1**

(ii) Prove that $\angle APE = \angle ABC$. **2**

(iii) Deduce that PQ is perpendicular to BC. **1**

(c) A particle moves in a straight line and its position at time t is given by

$$x = 5 + \sqrt{3}\sin 3t - \cos 3t.$$

(i) Express $\sqrt{3}\sin 3t - \cos 3t$ in the form $R\sin(3t - \alpha)$, where α is in radians. **2**

(ii) The particle is undergoing simple harmonic motion. Find the amplitude and the centre of the motion. **2**

(iii) When does the particle first reach its maximum speed after time $t = 0$? **1**

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) There are five matches on each weekend of a football season. Megan takes part in a competition in which she earns one point if she picks more than half of the winning teams for a weekend, and zero points otherwise. The probability that Megan correctly picks the team that wins any given match is $\frac{2}{3}$.

(i) Show that the probability that Megan earns one point for a given weekend is 0.7901, correct to four decimal places. **2**

(ii) Hence find the probability that Megan earns one point every week of the eighteen-week season. Give your answer correct to two decimal places. **1**

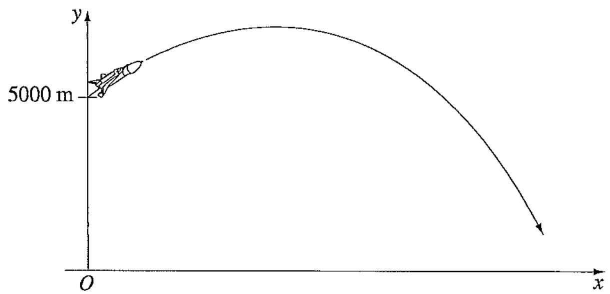
(iii) Find the probability that Megan earns at most 16 points during the eighteen-week season. Give your answer correct to two decimal places. **2**

Question 6 continues on page 9

Question 6 (continued)

Marks

(b) An experimental rocket is at a height of 5000 m, ascending with a velocity of $200\sqrt{2}$ m s^{-1} at an angle of 45° to the horizontal, when its engine stops.



After this time, the equations of motion of the rocket are:

$$x = 200t$$

$$y = -4.9t^2 + 200t + 5000,$$

where t is measured in seconds after the engine stops. (Do NOT show this.)

(i) What is the maximum height the rocket will reach, and when will it reach this height? **2**

(ii) The pilot can only operate the ejection seat when the rocket is descending at an angle between 45° and 60° to the horizontal. What are the earliest and latest times that the pilot can operate the ejection seat? **3**

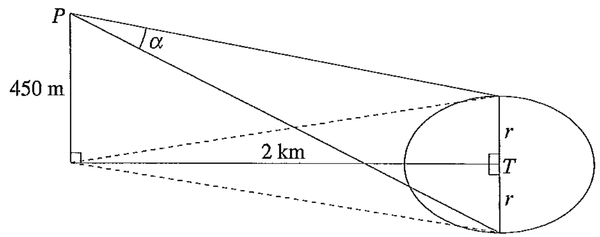
(iii) For the parachute to open safely, the pilot must eject when the speed of the rocket is no more than 350 m s^{-1}. What is the latest time at which the pilot can eject safely? **2**

End of Question 6

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) An oil tanker at T is leaking oil which forms a circular oil slick. An observer is measuring the oil slick from a position P, 450 metres above sea level and 2 kilometres horizontally from the centre of the oil slick.



(i) At a certain time the observer measures the angle, α, subtended by the diameter of the oil slick, to be 0.1 radians. What is the radius, r, at this time? **2**

(ii) At this time, $\dfrac{d\alpha}{dt} = 0.02$ radians per hour. Find the rate at which the radius of the oil slick is growing. **2**

(b) Let $f(x) = Ax^3 - Ax + 1$, where $A > 0$.

(i) Show that $f(x)$ has stationary points at $x = \pm\dfrac{\sqrt{3}}{3}$. **1**

(ii) Show that $f(x)$ has exactly one zero when $A < \dfrac{3\sqrt{3}}{2}$. **2**

(iii) By observing that $f(-1) = 1$, deduce that $f(x)$ does not have a zero in the interval $-1 \le x \le 1$ when $0 < A < \dfrac{3\sqrt{3}}{2}$. **1**

(iv) Let $g(\theta) = 2\cos\theta + \tan\theta$, where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. **3**

By calculating $g'(\theta)$ and applying the result in part (iii), or otherwise, show that $g(\theta)$ does not have any stationary points.

(v) Hence, or otherwise, deduce that $g(\theta)$ has an inverse function. **1**

End of paper

2005 HIGHER SCHOOL CERTIFICATE SOLUTIONS

MATHEMATICS EXTENSION 1

QUESTION 1

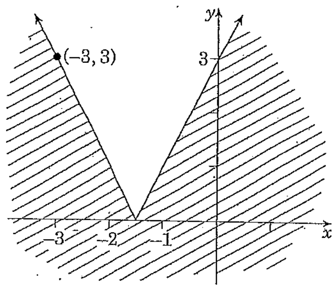
(a) $\displaystyle\int \frac{1}{x^2+49}\,dx = \int \frac{1}{7^2+x^2}\,dx$

$\displaystyle = \frac{1}{7}\tan^{-1}\frac{x}{7}+C$ (from standard integrals).

(b) $y \le |2x+3|$.

Now $y = |2x+3|$ has x-intercept: $x = -\dfrac{3}{2} = -1{\cdot}5.$

y-intercept: $y = 3.$



Test (0, 0) in $y \le |2x+3|$:

LHS $= y = 0.$ RHS $= |2x+3| = |2(0)+3| = 3.$

$\therefore$ LHS $\le$ RHS

$\therefore$ (0, 0) is in the region shaded.

(c) $y = \cos^{-1}\left(\dfrac{x}{4}\right)$. Domain: $-1 \le \dfrac{x}{4} \le 1$

$-4 \le x \le 4$

Range: $0 \le y \le \pi$

(d) Given $u = 2x^2+1.$

$\dfrac{du}{dx} = 4x$

$\therefore \quad du = 4x.dx.$

Now $\displaystyle\int x(2x^2+1)^{\frac{5}{4}}\,dx = \frac{1}{4}\int (2x^2+1)^{\frac{5}{4}}.4x\,dx$

$\displaystyle = \frac{1}{4}\int u^{\frac{5}{4}}.\,du$

$\displaystyle = \frac{1}{4}\left(\frac{u^{\frac{9}{4}}}{\frac{9}{4}}\right)+C$

$\displaystyle = \frac{1}{4}.\frac{4}{9}.u^{\frac{9}{4}}+C$

$\displaystyle = \frac{1}{9}(2x^2+1)^{\frac{9}{4}}+C.$

(e) *METHOD 1*

$A = (-1, 8) \quad B = (x, y) \quad P = (1, 4)$

$P = \left(\dfrac{mx_2+nx_1}{m+n}, \dfrac{my_2+ny_1}{m+n}\right)$

where $m:n = 2:3$

$A(-1, 8) = (x_1, y_1)$

$B(x, y) = (x_2, y_2).$

Equating x-coordinates of P:

$1 = \dfrac{2x+3(-1)}{2+3}$

$1 = \dfrac{2x-3}{5}$

$5 = 2x-3$

$8 = 2x$

$x = 4.$

Equating y-coordinates of P:

$4 = \dfrac{2y+3(8)}{2+3}$

$4 = \dfrac{2y+24}{5}$

$20 = 2y+24$

$-4 = 2y$

$y = -2.$

$\therefore$ B is $(4, -2)$.

METHOD 2

If P is the internal divisor $(2:3)$ of AB then B is the external divisor $(5:3)$ of AP.
(See diagram in Method 3 below.)

$B = \left(\dfrac{mx_2-nx_1}{m-n}, \dfrac{my_2-ny_1}{m-n}\right)$

where $m:n = 5:3$

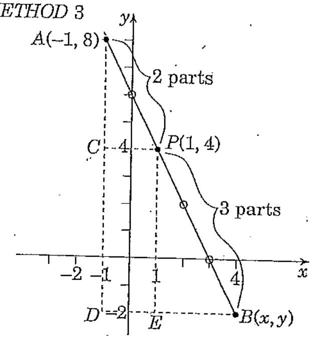
$A(-1, 8) = (x_1, y_1)$

$P(1, 4) = (x_2, y_2).$

$x = \dfrac{5\times1-3\times(-1)}{5-3} = \dfrac{8}{2} = 4.$ $\qquad y = \dfrac{5\times4-3\times8}{5-3} = \dfrac{-4}{2} = -2.$

$\therefore$ B is $(4, -2)$.

METHOD 3



$\dfrac{AP}{PB} = \dfrac{AC}{CD} = \dfrac{DE}{EB} = \dfrac{2}{3}$ (similar Δs)

From the diagram, $AC = 4$ and $DE = 2$

$\therefore \quad \dfrac{4}{CD} = \dfrac{2}{3} \Rightarrow CD = 6$

and $\dfrac{2}{EB} = \dfrac{2}{3} \Rightarrow EB = 3.$

$\therefore \quad x = 1+EB = 1+3 = 4.$ and $y = 4-CD = 4-6 = -2.$

$\therefore$ B is $(4, -2)$.

(f) *METHOD 1*

$y = 3x+5$ has $m_1 = 3$

$y = mx+4$ has $m_2 = m.$

If θ is the angle between the lines

$\tan\theta = \left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$ and $\theta = 45°.$

$\therefore \quad \tan 45° = \left|\dfrac{3-m}{1+3m}\right|$

$1 = \left|\dfrac{3-m}{1+3m}\right|.$

$\therefore \quad \dfrac{3-m}{1+3m} = 1$ or $\dfrac{3-m}{1+3m} = -1$

$3-m = 1+3m \qquad 3-m = -1-3m$

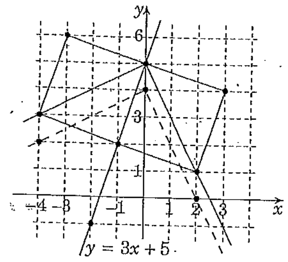
$2 = 4m \qquad 2m = -4$

$m = \dfrac{1}{2} \qquad m = -2$

$\therefore m = \dfrac{1}{2}$ or $-2.$

METHOD 2

The $45°$ angle can be constructed as the diagonal of a square. This gives the required gradients. The actual lines are not needed but can easily be found using parallel lines.



The diagonal on the left has gradient

$m = \dfrac{2}{4} = \dfrac{1}{2}.$

The diagonal on the right has gradient

$m = \dfrac{-4}{2} = -2.$

QUESTION 2

(a) $\dfrac{d}{dx}(2\sin^{-1}5x) = \dfrac{2}{\sqrt{1-(5x)^2}}\times 5 = \dfrac{10}{\sqrt{1-25x^2}}.$

(b) $\displaystyle\left(2x-\frac{1}{x^2}\right)^{12} = \sum_{r=0}^{12}\binom{12}{r}(2x)^{12-r}(-x^{-2})^r$

$\displaystyle = \sum_{r=0}^{12}\binom{12}{r}(-1)^r 2^{12-r}x^{12-3r}.$

When the term is independent of x,

$x^{12-3r} = x^0$

$\therefore \quad 12-3r = 0$

$r = 4.$

Hence the required term is

$\dbinom{12}{4}(-1)^4 2^{12-4}x^{12-3\times4} = \dbinom{12}{4}2^8x^0 = 126\,720.$

(c) (i) Let $y = e^{3x}(\cos x - 3\sin x)$

$\dfrac{dy}{dx} = 3e^{3x}(\cos x - 3\sin x) + e^{3x}(-\sin x - 3\cos x)$

$= e^{3x}(3\cos x - 9\sin x - \sin x - 3\cos x)$

$= -10e^{3x}\sin x.$

(ii) Hence $\displaystyle\int e^{3x}\sin x\,dx$

$\displaystyle = -\frac{1}{10}\int -10e^{3x}\sin x\,dx$

$= -\dfrac{1}{10}e^{3x}(\cos x - 3\sin x) + C.$

(d) (i) $T = 3+Ae^{-kt}$ so $T-3 = Ae^{-kt}$ —①

$\dfrac{dT}{dt} = -kAe^{-kt} = -k(T-3)$ using ① as required.

(ii) *METHOD* 1

When $t = 0$, $T = 25$,

$\therefore \quad 25 = 3 + Ae^{-k(0)}$

$A = 22.$

So $\quad T = 3 + 22e^{-kt}.$

When $t = 10$, $T = 11$,

$\therefore \quad 11 = 3 + 22e^{-k(10)}$

$22e^{-10k} = 8$

$e^{-10k} = \dfrac{4}{11}$

$-10k = \log_e \dfrac{4}{11}$

$k = -\dfrac{1}{10}\log_e \dfrac{4}{11}$

$= 0{\cdot}101\,16\ldots$

When $t = 15$, $\quad T = 3 + 22e^{-k(15)}$

$= 3 + 22e^{-0{\cdot}101\,16\ldots \times 15}$

$= 7{\cdot}824\,18\ldots$

$= 7{\cdot}8 \left(\text{nearest } \dfrac{1}{10}\,^\circ\text{C}\right).$

$\therefore$ The temperature of the salad is $7{\cdot}8^\circ$C.

METHOD 2

After 10 minutes the temperature difference of $25 - 3 = 22^\circ$C has reduced to $11 - 3 = 8^\circ$C, or $\dfrac{8}{22} = \dfrac{4}{11}$ of its original value. Therefore after 5 minutes it would have reduced to $\sqrt{\dfrac{4}{11}}$ of its original value, and after 15 minutes to $\sqrt{\dfrac{4}{11}}^{\,3}$ of its original value. (This is a geometric series.)

Therefore the temperature is

$T = 3 + 22 \times \sqrt{\dfrac{4}{11}}^{\,3}$

$= 7{\cdot}824\,18\ldots$

$= 7{\cdot}8^\circ\text{C} \quad \left(\text{nearest } \dfrac{1}{10}\,^\circ\text{C}\right).$

QUESTION 3

(a) (i) $g(x) = x^2 - \log_e(x+1)$.

The function $g(x)$ is continuous for $x > -1$.

$g(0{\cdot}7) = (0{\cdot}7)^2 - \log_e(0{\cdot}7+1)$

$= -0{\cdot}040\,62\ldots$

$< 0.$

$g(0{\cdot}9) = (0{\cdot}9)^2 - \log_e(0{\cdot}9+1)$

$= 0{\cdot}168\,146\,113\ldots$

$> 0.$

As $g(x)$ changes sign in the interval $0{\cdot}7 \le x \le 0{\cdot}9$, there is a zero between $0{\cdot}7$ and $0{\cdot}9$.

(ii) $g\left(\dfrac{0{\cdot}7+0{\cdot}9}{2}\right) = g(0{\cdot}8)$

$= (0{\cdot}8)^2 - \log_e(0{\cdot}8+1)$

$= 0{\cdot}052\,21\ldots$

$> 0.$

$\therefore$ The zero lies between $0{\cdot}7$ and $0{\cdot}8$.

$g\left(\dfrac{0{\cdot}7+0{\cdot}8}{2}\right) = g(0{\cdot}75)$

$= (0{\cdot}75)^2 - \log_e(0{\cdot}75+1)$

$= 0{\cdot}002\,88\ldots$

$> 0.$

$\therefore$ The zero lies between $0{\cdot}7$ and $0{\cdot}75$.

Therefore, correct to 1 decimal place, the zero is $0{\cdot}7$.

(b) (i) Using $\sin(A+B) = \sin A\cos B + \cos A\sin B$
and $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\sin(5x+4x) + \sin(5x-4x)$

$= (\sin 5x\cos 4x + \cos 5x\sin 4x)$

$\quad + (\sin 5x\cos 4x - \cos 5x\sin 4x)$

$= 2\sin 5x\cos 4x.$

(ii) $\displaystyle\int \sin 5x\cos 4x \; dx$

$= \dfrac{1}{2}\displaystyle\int 2\sin 5x\cos 4x \; dx$

$= \dfrac{1}{2}\displaystyle\int \sin(5x+4x) + \sin(5x-4x) \; dx,$ from (i)

$= \dfrac{1}{2}\displaystyle\int \sin 9x + \sin x \; dx$

$= \dfrac{1}{2}\left(-\dfrac{1}{9}\cos 9x - \cos x\right) + C$

$= -\dfrac{1}{18}\cos 9x - \dfrac{1}{2}\cos x + C.$

(c) $\quad f(x) = x^2 + 5x$

$f(x+h) = (x+h)^2 + 5(x+h)$

$= x^2 + 2xh + h^2 + 5x + 5h.$

$\therefore f'(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \displaystyle\lim_{h\to 0} \frac{(x^2 + 2xh + h^2 + 5x + 5h) - (x^2 + 5x)}{h}$

$= \displaystyle\lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 5x + 5h - x^2 - 5x}{h}$

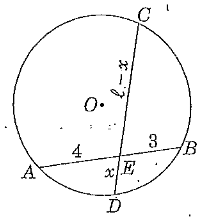
$= \displaystyle\lim_{h\to 0} \frac{2xh + h^2 + 5h}{h}$

$= \displaystyle\lim_{h\to 0} \frac{h(2x + h + 5)}{h}$

$= \displaystyle\lim_{h\to 0} (2x + h + 5)$

$= 2x + 5.$

(d) (i)



$AB = 7 \quad$ (given)

$\therefore \quad EB = AB - AE$

$= 7 - 4$

$= 3.$

$CD = \ell \quad$ (given)

$\therefore \quad CE = CD - DE$

$= \ell - x.$

Now $AE.EB = CE.ED$ (products of intercepts of intersecting chords)

$\therefore \quad 4 \times 3 = x(\ell - x)$

$12 = x\ell - x^2$

$\therefore x^2 - \ell x + 12 = 0.$

(ii) *METHOD* 1

We need to minimise the length of $CD = \ell$.

$x^2 - \ell x + 12 = 0$

$\ell x = x^2 + 12$

$\ell = x + \dfrac{12}{x}$

$\dfrac{d\ell}{dx} = 1 - \dfrac{12}{x^2}$

$\dfrac{d^2\ell}{dx^2} = \dfrac{24}{x^3}.$

For minimum value, $\dfrac{d\ell}{dx} = 0$ and $\dfrac{d^2\ell}{dx^2} > 0.$

If $\dfrac{d\ell}{dx} = 0$: $\quad 1 - \dfrac{12}{x^2} = 0$

$x^2 - 12 = 0$

$x^2 = 12$

$x = \sqrt{12} \quad$ (since $x > 0$)

$= 2\sqrt{3}$

then $\dfrac{d^2\ell}{dx^2} = \dfrac{24}{(2\sqrt{3})^3} > 0.$

Therefore a minimum occurs when $x = 2\sqrt{3}$.

The value of ℓ is given by $\ell = x + \dfrac{12}{x}$

$= 2\sqrt{3} + \dfrac{12}{2\sqrt{3}}$

$= 2\sqrt{3} + \dfrac{6}{\sqrt{3}}$

$= 2\sqrt{3} + 2\sqrt{3}$

$= 4\sqrt{3}.$

Therefore the length of the shortest chord is $4\sqrt{3}$.

METHOD 2

Treating $x^2 - \ell x + 12 = 0$ as a quadratic equation in x, the discriminant is

$\Delta = \ell^2 - 4 \times 12$

$= \ell^2 - 48.$

Since this equation has solutions, it follows that $\quad \Delta \ge 0$

$\ell^2 - 48 \ge 0$

$\ell^2 \ge 48$

$\ell \ge \sqrt{48} \qquad$ (NB: $\ell > 0$)

$= 4\sqrt{3}.$

Hence the minimum value of ℓ, which is the length of the shortest chord, is $4\sqrt{3}$.

QUESTION 4

(a) Let $\quad u = \sin x$

$du = \cos x \; dx.$

If $x = 0$, $\quad u = \sin 0 = 0.$

If $x = \dfrac{\pi}{4}$, $\quad u = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}.$

$\therefore \quad \displaystyle\int_0^{\frac{\pi}{4}} \cos x \sin^2 x \; dx = \int_0^{\frac{1}{\sqrt{2}}} u^2 \; du$

$= \left[\dfrac{1}{3}u^3\right]_0^{\frac{1}{\sqrt{2}}}$

$= \dfrac{1}{3} \times \left(\dfrac{1}{\sqrt{2}}\right)^3 - \dfrac{1}{3}(0)$

$= \dfrac{1}{6\sqrt{2}}$

$= \dfrac{\sqrt{2}}{12}.$

(b) *METHOD* 1 $\left(\text{Using } t = \tan\dfrac{\theta}{2}\right)$

$\sin\theta = \dfrac{2t}{1+t^2}$ implies $\operatorname{cosec}\theta = \dfrac{1+t^2}{2t}.$

$\tan\theta = \dfrac{2t}{1-t^2}$ implies $\cot\theta = \dfrac{1-t^2}{2t}.$

$\operatorname{cosec}\theta + \cot\theta = \dfrac{1+t^2}{2t} + \dfrac{1-t^2}{2t}$

$= \dfrac{2}{2t}$

$= \dfrac{1}{\tan\dfrac{\theta}{2}}$

$= \cot\dfrac{\theta}{2}.$

METHOD 2 (Using the double-angle formulae)

$\cos\theta = 2\cos^2\dfrac{\theta}{2} - 1$, $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}.$

$\operatorname{cosec}\theta + \cot\theta = \dfrac{1}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta}$

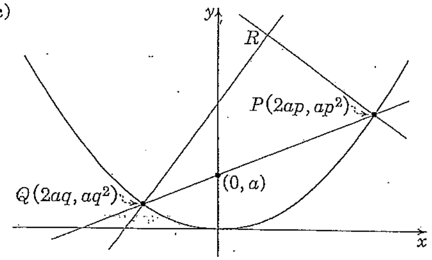
$= \dfrac{1 + \cos\theta}{\sin\theta}$

$= \dfrac{2\cos^2\dfrac{\theta}{2}}{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}$

$= \dfrac{\cos\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}}$

$= \cot\dfrac{\theta}{2}.$

(c)



This diagram is drawn for $a > 0$, but $a < 0$ is also possible.

(i) $x + py = 2ap + ap^3$ — ①

$x + qy = 2aq + aq^3$ — ②

① − ②:

$py - qy = 2ap + ap^3 - 2aq - aq^3$

$(p-q)y = 2a(p-q) + a(p-q)(p^2 + pq + q^2)$

$y = 2a + a(p^2 + pq + q^2) \quad (p \neq q)$

$= a(2 + p^2\ pq + q^2).$

Substitute in ①:

$x + pa(2 + p^2 + pq + q^2) = 2ap + ap^3$

$x + \cancel{2ap} + \cancel{ap^3} + ap^2q + apq^2 = \cancel{2ap} + \cancel{ap^3}$

$x = -apq(p+q).$

$\therefore R$ is $\left(-apq[p+q], a[p^2\ pq + q^2 + 2]\right).$

(ii) Substitute $(0, a)$ in $y = \frac{1}{2}(p+q)x - apq$

$a = \frac{1}{2}(p+q).0 - apq$

$a = -apq$

$pq = -1 \quad$ (since $a \neq 0$).

(iii) $x = -apq(p+q)$, from (i)

$= a(p+q)$, using $pq = -1$.

$\therefore p + q = \frac{x}{a}.$ — ①

$y = a(p^2 + pq + q^2 + 2)$, from (i)

$= a(p^2 + 2pq + q^2 - pq + 2)$

$= a\left([p+q]^2 + 3\right)$, using $pq = -1$

$= a\left(\left[\frac{x}{a}\right]^2 + 3\right)$, using ①

$= \frac{1}{a}(x^2 + 3a^2).$

$\therefore x^2 + 3a^2 = ay$

$x^2 = a(y - 3a).$

(d) $4^n - 1 - 7n > 0.$

When $n = 2$, LHS $= 4^2 - 1 - 7 \times 2$

$= 1$

> 0, as required.

Therefore the result is true when $n = 2$.

Let $k (\geq 2)$ be a value for which the result is true. Then $4^k - 1 - 7k > 0$

$4^k > 1 + 7k.$ — ①

When $n = k + 1$,

LHS $= 4^{k+1} - 1 - 7(k+1)$

$= 4 \times 4^k - 7k - 8$

$> 4(1 + 7k) - 7k - 8$, using ①

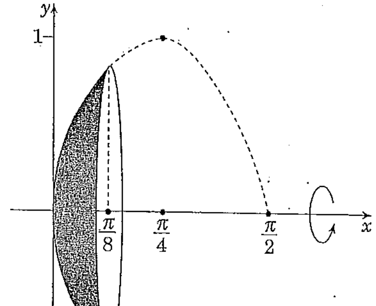
$= 21k - 4$

> 0, since $k \geq 2$.

Thus for $k \geq 2$, when the result is true for $n = k$ it is also true for $n = k + 1$. Since it is true for $n = 2$, the result is also true for $n = 2 + 1 = 3$, and again for $n = 3 + 1 = 4$ and so on. Hence the result is true for all integers $n \geq 2$.

QUESTION 5

(a)



$V = \int_a^b \pi y^2\, dx \quad$ where $\begin{cases} a = 0 \\ b = \frac{\pi}{8} \end{cases}$

$= \pi \int_0^{\frac{\pi}{8}} (\sin 2x)^2\, dx \quad$ using $y = \sin 2x$.

$= \frac{\pi}{2} \int_0^{\frac{\pi}{8}} 2\sin^2 2x\, dx$

$= \frac{\pi}{2} \int_0^{\frac{\pi}{8}} (1 - \cos 4x)\, dx$

$= \frac{\pi}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{8}}$

$= \frac{\pi}{2}\left[\frac{\pi}{8} - \frac{1}{4}\sin\left(4 \times \frac{\pi}{8}\right) - \left(0 - \frac{1}{4}\sin 0\right)\right]$

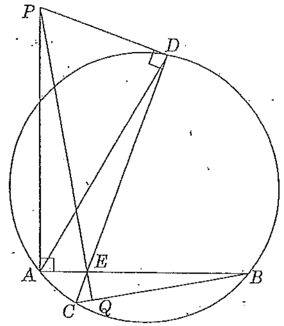
$= \frac{\pi}{2}\left(\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right)$

$= \frac{\pi}{2}\left(\frac{\pi}{8} - \frac{1}{4}\right)$

$= \frac{\pi^2}{16} - \frac{\pi}{8}.$

Therefore the volume of the solid is $\left(\frac{\pi^2}{16} - \frac{\pi}{8}\right)$ unit^3.

(b) (i)



$\angle PDE + \angle PAE = 90° + 90° \quad$ (given)

$= 180°.$

But these are opposite angles of the quadrilateral $DPAE$.

Therefore, $DPAE$ is a cyclic quadrilateral (opposite $\angle$s supplementary).

(ii) As $DPAE$ is a cyclic quadrilateral, a circle passes through D, P, A and E.

$\angle APE = \angle ADE \quad$ ($\angle$s in the same segment of circle $DPAE$, on AE)

$= \angle ADC \quad$ (same $\angle$)

$= \angle ABC \quad$ ($\angle$s in the same segment of circle $ADBC$, on AC).

(iii) Since $\angle APE = \angle ABC$,

then $\angle APQ = \angle ABQ \quad$ (same $\angle$s)

$\therefore$ $APBQ$ is a cyclic quadrilateral (equal $\angle$s standing on the same side of AQ)

$\therefore \quad \angle PQB = \angle PAB$ ($\angle$s standing on the same arc PB)

$\therefore \quad \angle PQB = 90°$

$\therefore \quad PQ \perp BC.$

(c) (i) Let $\sqrt{3}\sin 3t - \cos 3t = R\sin(3t - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

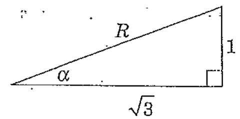
$\sqrt{3}\sin 3t - \cos 3t$

$= R(\sin 3t \cos\alpha - \cos 3t \sin\alpha)$

$= R\cos\alpha \sin 3t - R\sin\alpha \cos 3t.$

Equating terms: $R\cos\alpha = \sqrt{3}$ — ①

$R\sin\alpha = 1$ — ②



② ÷ ①: $\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{\sqrt{3}}$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}.$

①^2 + ②^2:

$R^2\cos^2\alpha + R^2\sin^2\alpha = (\sqrt{3})^2 + 1^2$

$R^2 = 4$

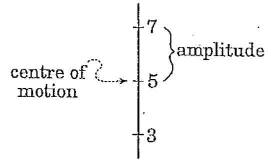
$R = 2.$

Therefore $\sqrt{3}\sin 3t - \cos 3t = 2\sin\left(3t - \frac{\pi}{6}\right).$

(ii) $x = 5 + \sqrt{3}\sin 3t - \cos 3t$

$= 5 + 2\sin\left(3t - \frac{\pi}{6}\right)$, from (i).

As $-2 \leq 2\sin\left(3t - \frac{\pi}{6}\right) \leq 2$, the particle oscillates between $x = 3$ and $x = 7$.



Therefore the centre of motion is 5 and the amplitude is 2.

(iii) The particle is at maximum speed when it passes through the centre of motion, that is, when $x = 5$.

$\therefore \quad 5 = 5 + 2\sin\left(3t - \frac{\pi}{6}\right), \quad t > 0$

$\sin\left(3t - \frac{\pi}{6}\right) = 0$, where $3t - \frac{\pi}{6} > -\frac{\pi}{6}$

$3t - \frac{\pi}{6} = 0 \quad$ (as we need the *first* time only)

$3t = \frac{\pi}{6}$

$t = \frac{\pi}{18}.$

Therefore it reaches maximum speed after $\frac{\pi}{18}$ seconds.

QUESTION 6

(a) P(correct) $= \frac{2}{3}$, P(not correct) $= \frac{1}{3}$.

(i) P(earns 1 point)

$=$ P(5 correct) + P(4 correct) + P(3 correct)

$= {}^5C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^0 + {}^5C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^1 + {}^5C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2$

$= \frac{32}{243} + \frac{80}{243} + \frac{80}{243}$

$= \frac{64}{81}$

$= 0{\cdot}790\,1234\ldots$

$= 0{\cdot}7901$ (4 d.p. as required).

(ii) $P(1 \text{ point every week for 18 weeks})$
$= 0{\cdot}7901^{18}$
$= 0{\cdot}014\,39\ldots$
$= 0{\cdot}01$ (2 d.p. as required).

(iii) $P(\text{at most 16 points})$
$= 1 - P(18 \text{ points}) - P(17 \text{ points})$
$= 1 - 0{\cdot}014 - {}^{18}C_{17}(0{\cdot}7901)^{17}(1 - 0{\cdot}7901)$
$= 1 - 0{\cdot}014 - 0{\cdot}069$
$= 0{\cdot}92$. (2 d.p.).

(b) $x = 200t$
$y = -4{\cdot}9t^2 + 200t + 5000.$

(i) For maximum height, vertical velocity = 0.

$$\frac{dy}{dt} = -9{\cdot}8t + 200$$
$$\frac{dy}{dt} = 0 \text{ when } 9{\cdot}8t = 200.$$
$$t = \frac{200}{9{\cdot}8}$$
$$= 20{\cdot}4081\ldots$$
$$= 20{\cdot}4 \text{ s} \left(\text{nearest } \frac{1}{10} \text{ second}\right).$$
$$y = -4{\cdot}9(20{\cdot}408)^2 + 200(20{\cdot}408) + 5000$$
$$= 7040{\cdot}8 \text{ m}.$$

Therefore the maximum height is 7041 metres when $t = 20{\cdot}4$ seconds.

(ii) The angle at any point is given by

$$\tan\theta = \frac{dy}{dx}$$
$$= \frac{dy}{dt} \div \frac{dx}{dt} \quad \left(\text{or } \frac{\dot{y}}{\dot{x}}\right)$$
$$= \frac{-9{\cdot}8t + 200}{200}.$$

Noting that the angles are negative when descending,

$$-\tan 60° \le \frac{-9{\cdot}8t + 200}{200} \le -\tan 45°$$
$$-\sqrt{3} \le \frac{-9{\cdot}8t + 200}{200} \le -1$$
$$1 \le \frac{9{\cdot}8t - 200}{200} \le \sqrt{3}$$
$$200 \le 9{\cdot}8t - 200 \le 200\sqrt{3}$$
$$400 \le 9{\cdot}8t \le 200(\sqrt{3} + 1)$$
$$\frac{400}{9{\cdot}8} \le t \le \frac{200(\sqrt{3} + 1)}{9{\cdot}8}$$

The earliest time is $\frac{400}{9{\cdot}8} = 40{\cdot}8$ seconds.

The latest time is $\frac{200(\sqrt{3} + 1)}{9{\cdot}8} = 55{\cdot}7$ s.

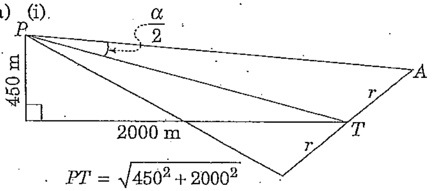
Note that due to symmetry, the earliest time could be found by doubling the time it takes to achieve the maximum height. This is because the angle of projection is 45°.

(iii)
$$v^2 = \dot{x}^2 + \dot{y}^2$$
$$350^2 = (-9{\cdot}8t + 200)^2 + 200^2$$
$$(9{\cdot}8t - 200)^2 = 82\,500$$
$$9{\cdot}8t - 200 = \sqrt{82\,500} \quad (\text{NB: } t \ge 0)$$
$$t = \frac{200 + \sqrt{82\,500}}{9{\cdot}8}$$
$$= 49{\cdot}7 \text{ s} \left(\text{nearest } \frac{1}{10} \text{ s}\right).$$

The latest time to eject is the smaller of 49·7 s and 55·7 s, which is 49·7 s.

QUESTION 7

(a) (i)



$$PT = \sqrt{450^2 + 2000^2}$$
$$= 2050 \text{ m}.$$

In ΔPAT,

$$\tan\frac{\alpha}{2} = \frac{r}{2050}$$
$$r = 2050\tan\frac{\alpha}{2}$$
$$= 2050\tan 0{\cdot}05$$
$$= 102{\cdot}5855\ldots$$
$$= 102{\cdot}6 \text{ m} \left(\text{nearest } \frac{1}{10} \text{ m}\right).$$

(ii)
$$\frac{dr}{dt} = \frac{dr}{d\alpha} \times \frac{d\alpha}{dt}.$$

Now $r = 2050\tan\frac{\alpha}{2}$, so

$$\frac{dr}{d\alpha} = \left(2050\sec^2\frac{\alpha}{2}\right) \times \frac{1}{2}$$
$$= 1025\sec^2\frac{\alpha}{2}$$

and $\frac{d\alpha}{dt} = 0{\cdot}02$ (given), so

$$\frac{dr}{dt} = 1025\sec^2\left(\frac{\alpha}{2}\right) \times 0{\cdot}02$$
$$= 1025\sec^2(0{\cdot}05) \times 0{\cdot}02, \quad \text{when } \alpha = 0{\cdot}1,$$
$$= 20{\cdot}5513\ldots$$
$$= 20{\cdot}6 \text{ m h}^{-1} \left(\text{nearest } \frac{1}{10} \text{ m h}^{-1}\right).$$

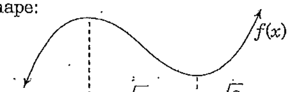
The radius of the oil slick is growing at a rate of 20·6 metres per hour.

(b) (i) $f(x) = Ax^3 - Ax + 1$ where $A > 0$.
Stationary points occur when $f'(x) = 0$.

$$f'(x) = 3Ax^2 - A$$
$$\therefore \text{ if } f'(x) = 0, \quad 3Ax^2 - A = 0$$
$$3x^2 - 1 = 0$$
$$x^2 = \frac{1}{3}$$

$$x = \pm\frac{1}{\sqrt{3}}$$
$$= \pm\frac{\sqrt{3}}{3}.$$

(ii) As $f(x)$ is of degree 3 with $A > 0$ and two stationary points, it will have the following shape:



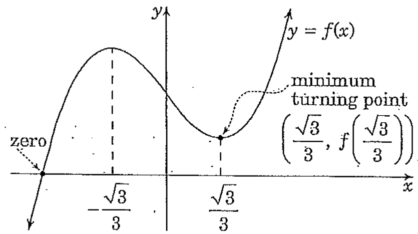
There will be exactly one zero if either the minimum turning point lies above the x-axis $\left[f\left(\frac{\sqrt{3}}{3}\right) > 0\right]$ or the maximum turning point lies below the x-axis $\left[f\left(-\frac{\sqrt{3}}{3}\right) < 0\right]$.

$$\text{Now } f\left(\frac{\sqrt{3}}{3}\right) = A\left(\frac{\sqrt{3}}{3}\right)^3 - A\left(\frac{\sqrt{3}}{3}\right) + 1$$
$$= A\left(\frac{3\sqrt{3}}{27} - \frac{\sqrt{3}}{3}\right) + 1$$
$$= A\left(\frac{\sqrt{3} - 3\sqrt{3}}{9}\right) + 1$$
$$= \frac{-2\sqrt{3}}{9}A + 1.$$

Now when $0 < A < \frac{3\sqrt{3}}{2}$,

$$f\left(\frac{\sqrt{3}}{3}\right) > \frac{-2\sqrt{3}}{9} \times \frac{3\sqrt{3}}{2} + 1$$
$$= 0.$$

Hence the minimum turning point lies above the x-axis, so there is exactly one zero.



(iii) $f(-1) = A(-1)^3 - A(-1) + 1$
$= -A + A + 1$
$= 1$ as stated.

From the shape of the graph shown in (ii) it is clear that, when $0 < A < \frac{3\sqrt{3}}{2}$, the only zero is smaller than any x-value for which $f(x) > 0$. Since $f(-1) > 0$, it follows that the zero is less than -1, and so not in the interval $-1 \le x \le 1$.

(iv) $g(\theta) = 2\cos\theta + \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$
$g'(\theta) = -2\sin\theta + \sec^2\theta.$

Stationary points occur when $g'(\theta) = 0$.

ie.
$$-2\sin\theta + \sec^2\theta = 0$$
$$-2\sin\theta + \frac{1}{\cos^2\theta} = 0$$
$$-2\sin\theta + \frac{1}{1 - \sin^2\theta} = 0$$
$$-2\sin\theta(1 - \sin^2\theta) + 1 = 0$$
$$-2\sin\theta + 2\sin^3\theta + 1 = 0$$
$$2\sin^3\theta - 2\sin\theta + 1 = 0.$$

This is of the form $Ax^3 - Ax + 1 = 0$, where $A = 2$ and $x = \sin\theta$.

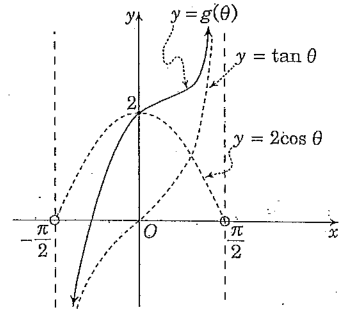
Noting that $A < \frac{3\sqrt{3}}{2}$ and $-1 \le x \le 1$ (since $-1 \le \sin\theta \le 1$), we have shown in part (iii) that this equation has no solutions in this interval.

Hence $2\sin^3\theta - 2\sin\theta + 1 = 0$ has no solutions, and so $g(\theta)$ has no stationary points.

(v) In the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $g(\theta)$ is continuous and differentiable and has no stationary points. Hence it is either monotonic increasing or monotonic decreasing, and so is a one-to-one function and has an inverse function.

Note that $g'(0) = -2\sin\theta + \sec^2\theta$
$= 1.$

Since $g'(0) > 0$, it follows that $g'(\theta) > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so that $g(\theta)$ is monotonic increasing. Also note that $g(\theta)$ can be sketched as the sum of its components.



The sketch shows that $g(\theta)$ is clearly monotonic increasing for $-\frac{\pi}{2} < \theta < 0$, and hence for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

END OF EXTENSION 1 SOLUTIONS