BOARD OF STUDIES
NEW SOUTH WALES

2008

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Total marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) The polynomial x^3 is divided by $x+3$. Calculate the remainder. **2**

(b) Differentiate $\cos^{-1}(3x)$ with respect to x. **2**

(c) Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\,dx$. **2**

(d) Find an expression for the coefficient of x^8y^4 in the expansion of $(2x+3y)^{12}$. **2**

(e) Evaluate $\displaystyle\int_{0}^{\frac{\pi}{4}} \cos\theta\,\sin^2\theta\,d\theta$. **2**

(f) Let $f(x)=\log_e[(x-3)(5-x)]$. **2**

What is the domain of $f(x)$?

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $u=\log_e x$ to evaluate $\displaystyle\int_{e}^{e^2} \frac{1}{x(\log_e x)^2}\,dx$. **3**

(b) A particle moves on the x-axis with velocity v. The particle is initially at rest at $x=1$. Its acceleration is given by $\ddot{x}=x+4$. **3**

Using the fact that $\ddot{x}=\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$, find the speed of the particle at $x=2$.

(c) The polynomial $p(x)$ is given by $p(x)=ax^3+16x^2+cx-120$, where a and c are constants. **3**

The three zeros of $p(x)$ are -2, 3 and α.

Find the value of α.

(d) The function $f(x)=\tan x-\log_e x$ has a zero near $x=4$. **3**

Use one application of Newton's method to obtain another approximation to this zero. Give your answer correct to two decimal places.

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

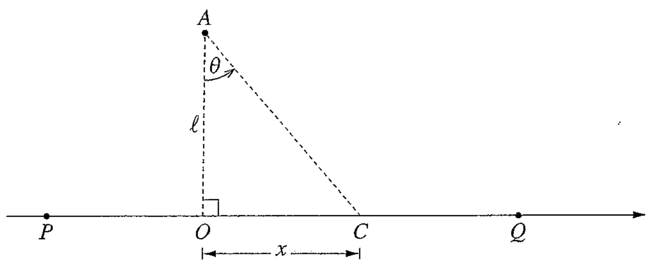
(a) (i) Sketch the graph of $y = |2x - 1|$. **1**

(ii) Hence, or otherwise, solve $|2x - 1| \le |x - 3|$. **3**

(b) Use mathematical induction to prove that, for integers $n \ge 1$, **3**

$$1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n+2) = \frac{n}{6}(n+1)(2n+7).$$

(c)



A race car is travelling on the x-axis from P to Q at a constant velocity, v. A spectator is at A which is directly opposite O, and $OA = \ell$ metres. When the car is at C, its displacement from O is x metres and $\angle OAC = \theta$, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(i) Show that $\dfrac{d\theta}{dt} = \dfrac{v\ell}{\ell^2 + x^2}$. **2**

(ii) Let m be the maximum value of $\dfrac{d\theta}{dt}$. **1**

Find the value of m in terms of v and ℓ.

(iii) There are two values of θ for which $\dfrac{d\theta}{dt} = \dfrac{m}{4}$. **2**

Find these two values of θ.

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) A turkey is taken from the refrigerator. Its temperature is 5°C when it is placed in an oven preheated to 190°C.

Its temperature, T°C, after t hours in the oven satisfies the equation

$$\frac{dT}{dt} = -k(T - 190).$$

(i) Show that $T = 190 - 185e^{-kt}$ satisfies both this equation and the initial condition. **2**

(ii) The turkey is placed into the oven at 9 am. At 10 am the turkey reaches a temperature of 29°C. The turkey will be cooked when it reaches a temperature of 80°C. **3**

At what time (to the nearest minute) will it be cooked?

(b) Barbara and John and six other people go through a doorway one at a time.

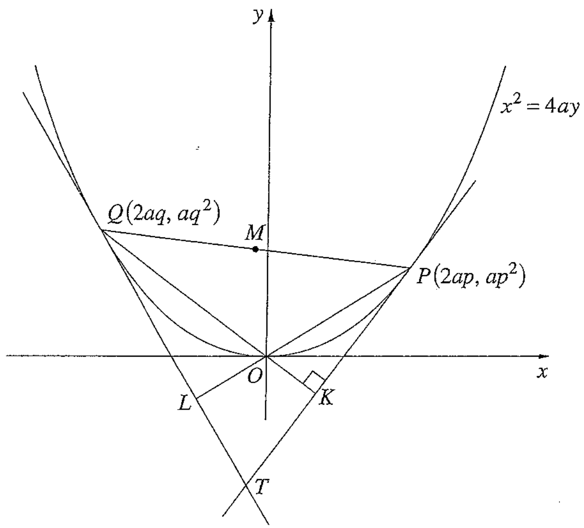
(i) In how many ways can the eight people go through the doorway if John goes through the doorway after Barbara with no-one in between? **1**

(ii) Find the number of ways in which the eight people can go through the doorway if John goes through the doorway after Barbara. **1**

Question 4 continues on page 7

Question 4 (continued) **Marks**

(c)



The points $P(2ap, ap^2)$, $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The tangents to the parabola at P and Q intersect at T. The chord QO produced meets PT at K, and $\angle PKQ$ is a right angle.

(i) Find the gradient of QO, and hence show that $pq = -2$. **2**

(ii) The chord PO produced meets QT at L. Show that $\angle PLQ$ is a right angle. **1**

(iii) Let M be the midpoint of the chord PQ. By considering the quadrilateral $PQLK$, or otherwise, show that $MK = ML$. **2**

End of Question 4

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Let $f(x) = x - \frac{1}{2}x^2$ for $x \le 1$. This function has an inverse, $f^{-1}(x)$.

(i) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. (Use the same scale on both axes.) **2**

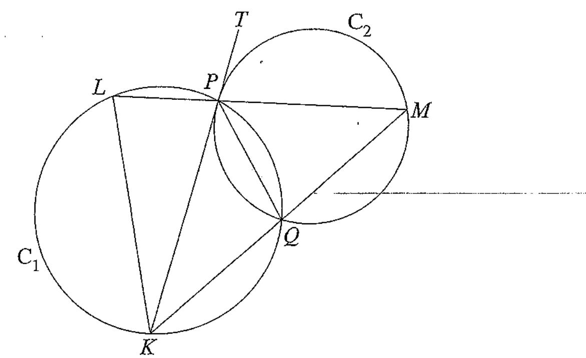
(ii) Find an expression for $f^{-1}(x)$. **3**

(iii) Evaluate $f^{-1}\left(\frac{3}{8}\right)$. **1**

(b) A particle is moving in simple harmonic motion in a straight line. Its maximum speed is 2 m s^{-1} and its maximum acceleration is 6 m s^{-2}. **3**

Find the amplitude and the period of the motion.

(c) **3**



Two circles C_1 and C_2 intersect at P and Q as shown in the diagram. The tangent TP to C_2 at P meets C_1 at K. The line KQ meets C_2 at M. The line MP meets C_1 at L.

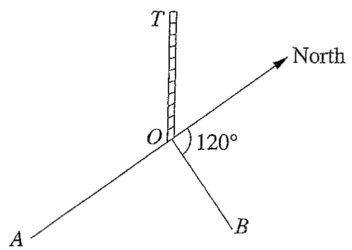
Copy or trace the diagram into your writing booklet.

Prove that ΔPKL is isosceles.

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) From a point A due south of a tower, the angle of elevation of the top of the tower T, is 23°. From another point B, on a bearing of 120° from the tower, the angle of elevation of T is 32°. The distance AB is 200 metres.



(i) Copy or trace the diagram into your writing booklet, adding the given information to your diagram. **1**

(ii) Hence find the height of the tower. **3**

(b) It can be shown that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ for all values of θ. (Do NOT prove this.) **3**

Use this result to solve $\sin 3\theta + \sin 2\theta = \sin\theta$ for $0 \le \theta \le 2\pi$.

(c) Let p and q be positive integers with $p \le q$.

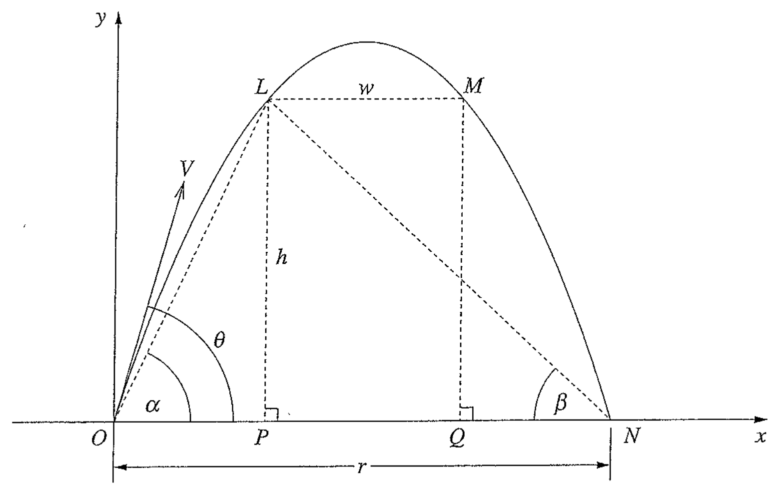
(i) Use the binomial theorem to expand $(1+x)^{p+q}$, and hence write down the term of $\dfrac{(1+x)^{p+q}}{x^q}$ which is independent of x. **2**

(ii) Given that $\dfrac{(1+x)^{p+q}}{x^q} = (1+x)^p\left(1+\dfrac{1}{x}\right)^q$, apply the binomial theorem and the result of part (i) to find a simpler expression for **3**

$$1 + \binom{p}{1}\binom{q}{1} + \binom{p}{2}\binom{q}{2} + \cdots + \binom{p}{p}\binom{q}{p}.$$

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.



A projectile is fired from O with velocity V at an angle of inclination θ across level ground. The projectile passes through the points L and M, which are both h metres above the ground, at times t_1 and t_2 respectively. The projectile returns to the ground at N.

The equations of motion of the projectile are

$$x = Vt\cos\theta \text{ and } y = Vt\sin\theta - \frac{1}{2}gt^2. \text{ (Do NOT prove this.)}$$

(a) Show that $t_1 + t_2 = \dfrac{2V}{g}\sin\theta$ AND $t_1 t_2 = \dfrac{2h}{g}$. **2**

Question 7 continues on page 11

Marks

Question 7 (continued)

Let $\angle LON = \alpha$ and $\angle LNO = \beta$. It can be shown that

$$\tan\alpha = \frac{h}{Vt_1 \cos\theta} \text{ and } \tan\beta = \frac{h}{Vt_2 \cos\theta}. \text{ (Do NOT prove this.)}$$

(b) Show that $\tan\alpha + \tan\beta = \tan\theta$. **2**

(c) Show that $\tan\alpha \tan\beta = \dfrac{gh}{2V^2 \cos^2\theta}$. **1**

Let $ON = r$ and $LM = w$.

(d) Show that $r = h(\cot\alpha + \cot\beta)$ and $w = h(\cot\beta - \cot\alpha)$. **2**

Let the gradient of the parabola at L be $\tan\phi$.

(e) Show that $\tan\phi = \tan\alpha - \tan\beta$. **3**

(f) Show that $\dfrac{w}{\tan\phi} = \dfrac{r}{\tan\theta}$. **2**

End of paper

HSC 2008 EXT 1 – SOLUTIONS (1)

1)

(a) $P(-3) = (-3)^3 = -27$

$$
\begin{array}{r}
x^2 - 3x + 9 \\
x+3 \overline{\smash{)}\, x^3 \qquad\qquad} \\
x^3 + 3x^2 \\
\hline
-3x^2 \\
-3x^2 - 9x \\
\hline
9x \\
9x + 27 \\
\hline
-27
\end{array}
$$

or

$x^3 = (x+3)(x^2 - 3x + 9) \boxed{-27}$

(b) $\frac{d}{dx}[\cos^{-1}(3x)] = \frac{-1}{\sqrt{1-(3x)^2}} \cdot 3 = \frac{-3}{\sqrt{1-9x^2}}$

(c) $\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-1}^{1}$

$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$

$= \frac{\pi}{6} - (-\pi/6)$

$= \pi/3$

(d) $(2x+3y)^{12} = \cancel{(2x)^{12}(3y)^{0} + \binom{12}{1}(2x)^1(3y)^{11} + \binom{12}{2}(2x)^2(3y)^{10} + \dots + \binom{12}{8}(2x)^8(3y)^4 + \dots + (2x)^{12}(3y)^{0}}$

$= \sum_{r=0}^{12} \binom{12}{r}(2x)^r(3y)^{12-r}$

$\Rightarrow T_9 = \binom{12}{8}(2x)^8(3y)^4 = \left[\binom{12}{8}2^8 3^4\right]x^8y^4$

$\Rightarrow$ Coeff. $= \binom{12}{8}2^8 3^4 = 10{,}264{,}320$

e) $\int_0^{\pi/4} \cos\theta \sin^2\theta \, d\theta = \left[\frac{\sin^3\theta}{3}\right]_0^{\pi/4}$

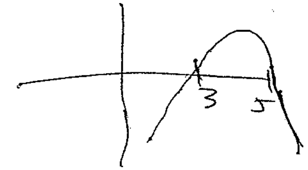
$= \frac{1}{3}\left[\sin^3(\pi/4) - \sin^3(0)\right]$

$= \frac{1}{3}\left[\left(\frac{1}{\sqrt{2}}\right)^3 - 0\right] = \frac{1}{6\sqrt{2}}$

(f) $f(x) = \ln[(x-3)(5-x)]$

Need $(x-3)(5-x) > 0$

$\Rightarrow 3 < x < 5$



2) (a) Let $I = \int_e^{e^2} \frac{1}{x(\ln x)^2}\,dx$

Let $u = \ln x$

$\Rightarrow du = \frac{1}{x}dx$

when $x = e$ $u = \ln e = 1$

when $x = e^2$ $u = \ln e^2 = 2\ln e = 2$

$\therefore I = \int_1^2 \frac{1}{u^2}\,du$

$= \int_1^2 u^{-2}\,du$

$= \left[\frac{u^{-1}}{-1}\right]_1^2$

$= \left[\frac{1}{u}\right]_2^1$

$= 1 - \frac{1}{2} = \frac{1}{2}$

(b) $t = 0 \Rightarrow x = 1$ & $v = 0$ ——— (*)

$\ddot{x} = x + 4$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = x + 4$

$\Rightarrow \frac{1}{2}v^2 = \int(x+4)\,dx = \frac{x^2}{2} + 4x + c$

But, $0 = \frac{1}{2} + 4 + c$ (by (*))

$\Rightarrow c = -\frac{9}{2}$

$\therefore \frac{1}{2}v^2 = \frac{x^2}{2} + 4x - \frac{9}{2}$

$\Rightarrow v^2 = x^2 + 8x - 9$

$\therefore$ When $x = 2$ $v^2 = 4 + 16 - 9 = 11$

$\therefore |v| = \sqrt{11}$

ie, speed $= \sqrt{11}$

(c) $p(x) = ax^3 + 16x^2 + cx - 120$

$p(-2) = p(3) = p(\alpha) = 0$

$$\therefore \begin{cases} -8a + 64 - 2c - 120 = 0 \\ 27a + 144 + 3c - 120 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 8a + 2c = -56 \\ 27a + 3c = -24 \end{cases}$$

$$\Rightarrow \begin{cases} 4a + c = -28 \\ 9a + c = -8 \end{cases}$$

$$\Rightarrow \quad 5a = 20$$

$$\Rightarrow a = 4$$

$$\Rightarrow c = -8 - 36 = -44$$

$$\therefore p(x) = 4x^3 + 16x^2 - 44x - 120$$

And, $-2 + 3 + \alpha = \frac{-16}{4} = -4$

$$\Rightarrow \alpha = -5$$

(d) $f(x) = \tan x - \ln x \Rightarrow f'(x) = \sec^2 x - \frac{1}{x}$

$x_0 = 4$

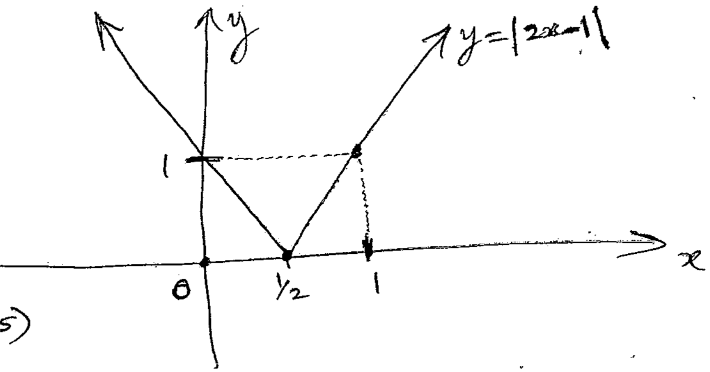
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 4 - \left[\frac{\tan(4) - \ln(4)}{\sec^2(4) - \frac{1}{4}}\right]$$

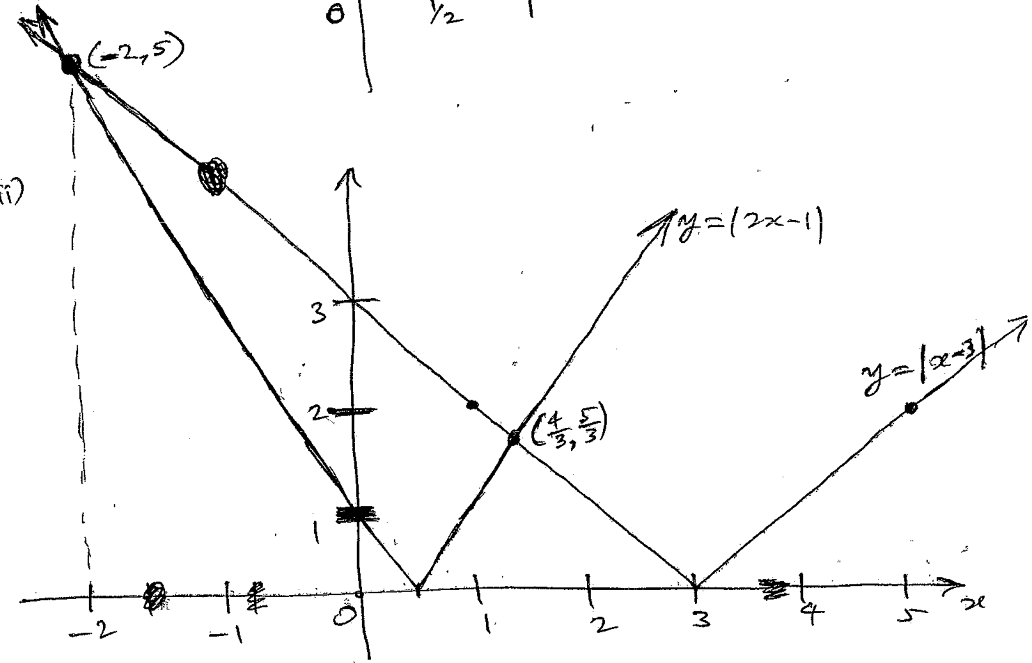
$$\approx 4 - (-0.109288496)$$

$$\approx \boxed{4.11} \quad (\text{to } 2\,dp)$$

3(a) (i)



(ii)



From graph $|2x-1| \le |x-3|$ occurs when

$$-2 \le x \le 4/3$$

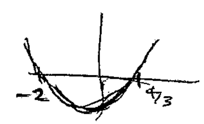
OR $|2x-1| \le |x-3| \iff (2x-1)^2 \le (x-3)^2$

$$\iff 4x^2 - 4x + 1 \le x^2 - 6x + 9$$

$$\iff 3x^2 + 2x - 8 \le 0$$

$$\iff (3x-4)(x+2) \le 0$$

$$\therefore -2 \le x \le 4/3$$



(b) Let $P(n)$ be the propn that

$$1\times 3 + 2\times 4 + 3\times 5 + \dots + n(n+2) = \frac{n}{6}(n+1)(2n+7)$$

Then $P(1)$ says that $1\times 3 = \frac{1}{6}(1+1)(2+7)$

which is true since $\frac{1}{6}(1+1)(2+7) = \frac{1}{6}(2)(9) = 3$

Assume $P(K)$ is true (for some $K \geq 1$)

Then $1\times 3 + 2\times 4 + \dots + K(K+2) = \frac{K}{6}(K+1)(2K+7)$

$\therefore 1\times 3 + 2\times 4 + \dots + K(K+2) + (K+1)(K+3) = \frac{K}{6}(K+1)(2K+7) + (K+1)(K+3)$

$$= \frac{K+1}{6}\left[K(2K+7) + 6(K+3)\right]$$

$$= \frac{K+1}{6}\left[2K^2 + 13K + 18\right]$$

$$= \frac{K+1}{6}\left[(K+2)(2K+9)\right]$$

$$= \frac{K+1}{6}\left[(K+1)+1\right]\left[2(K+1)+7\right]$$

So, $P(K)$ true $\Rightarrow P(K+1)$ true

And, $P(1)$ is true

$\therefore$ by induction $P(n)$ is true for $n \geq 1$. //

(c) (i) $\tan\theta = \frac{x}{l}$

$\therefore \theta = \tan^{-1}\frac{x}{l}$ [noting that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$]

$$\Rightarrow \frac{d\theta}{dt} = \frac{d\theta}{dx}\cdot\frac{dx}{dt} = v\frac{d\theta}{dx} = v\frac{d}{dx}\left(\tan^{-1}\frac{x}{l}\right)$$

$$= \frac{vl}{l^2+x^2} \quad //$$

(ii) $\max\left(\frac{d\theta}{dt}\right) = \max\left(\frac{vl}{l^2+x^2}\right) = \frac{vl}{l^2+0} = \frac{vl}{l^2} = \frac{v}{l}$

So $m = \frac{v}{l}$ [NOTING v & l are POSITIVE CONSTANTS]

(Max occurs when l^2+x^2 is a minimum)

[MORE FORMALLY: Let $y = \frac{vl}{l^2+x^2}$ ($y_{max} = m$)

$$\frac{dy}{dx} = vl\frac{d}{dx}(l^2+x^2)^{-1} = -\frac{2vlx}{(l^2+x^2)^2}$$

$\therefore \frac{dy}{dx} = 0$ when $x = 0$

And, $\begin{cases} \frac{dy}{dx} > 0 & \text{if } x < 0 \\ \frac{dy}{dx} < 0 & \text{if } x > 0 \end{cases} \Rightarrow$ (ABS.) MAX @ $x = 0$]

(iii) $\frac{d\theta}{dt} = \frac{m}{4} \Leftrightarrow \frac{vl}{l^2+x^2} = \frac{v}{4l}$

$\Leftrightarrow 4vl^2 = v(l^2+x^2)$

$\Leftrightarrow 3vl^2 = vx^2$

$\Leftrightarrow 3l^2 = x^2$

$\Leftrightarrow \frac{x^2}{l^2} = 3$

$\Leftrightarrow \frac{x}{l} = \pm\sqrt{3}$

$\therefore \theta = \tan^{-1}(-\sqrt{3})$ or $\tan^{-1}\sqrt{3}$

$\therefore \boxed{\theta = \pm\frac{\pi}{3}}$

4 (a) When $t = 0$, $T = 5$ ie, $T(0) = 5$ ---(1)

$\frac{dT}{dt} = -K(T-190)$ ---(2)

(i) Let $T = 190 - 185e^{-Kt}$ ---(*)

Then $T(0) = 190 - 185e^0 = 5$

and $\frac{dT}{dt} = -185(-K)e^{-Kt} = -K(-185e^{-Kt}) = -K(T-190)$

$\therefore$ (*) satisfies (1) & (2) //

(ii) $t = 0$ corresponds to 9 am $T(0) = 5$

$t = 1$ ---------- 10am $T(1) = 29$

$t = \tau$ ---------- (9am + τ hrs) $T(\tau) = 80$

So, $29 = 190 - 185e^{-K}$

$\Rightarrow 185e^{-K} = 161 \Rightarrow e^{-K} = \frac{161}{\dots}$

(7)

$\therefore \ T = 190 - 185\left(\frac{161}{185}\right)^t$

So, $80 = 190 - 185\left(\frac{161}{185}\right)^\tau$

$\therefore \ \left(\frac{161}{185}\right)^\tau = \frac{110}{185}$

$\therefore \ \tau \log\left(\frac{161}{185}\right) = \log\left(\frac{110}{185}\right)$

$\Rightarrow \ \tau = \dfrac{\log\left(\frac{110}{185}\right)}{\log\left(\frac{161}{185}\right)} = 3.741417751\ldots$

$\Rightarrow$ time taken = 3 hours 44 mins (nearest minute)

$\therefore$ cooked @ 12.44pm

(b) $BJ\underbrace{ACDEFG}_{6}$

(i) BJ can be treated as 1 object

$\therefore$ #ways to arrange 7 objects = $7! = 5040$

OR

BJ	6	5	4	3	2	1	6!
6	BJ	5	4	3	2	1	6!
⋮							⋮
6	5	4	3	2	1	BJ	6!

7 ways $\Rightarrow 7 * 6! = 7!$ ways

(ii)

B	7	6	5	4	3	2	J	$7! = 7*6!$
6	B	6	5	4	3	2	J	$6*6!$
6	5	B	5	4	3	2	J	$5*6!$
6	5	4	B	4	3	2	J	$4*6!$
⋮								
6	5	4	3	2	1	B	J	$1*6!$

#ways $= (1+2+3+4+5+6+7)6! = 28 * 6! = 20160$

OR

If there are no restrictions then the #ways $= 8!$

In half of these B goes before J and half have J before B.

$\therefore$ #ways $= \frac{8!}{2} = 20160$

(8)

(c) (i) $m_{QO} = \dfrac{aq^2 - 0}{2aq - 0} = \dfrac{aq^2}{2aq} = \dfrac{q}{2}$ [since $a \neq 0, q \neq 0$]

$m_{PK} = p$ $\left[\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a} = \frac{2ap}{2a} = p\right]$

$m_{QK} = \frac{q}{2}$

$\Rightarrow p \cdot \frac{q}{2} = -1$ (since $PK \perp QK$)

$\Rightarrow pq = -2$ //

(ii) $m_{PO} = \frac{p}{2} \Rightarrow m_{PL} = \frac{p}{2}$

$m_{QL} = q$

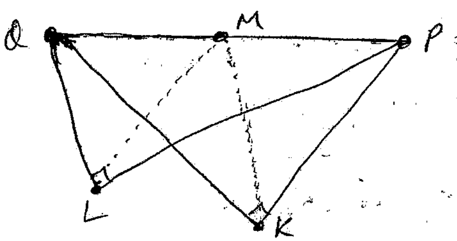
as in (i)

$\Rightarrow m_{PL} \cdot m_{QL} = \frac{p}{2} \cdot q = \frac{pq}{2} = \frac{-2}{2} = -1$ (by (i))

$\therefore PL \perp QL$

$\therefore \angle PLQ = 90°$ //

(iii)



PQLK is a cyclic quadrilateral (since $\angle PKQ = \angle PLQ$) [angles subtended on same side equal]

Further PQ is diameter of the circle (since $\angle PKQ = 90°$)

$\therefore$ M is centre of circle (M = midpt of diameter PQ)

$\therefore MK = ML$ (both radii of the circle) //

(5) (a) $f(x) = x - \frac{1}{2}x^2$ for $x \leq 1$

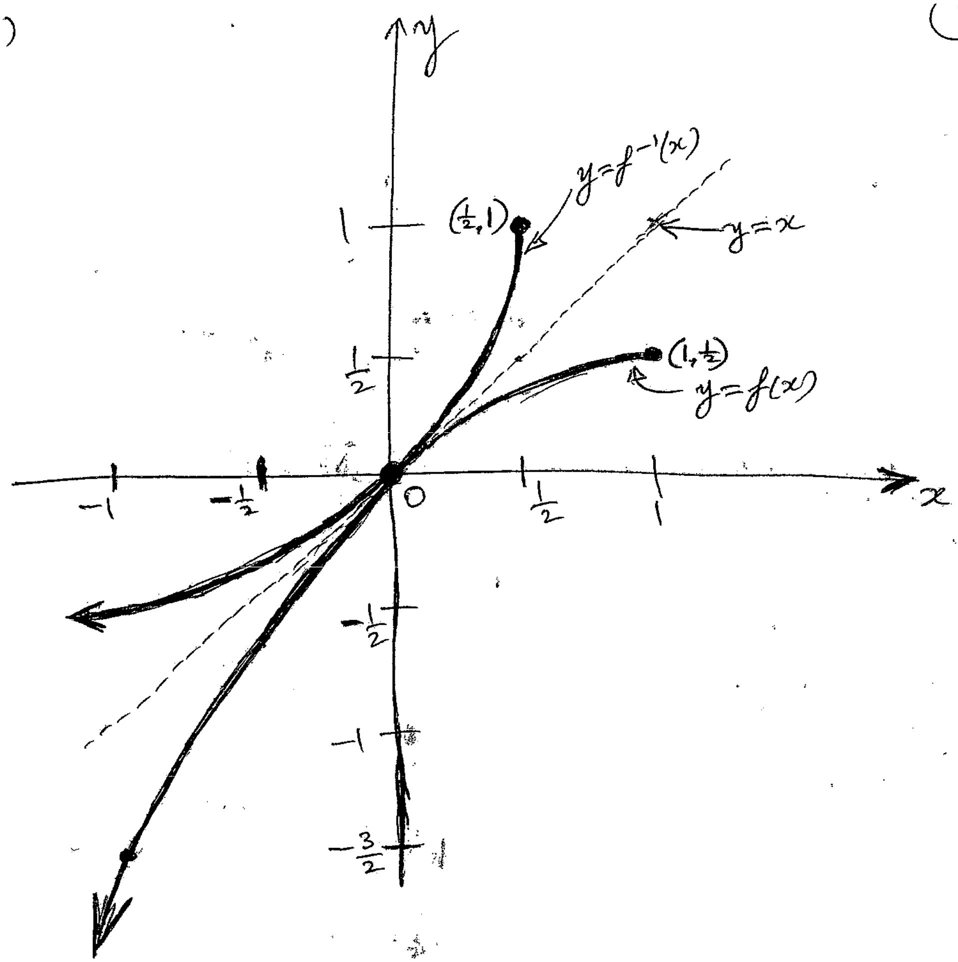
[$f'(x) = 1 - x \Rightarrow f'(x) > 0$ for $x < 1$]

$f(x) = \frac{x}{2}(2 - x)$

max $f(x)$ occurs when $x = \dfrac{-1}{2(-\frac{1}{2})} = 1$

$\Rightarrow$ max $f(x) = 1 - \frac{1}{2} = \frac{1}{2}$

(i)



(ii) Let $y = x - \frac{1}{2}x^2$ $\quad (\Rightarrow \frac{1}{2}x^2 - x + y = 0)$ $\quad$ [for $x \le 1$]

$\Rightarrow x^2 - 2x + 2y = 0$

$\Rightarrow x = \dfrac{2 \pm \sqrt{4-8y}}{2} = \dfrac{2 \pm 2\sqrt{1-2y}}{2} = 1 \pm \sqrt{1-2y}$

But $x \le 1 \Rightarrow x = 1 - \sqrt{1-2y}$ $\quad \therefore f^{-1}(y) = 1 - \sqrt{1-2y}$

$\therefore \boxed{f^{-1}(x) = 1 - \sqrt{1-2x}}$ $\quad$ (Note that $x \le \frac{1}{2}$)

(iii) $f^{-1}\left(\frac{3}{8}\right) = 1 - \sqrt{1 - 3/4} = 1 - \sqrt{\frac{1}{4}} = 1 - \frac{1}{2} = \frac{1}{2}$

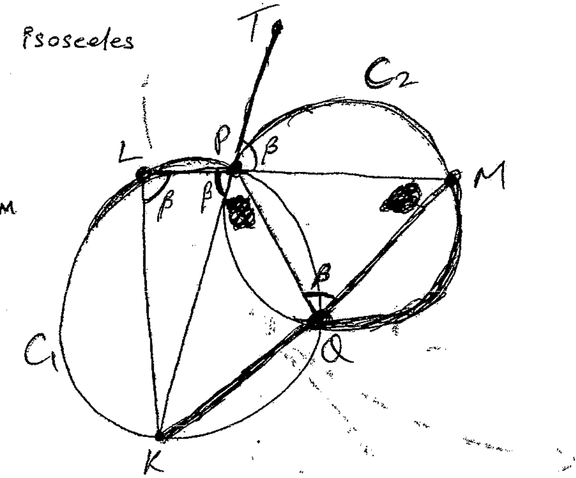
(b) $x = A\cos(nt+\alpha) \Rightarrow \dot{x} = -nA\sin(nt+\alpha) \Rightarrow \ddot{x} = -n^2A\cos(nt+\alpha)$

$\max|\dot{x}| = nA = 2$ & $\max \ddot{x} = n^2A = 6 \Rightarrow \dfrac{n^2A}{nA} = \dfrac{6}{2} \Rightarrow n = 3 \Rightarrow A = \dfrac{2}{3}$

$\therefore$ Amplitude $= \dfrac{2}{3}$ m

& Period $= \dfrac{2\pi}{n} = \dfrac{2\pi}{3}$ secs

(c) RTP ΔPKL is isosceles



$\angle TPM = \angle PQM$
(angle between tangent TP & chord PM equals angle in alt. segment)

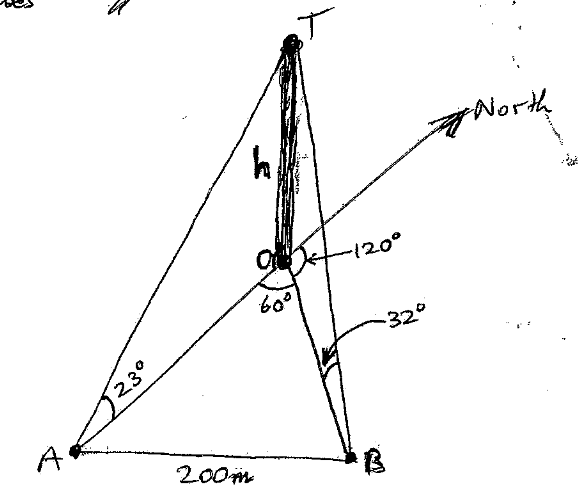
$\angle PQM = \angle PLK$
(ext. $\angle$ of cyclic quad (PLKQ) = int. opp. $\angle$)

$\angle TPM = \angle KPL$
(vert. opp. $\angle$s are equal)

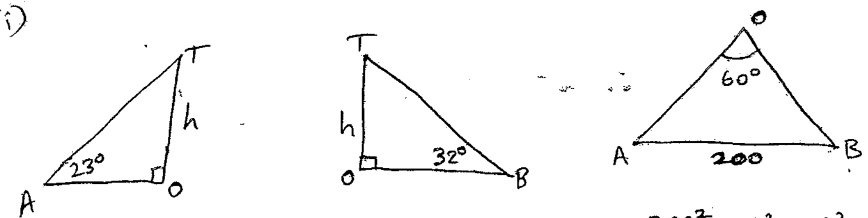
$\therefore \angle PLK = \angle KPL$

$\therefore \Delta PKL$ is isosceles //

6 (a) (i)



(ii)



$\therefore \dfrac{OA}{h} = \cot 23°$

$\Rightarrow OA = h\cot 23°$

$\dfrac{OB}{h} = \cot 32°$

$\Rightarrow OB = h\cot 32°$

$200^2 = OA^2 + OB^2 - 2(OA)(OB)\cos 60°$

$\therefore 40000 = h^2(\cot 23°)^2 + h^2(\cot 32°)^2 - (h\cot 23°)(h\cot 32°)$

$\therefore \quad h^2\left[(\cot 23^\circ)^2+(\cot 32^\circ)^2-(\cot 23^\circ)(\cot 32^\circ)\right]=40000$

$\therefore \quad h^2=\dfrac{40000}{(\cot 23^\circ)^2+(\cot 32^\circ)^2-(\cot 23^\circ)(\cot 32^\circ)}\approx\dfrac{40000}{4.340959088}$

$\therefore \quad h^2\approx 9214.553556$

$\Rightarrow h\approx 95.99246614$

$\therefore$ Height $=$ 96m

(b) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ ______ (1)

$\sin 3\theta + \sin 2\theta = \sin\theta$ ______ (*)

$\Leftrightarrow 3\sin\theta - 4\sin^3\theta + \sin 2\theta = \sin\theta$ by (1)

$\therefore \quad 2\sin\theta - 4\sin^3\theta + 2\sin\theta\cos\theta = 0$

$\therefore 2\sin\theta(1-2\sin^2\theta+\cos\theta)=0$

$\therefore 2\sin\theta\left[1-2(1-\cos^2\theta)+\cos\theta\right]=0$

$\therefore 2\sin\theta\left[2\cos^2\theta+\cos\theta-1\right]=0$

$\therefore 2\sin\theta(2\cos\theta-1)(\cos\theta+1)=0$

$\therefore \sin\theta=0$ OR $\cos\theta=\frac{1}{2}$ OR $\cos\theta=-1$

$\therefore \theta=0,\pi,2\pi$ OR $\theta=\frac{\pi}{3},\frac{5\pi}{3}$ OR $\theta=\pi$

$\therefore \theta=0,\frac{\pi}{3},\pi,\frac{5\pi}{3},2\pi$

(c) $p,q\in\mathbb{Z}^+,\ p\le q$

(i) $(1+x)^{p+q}=\sum_{r=0}^{p+q}\binom{p+q}{r}1^{p+q-r}x^r$

$\therefore \boxed{(1+x)^{p+q}=\sum_{r=0}^{p+q}\binom{p+q}{r}x^r}$

$\therefore \dfrac{(1+x)^{p+q}}{x^q}=\sum_{r=0}^{p+q}\binom{p+q}{r}x^{r-q}$

$\Rightarrow T_{r+1}=\binom{p+q}{r}x^{r-q}$

Indpt of $x \Rightarrow r=q$

$\therefore$ term indpt of x $=T_{q+1}=\boxed{\binom{p+q}{q}}=\dfrac{(p+q)!}{p!\,q!}$

(ii) $\dfrac{(1+x)^{p+q}}{x^q}=(1+x)^p\left(1+\frac{1}{x}\right)^q$ ------ (*)

Now, $(1+x)^p=\sum_{k=0}^{p}\binom{p}{k}x^k$

And, $\left(1+\frac{1}{x}\right)^q=\sum_{n=0}^{q}\binom{q}{n}\left(\frac{1}{x}\right)^n=\sum_{n=0}^{q}\binom{q}{n}x^{-n}$

$\therefore (1+x)^p\left(1+\frac{1}{x}\right)^q=\left\{\sum_{k=0}^{p}\binom{p}{k}x^k\right\}\left\{\sum_{n=0}^{q}\binom{q}{n}x^{-n}\right\}$

$=\left\{\binom{p}{0}+\binom{p}{1}x+\dots+\binom{p}{k}x^k+\dots+x^p\right\}\left\{\binom{q}{0}+\binom{q}{1}x^{-1}+\dots+\binom{q}{n}x^{-n}+\dots+x^{-q}\right\}$

Term indpt of x in this expression is:

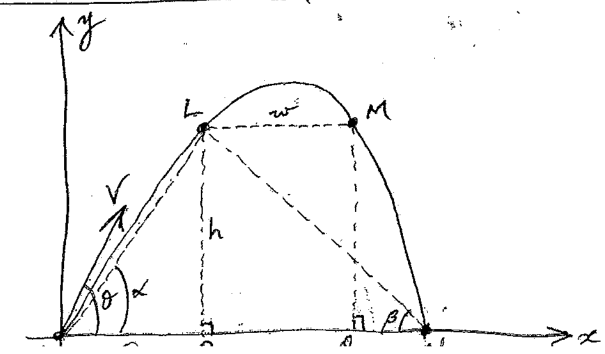
$\binom{p}{0}\binom{q}{0}+\binom{p}{1}\binom{q}{1}+\dots+\binom{p}{p}\binom{q}{p}$ (noting that $p\le q$)

$=1+\binom{p}{1}\binom{q}{1}+\binom{p}{2}\binom{q}{2}+\dots+\binom{p}{p}\binom{q}{p}$

$\therefore \boxed{1+\binom{p}{1}\binom{q}{1}+\binom{p}{2}\binom{q}{2}+\dots+\binom{p}{p}\binom{q}{p}=\binom{p+q}{q}}$ by (i) & (*)

7) $x=Vt\cos\theta$

$y=Vt\sin\theta-\frac{1}{2}gt^2$



(13)

(a) RTP $t_1 + t_2 = \frac{2V}{g}\sin\theta$ AND $t_1 t_2 = \frac{2h}{g}$

t_1, t_2 are roots of $Vt\sin\theta - \frac{1}{2}gt^2 = h$

ie, $\frac{1}{2}gt^2 - (V\sin\theta)t + h = 0$

$$\therefore t_1 + t_2 = \frac{V\sin\theta}{\frac{1}{2}g} = \frac{2V}{g}\sin\theta \quad //$$

$$\text{and } t_1 t_2 = \frac{h}{\frac{1}{2}g} = \frac{2h}{g} \quad //$$

Now,

$$\tan\alpha = \frac{h}{Vt_1\cos\theta} \text{ and } \tan\beta = \frac{h}{Vt_2\cos\theta} \quad \text{------(*)}$$

(b) RTP $\tan\alpha + \tan\beta = \tan\theta$

$$\tan\alpha + \tan\beta = \frac{h}{Vt_1\cos\theta} + \frac{h}{Vt_2\cos\theta}$$

$$= \frac{h}{V\cos\theta}\left(\frac{1}{t_1} + \frac{1}{t_2}\right)$$

$$= \frac{h}{V\cos\theta}\left(\frac{t_1 + t_2}{t_1 t_2}\right)$$

$$= \frac{h}{V\cos\theta}\left(\frac{2V}{g}\sin\theta \div \frac{2h}{g}\right) \quad \text{by (a)}$$

$$= \frac{\cancel{h}}{\cancel{V}\cos\theta}\left(\frac{\cancel{2V}\sin\theta}{\cancel{g}} * \frac{\cancel{g}}{\cancel{2h}}\right)$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta \quad //$$

(c) RTP $\tan\alpha\tan\beta = \frac{gh}{2V^2\cos^2\theta}$

$$\tan\alpha\tan\beta = \frac{h}{Vt_1\cos\theta}\cdot\frac{h}{Vt_2\cos\theta} = \frac{h^2}{V^2 t_1 t_2\cos^2\theta}$$

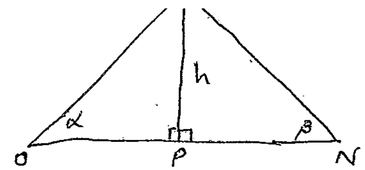
$$= \frac{h^2}{V^2\frac{2h}{g}\cos^2\theta} \quad [\text{by (a)}]$$

$$= \frac{gh}{2V^2\cos^2\theta} \quad //$$

Also, ON = r and LM = w

(d) RTP $r = h(\cot\alpha + \cot\beta)$ and $w = h(\cot\beta - \cot\alpha)$

(14)



$$\frac{OP}{h} = \cot\alpha \Rightarrow OP = h\cot\alpha$$

$$\frac{PN}{h} = \cot\beta \Rightarrow PN = h\cot\beta$$

$$\therefore r = ON = OP + PN = h\cot\alpha + h\cot\beta = h(\cot\alpha + \cot\beta) \quad //$$

By symmetry, QN = OP

$$\therefore PQ = ON - 2OP$$

$$= h(\cot\alpha + \cot\beta) - 2h\cot\alpha$$

$$= h(\cot\beta - \cot\alpha) \quad //$$

Finally, grad. of parabola @ L is $\tan\phi$

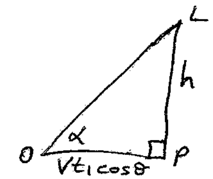
(e) RTP $\tan\phi = \tan\alpha - \tan\beta$

Now, $\dot{x} = V\cos\theta$ and $\dot{y} = V\sin\theta - gt$

$$\therefore \tan\phi = \frac{V\sin\theta - gt_1}{V\cos\theta}$$

$$= \tan\theta - \frac{g}{V\cos\theta}t_1$$

But, $\tan\alpha = \frac{h}{Vt_1\cos\theta} \Rightarrow t_1 = \frac{h}{V\cos\theta\tan\alpha}$



$$\therefore \tan\phi = \tan\theta - \frac{gh}{V^2\cos^2\theta\tan\alpha}$$

$$= \tan\theta - \frac{2\cancel{\tan\alpha}\tan\beta}{\cancel{\tan\alpha}} \quad \text{by (c)}$$

$$= \tan\alpha + \tan\beta - 2\tan\beta \quad \text{by (b)}$$

$$= \tan\alpha - \tan\beta \quad //$$

(f) RTP $\frac{w}{\tan\phi} = \frac{r}{\tan\theta}$

$$\frac{w}{\tan\phi} = \frac{h(\cot\beta - \cot\alpha)}{\tan\alpha - \tan\beta} \quad \text{by (d), (e)}$$

$$\frac{r}{\tan\theta} = \frac{h(\cot\alpha + \cot\beta)}{\tan\alpha + \tan\beta} \quad \text{by (b), (d)}$$

$$\therefore \frac{w}{\tan\phi} = \frac{h(\cot\beta - \cot\alpha)}{\tan\alpha - \tan\beta}\cdot\frac{\tan\alpha + \tan\beta}{\tan\alpha + \tan\beta} = \frac{h(\cancel{1} - \cancel{1} + \tan\alpha\cot\beta - \cot\alpha\tan\beta)}{\tan^2\alpha - \tan^2\beta} \quad \text{...(1)}$$

$$\text{and } \frac{r}{\tan\theta} = \frac{h(\cot\alpha + \cot\beta)}{\tan\alpha + \tan\beta}\cdot\frac{\tan\alpha - \tan\beta}{\tan\alpha - \tan\beta} = \frac{h(\cancel{1} - \cancel{1} + \tan\alpha\cot\beta - \cot\alpha\tan\beta)}{\tan^2\alpha - \tan^2\beta} \quad \text{...(2)}$$