BOARD OF STUDIES

NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

1999

MATHEMATICS

2/3 UNIT (COMMON)

Time allowed—Three hours
(Plus 5 minutes reading time)

Directions to Candidates

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Answer each question in a SEPARATE Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

QUESTION 1 Use a SEPARATE Writing Booklet. **Marks**

(a) The points A and B have coordinates $(3, -4)$ and $(7, 2)$ respectively. Find the coordinates of the midpoint of AB. **2**

(b) Find the value of e^3, correct to three significant figures. **2**

(c) Solve $3 - 2x \geq 7$. **2**

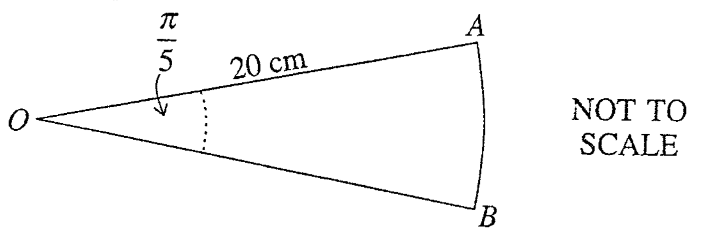
(d) Solve the simultaneous equations **2**

$$x + y = 1$$

$$2x - y = 5.$$

(e) Find integers a and b such that $\left(5 - \sqrt{2}\right)^2 = a + b\sqrt{2}$. **2**

(f) **2**



In the diagram, AB is an arc of a circle with centre O. The radius OA is 20 cm. The angle AOB is $\dfrac{\pi}{5}$ radians. Find the area of the sector AOB.

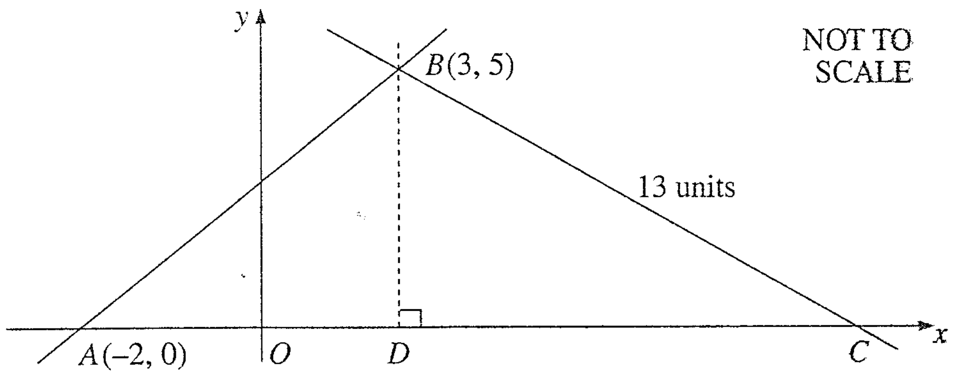
QUESTION 2 Use a SEPARATE Writing Booklet. **Marks**

(a) Find: **4**

(i) $\int \left(\frac{1}{x^2} + \frac{1}{x} \right) dx$

(ii) $\int \cos(2x+1)\, dx$.

(b) **8**



The diagram shows the points $A(-2, 0)$, $B(3, 5)$ and the point C which lies on the x axis. The point D also lies on the x axis such that BD is perpendicular to AC.

(i) Show that the gradient of AB is 1.

(ii) Find the equation of the line AB.

(iii) What is the size of $\angle BAC$?

(iv) The length of BC is 13 units. Find the length of DC.

(v) Calculate the area of ΔABC.

(vi) Calculate the size of $\angle ABC$, to the nearest degree.

QUESTION 3 Use a SEPARATE Writing Booklet. **Marks**

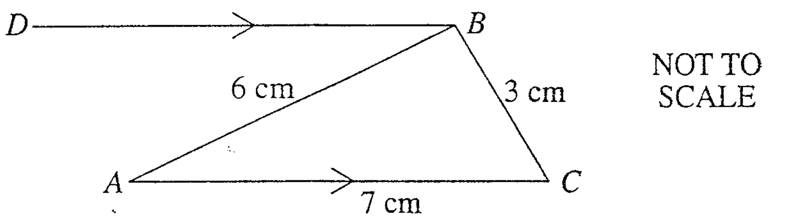
(a) Differentiate the following functions: **4**

(i) $x \tan x$

(ii) $\dfrac{e^x}{1+x}$.

(b) Find the equation of the normal to the curve $y = \sqrt{x+2}$ at the point (7, 3). **4**

(c) **4**



In the diagram, AC is parallel to DB, AB is 6 cm, BC is 3 cm and AC is 7 cm.

(i) Use the cosine rule to find the size of $\angle ACB$, to the nearest degree.

(ii) Hence find the size of $\angle DBC$, giving reasons for your answer.

QUESTION 4 Use a SEPARATE Writing Booklet. **Marks**

(a) An infinite geometric series has a first term of 8 and a limiting sum of 12. Calculate the common ratio. **2**

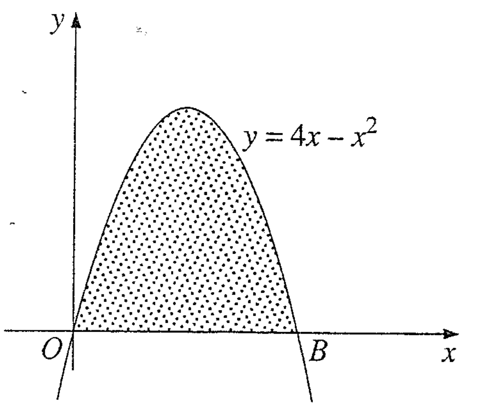
(b) A market gardener plants cabbages in rows. The first row has 35 cabbages. The second row has 39 cabbages. Each succeeding row has 4 more cabbages than the previous row. **5**

(i) Calculate the number of cabbages in the 12th row.

(ii) Which row would be the first to contain more than 200 cabbages?

(iii) The farmer plants only 945 cabbages. How many rows are needed?

(c) **5**



The diagram shows the graph of the function $y = 4x - x^2$.

(i) Find the x coordinate of the point B where the curve crosses the positive x axis.

(ii) Find the area of the shaded region contained by the curve $y = 4x - x^2$ and the x axis.

(iii) Write down a pair of inequalities that specify the shaded region.

QUESTION 5 Use a SEPARATE Writing Booklet. **Marks**

(a) Consider the curve given by $y = x^3 - 6x^2 + 9x + 1$. **8**

(i) Find $\frac{dy}{dx}$.

(ii) Find the coordinates of the two stationary points.

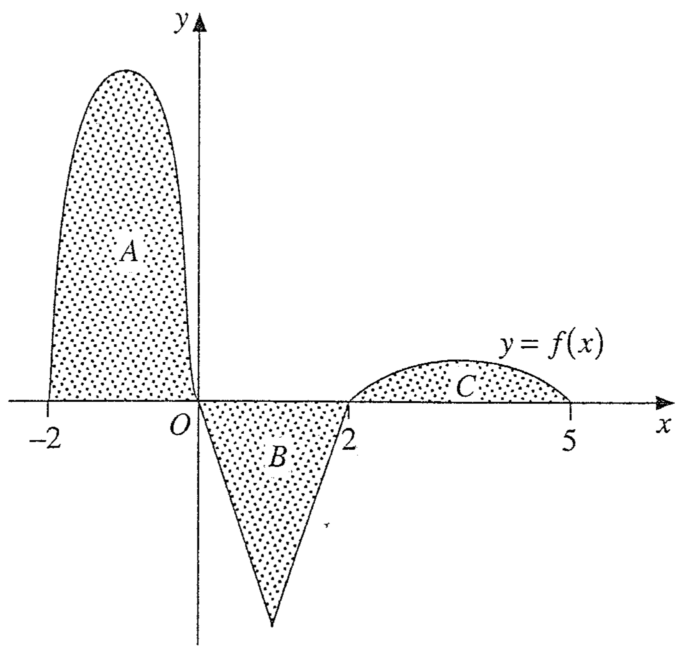
(iii) Determine the nature of the stationary points.

(iv) Sketch the curve for $x \geq 0$.

(b) Let $\log_a 2 = x$ and $\log_a 3 = y$.

Find an expression for $\log_a 12$ in terms of x and y. **2**

(c) **2**



The graph of the function f is shown in the diagram. The shaded areas are bounded by $y = f(x)$ and the x axis. The shaded area A is 8 square units, the shaded area B is 3 square units and the shaded area C is 1 square unit.

Evaluate $\int_{-2}^{5} f(x)\,dx$.

QUESTION 6 Use a SEPARATE Writing Booklet. **Marks**

(a) The mass M kg of a radioactive substance present after t years is given by $M = 10e^{-kt}$, where k is a positive constant. After 100 years the mass has reduced to 5 kg. **7**

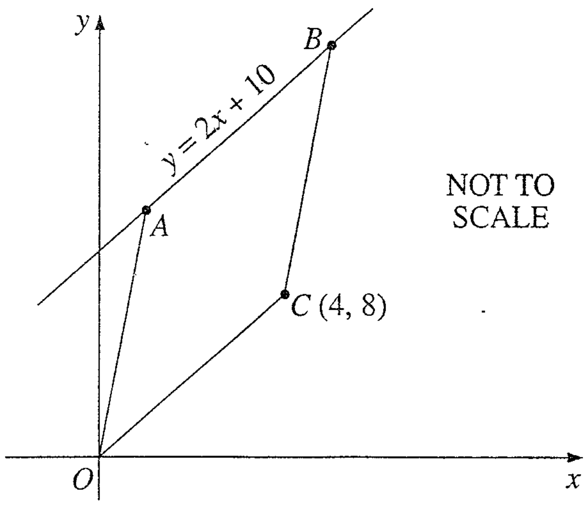
(i) What was the initial mass?

(ii) Find the value of k.

(iii) What amount of the radioactive substance would remain after a period of 1000 years?

(iv) How long would it take for the initial mass to reduce to 8 kg?

(b) **5**



The equation of AB is $y = 2x + 10$. The point C is $(4, 8)$.

Copy or trace the diagram into your Writing Booklet.

(i) Show that OC and AB are parallel.

(ii) State why $\angle ABO = \angle BOC$.

(iii) The line OB divides the quadrilateral $OABC$ into two congruent triangles. Prove that $OABC$ is a parallelogram.

QUESTION 7 Use a SEPARATE Writing Booklet. **Marks**

(a) Isabella invests $\$P$ at 8% per annum compounded annually. She intends to withdraw \$3000 at the end of each of the next six years to cover school fees. **5**

(i) Write down an expression for the amount $\$A_1$ remaining in the account following the withdrawal of the first \$3000.

(ii) Find an expression for the amount $\$A_2$ remaining in the account after the second withdrawal.

(iii) Calculate the amount $\$P$ that Isabella needs to invest if the account balance is to be \$0 at the end of six years.

(b) A particle P is moving along the x axis. Its position at time t seconds is given by **7**

$$x = 2\sin t - t, \qquad t \geq 0.$$

(i) Find an expression for the velocity of the particle.

(ii) In what direction is the particle moving at $t = 0$?

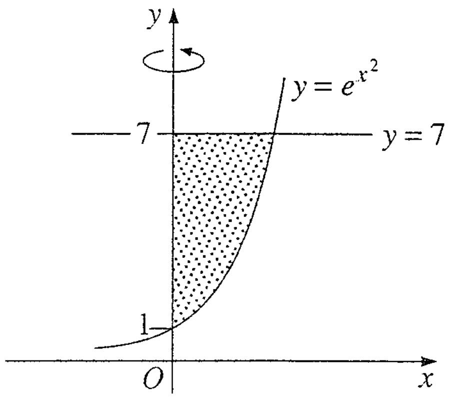
(iii) Determine when the particle first comes to rest.

(iv) When is the acceleration negative for $0 \leq t \leq 2\pi$?

(v) Calculate the total distance travelled by the particle in the first π seconds.

QUESTION 8 Use a SEPARATE Writing Booklet. **Marks**

(a) **5**



The shaded region bounded by $y = e^{x^2}$, $y = 7$ and the y axis is rotated around the y axis to form a solid.

(i) Show that the volume of the solid is given by $V = \pi \int_1^7 \log_e y \, dy$.

(ii) Copy and complete the table. Give your answers correct to 3 decimal places.

y	1	4	7
$\log_e y$			

(iii) Use the trapezoidal rule with 3 function values to approximate the volume, V.

(b) A box contains five cards. Each card is labelled with a number. The numbers on the cards are 0, 3, 3, 5, 5. **5**

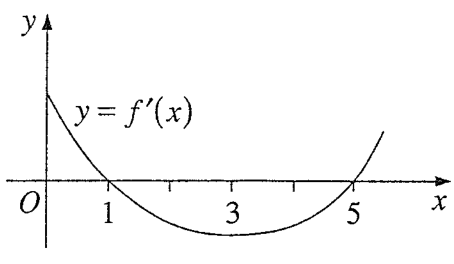
Cameron draws one card at random from the box and then draws a second card at random *without* replacing the first card drawn.

(i) What is the probability that he draws a '5', then a '3'?

(ii) What is the probability that the sum of the two numbers drawn is at least 8?

(iii) What is the probability that the second card drawn is labelled '3'?

(c) **2**

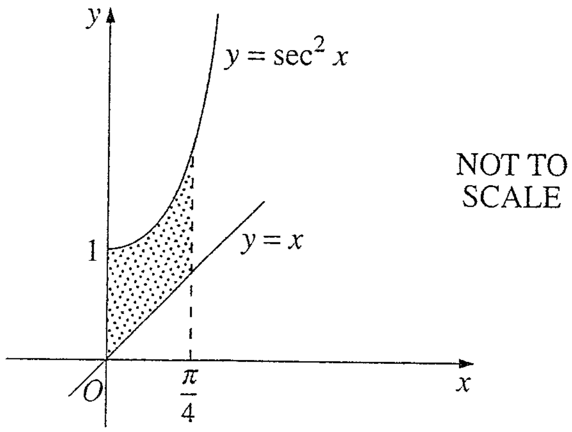


The diagram shows the graph of the gradient function of the curve $y = f(x)$.

For what value of x does $f(x)$ have a local minimum? Justify your answer.

QUESTION 9 Use a SEPARATE Writing Booklet. **Marks**

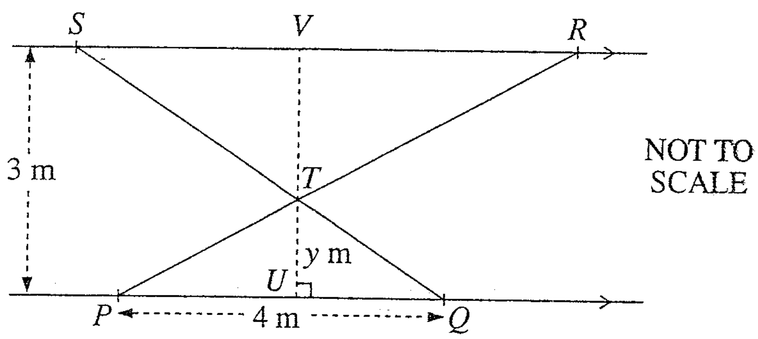
(a) 3



The diagram shows the graphs of the functions $y = \sec^2 x$ and $y = x$ between $x = 0$ and $x = \frac{\pi}{4}$.

Calculate the area of the shaded region.

(b) 9



In the diagram, PQ and SR are parallel railings which are 3 metres apart. The points P and Q are fixed 4 metres apart on the lower railing. Two crossbars PR and QS intersect at T as shown in the diagram. The line through T perpendicular to PQ intersects PQ at U and SR at V. The length of UT is y metres.

(i) By using similar triangles, or otherwise, show that $\dfrac{SR}{PQ} = \dfrac{VT}{UT}$.

(ii) Show that $SR = \dfrac{12}{y} - 4$.

(iii) Hence show that the total area A of ΔPTQ and ΔRTS is $A = 4y - 12 + \dfrac{18}{y}$.

(iv) Find the value of y that minimises A. Justify your answer.

Marks

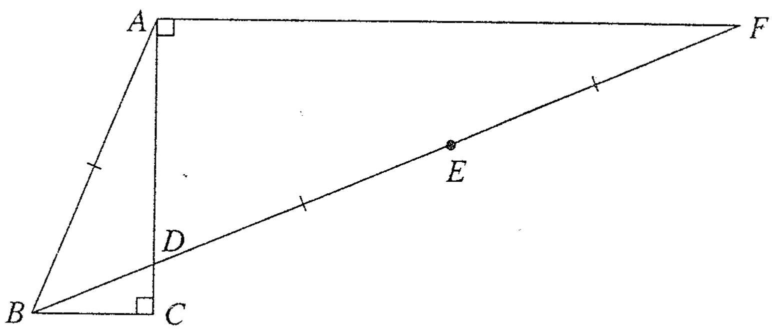
(a) (i) Show that $x = \frac{\pi}{3}$ is a solution of $\sin x = \frac{1}{2}\tan x$. **6**

(ii) On the same set of axes, sketch the graphs of the functions $y = \sin x$ and $y = \frac{1}{2}\tan x$ for $-\pi \le x \le \pi$.

(iii) Hence find all solutions of $\sin x = \frac{1}{2}\tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(iv) Use your graphs to solve $\sin x \le \frac{1}{2}\tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(b) **6**



In the diagram, $AC \perp BC$, $AC \perp AF$ and $AB = DE = EF$.

Copy or trace the diagram into your Writing Booklet.

(i) Show that $\angle DBC = \angle DFA$.

(ii) On your diagram, mark the point G on the line AF such that $EG \parallel AC$.

Show that $\Delta AGE \equiv \Delta FGE$.

(iii) Prove that $\angle ABD = 2\angle DBC$.

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

1999 HIGHER SCHOOL CERTIFICATE SOLUTIONS

2/3 UNIT (COMMON) MATHEMATICS

QUESTION 1

(a) $A(3,-4)$, $B(7,2)$

$$\text{Midpoint} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$
$$= \left(\frac{3+7}{2}, \frac{-4+2}{2}\right)$$
$$= (5,-1).$$

(b) $e^3 = 20{\cdot}085\,536\,92 = 20{\cdot}1$ (3 sig. figs).

(c)
$$3-2x \ge 7$$
$$-2x \ge 4$$
$$x \le -2.$$

(d)
$$x+y = 1 \quad \text{—①}$$
$$2x-y = 5 \quad \text{—②}$$
①+②:
$$3x = 6$$
$$x = 2.$$
Substitute in ①:
$$2+y = 1$$
$$y = -1.$$
Solution: $x=2$, $y=-1$.

(e)
$$\left(5-\sqrt{2}\right)^2 = a+b\sqrt{2}$$
$$\left(5-\sqrt{2}\right)^2 = (5)^2 - 2\times5\times\sqrt{2} + \left(\sqrt{2}\right)^2$$
$$= 25-10\sqrt{2}+2$$
$$= 27-10\sqrt{2}.$$
$$\therefore\ 27-10\sqrt{2} = a+b\sqrt{2}$$
$$a = 27,\ b = -10.$$

(f)
$$\text{Area} = \frac{1}{2}r^2\theta = \frac{1}{2}\times20^2\times\frac{\pi}{5}$$
$$= 40\pi\ \text{cm}^2$$
$$= 126\ \text{cm}^2\ \left(\text{nearest cm}^2\right).$$

QUESTION 2

(a) (i)
$$\int\left(\frac{1}{x^2}+\frac{1}{x}\right)dx = \int\left(x^{-2}+\frac{1}{x}\right)dx$$
$$= \frac{x^{-1}}{-1}+\ln x + c$$
$$= \frac{-1}{x}+\ln x + c.$$

(ii)
$$\int\cos(2x+1)\,dx = \frac{1}{2}\int 2\cos(2x+1)\,dx$$
$$= \frac{1}{2}\sin(2x+1)+C.$$

(b) (i) Gradient $AB = \dfrac{5-0}{3-(-2)} = \dfrac{5}{5} = 1.$

(ii)
$$y-y_1 = m(x-x_1)$$
$$y-0 = 1[x-(-2)]$$
$$y = x+2.$$

(iii) $\tan\angle BAC = \text{gradient } AB = 1.$

$\therefore\ \angle BAC = 45°.$

OR In ΔABD, $\angle D = 90°$, $AD = 5$, $BD = 5$.

$\therefore\ \Delta ABD$ is isosceles and right angled,

$\therefore\ \angle BAC = 45°.$

(iv) BD equals 5; BC, the hypotenuse of ΔBDC, equals 13; so by Pythagoras, DC equals 12.

(v)
$$\text{Area } \Delta ABC = \frac{1}{2}\times17\times5 \text{ sq. units}$$
$$= 42{\cdot}5 \text{ sq. units.}$$

(vi)
$$\cos\angle CBD = \frac{5}{13}$$
$$\therefore\ \angle CBD = 67°$$
$$\therefore\ \angle ABC = 45° + 67°$$
$$= 112°.$$

OR
$$\frac{\sin\angle ABC}{17} = \frac{\sin\angle BAC}{13}$$
$$\sin\angle ABC = \frac{17\sin 45}{13}$$
$$= \frac{17}{13\sqrt{2}}.$$
$$\therefore\ \angle ABC = 68° \text{ or } 180° - 68°.$$
But $\angle ABC$ is obtuse,
$$\therefore\ \angle ABC = 180° - 68°$$
$$= 112°.$$

QUESTION 3

(a) (i) $\frac{d}{dx}(x\tan x) = \tan x \times 1 + x \times \sec^2 x$
$= \tan x + x\sec^2 x.$

(ii) $\frac{d}{dx}\left(\frac{e^x}{1+x}\right) = \frac{(1+x)\times e^x - e^x \times 1}{(1+x)^2}$
$= \frac{e^x + xe^x - e^x}{(1+x)^2}$
$= \frac{xe^x}{(1+x)^2}.$

(b) $y = \sqrt{x+2} = (x+2)^{\frac{1}{2}}.$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(x+2)^{-\frac{1}{2}} \times 1$
$= \frac{1}{2\sqrt{x+2}}.$

Let m_1 = gradient of tangent at (7, 3)
and m_2 = gradient of normal at (7, 3).

$\therefore \quad m_1 = \frac{1}{2\sqrt{7+2}}$
$= \frac{1}{2\sqrt{9}}$
$= \frac{1}{6}.$

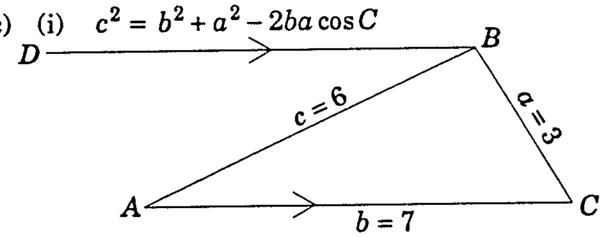
Now $m_2 = -\frac{1}{m_1}$ (for $\perp$ lines)
$= -6.$

Equation of normal is $y - y_1 = m_2(x - x_1)$

$\therefore \quad y - 3 = -6(x-7)$
$= -6x + 42$

$\therefore \quad y = -6x + 45$ or $6x + y - 45 = 0.$

(c) (i) $c^2 = b^2 + a^2 - 2ba\cos C$



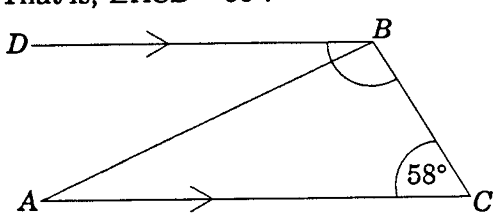
$6^2 = 7^2 + 3^2 - 2 \times 7 \times 3 \cos C$

$\cos C = \frac{7^2 + 3^2 - 6^2}{2 \times 7 \times 3}$

$\therefore \quad \angle C = 58{\cdot}411\,864\,49°$
$= 58°$ (nearest degree).

That is, $\angle ACB = 58°$.

(ii)



$\angle DBC + 58° = 180°$ (cointerior $\angle$s, $AC \| DB$)

$\therefore \quad \angle DBC = 180° - 58°$
$= 122°.$

QUESTION 4

(a) $a = 8, \quad S = 12$

Now $S = \frac{a}{1-r}, \quad \therefore \quad 12 = \frac{8}{1-r}$

$12(1-r) = 8$
$12 - 12r = 8$
$-12r = -4$
$r = \frac{-4}{-12}$
$= \frac{1}{3}.$

That is, the common ratio is $\frac{1}{3}$.

(b) 35
39 = 35+4

This is an arithmetic sequence with $a = 35$, $d = 4$.

(i) $T_n = a + (n-1)d$
$T_{12} = 35 + 11 \times 4$
$= 79.$

(ii) $T_n > 200, \qquad n = ?$

$\therefore \quad a + (n-1)d > 200$
$35 + (n-1)\times 4 > 200$
$35 + 4n - 4 > 200$
$4n > 169$
$n > 42\frac{1}{4}.$

Since n is a whole number, the 43rd row will be the first row to contain more than 200 cabbages.

(iii) $S_n = 945, \qquad S_n = \frac{n}{2}[2a + (n-1)d]$

$\therefore \qquad 945 = \frac{n}{2}[2 \times 35 + (n-1)\times 4]$
$= \frac{n}{2}(70 + 4n - 4)$
$= \frac{n}{2}(66 + 4n)$
$= 33n + 2n^2.$

$\therefore 2n^2 + 33n - 945 = 0$
$(2n + 63)(n - 15) = 0$
$\therefore n = 15$ or $-31{\cdot}5.$

But $n > 0$, so $n = 15$.

That is, 15 rows are needed for a total of 945 cabbages.

(c) (i) $y = 4x - x^2$

At B, $y = 0$, $\quad \therefore \quad 4x - x^2 = 0$
$x(4-x) = 0$
$x = 0$ or 4.

But $x = 0$ is the origin,
$\therefore$ at B, $x = 4$.

(ii) $$\text{Area} = \int_0^4 \left(4x - x^2\right) dx$$
$$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$$
$$= \left(2\times 4^2 - \frac{4^3}{3}\right) - (0)$$
$$= \left(32 - 21\tfrac{1}{3}\right)$$
$$= 10\tfrac{2}{3} \text{ square units.}$$

(iii) $y \ge 0,\ \ y \le 4x - x^2.$

(N.B. $y > 0,\ \ y < 4x - x^2$ is also correct.)

QUESTION 5

(a) $y = x^3 - 6x^2 + 9x + 1$

(i) $$\frac{dy}{dx} = 3x^2 - 12x + 9$$
$$= 3\left(x^2 - 4x + 3\right).$$

(ii) Stationary points when $\frac{dy}{dx} = 0$.

That is, $\left(x^2 - 4x + 3\right) = 0$

$(x-3)(x-1) = 0.$

$\therefore\ x = 3$ or $x = 1$.

Stationary points are $(3,\ 1)$ and $(1,\ 5)$.

(iii) $$\frac{d^2y}{dx^2} = 6x - 12$$

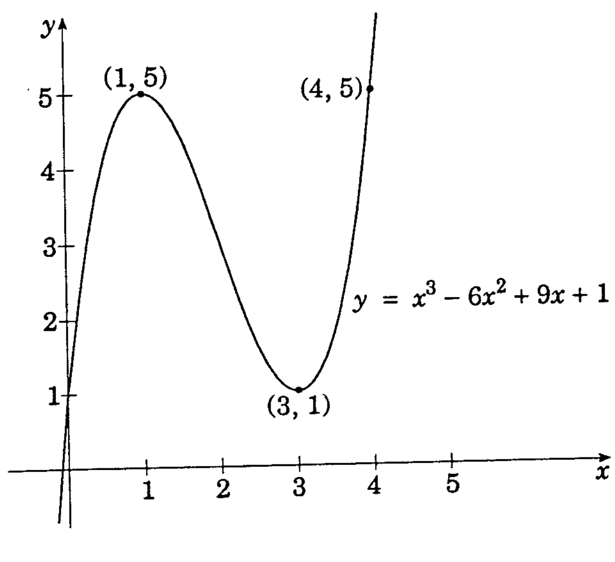
When $x = 1$, $\frac{d^2y}{dx^2} = 6 - 12 < 0$,

$\therefore$ the curve is concave down and $(1,\ 5)$ is a relative maximum.

When $x = 3$, $\frac{d^2y}{dx^2} = 18 - 12 > 0$,

$\therefore$ the curve is concave up and $(3,\ 1)$ is a relative minimum.

(iv)



(b) $\log_a 2 = x, \quad \log_a 3 = y$
$$\log_a 12 = \log_a 4\times 3$$
$$= \log_a 4 + \log_a 3$$
$$= \log_a 2^2 + \log_a 3$$
$$= 2\log_a 2 + \log_a 3$$
$$= 2x + y.$$

There are other possible answers (for example, $x\log_2 12$), but this is the simplest and most obvious.

(c) $$\int_{-2}^{5} f(x)\, dx = 8 - 3 + 1$$
$$= 6.$$

QUESTION 6

(a) $M = 10e^{-kt}$

(i) Initial mass occurs when $t = 0$,

that is, $M = 10e^{-k\times 0}$
$= 10.$

So the initial mass is 10 kg.

(ii) $t = 100,\ M = 5$
$$M = 10e^{-kt}$$
$$\therefore\quad 5 = 10e^{-100k}$$
$$e^{-100k} = 0{\cdot}5$$
$$\ln e^{-100k} = \ln 0{\cdot}5$$
$$-100k = \ln 0{\cdot}5$$
$$k = \frac{\ln 0{\cdot}5}{-100}$$
$$= 6{\cdot}931\,471\,806\times 10^{-3}$$
$$= 0{\cdot}006\,93 \text{ (3 sig. figs).}$$

(iii) $t = 1000,\ M = ?$
$$M = 10e^{-k\times 1000}$$
$$= 0{\cdot}009\,765\,625$$
$$= 0{\cdot}009\,766 \text{ kg (4 sig. figs)}$$
$$= 9{\cdot}766 \text{ g.}$$

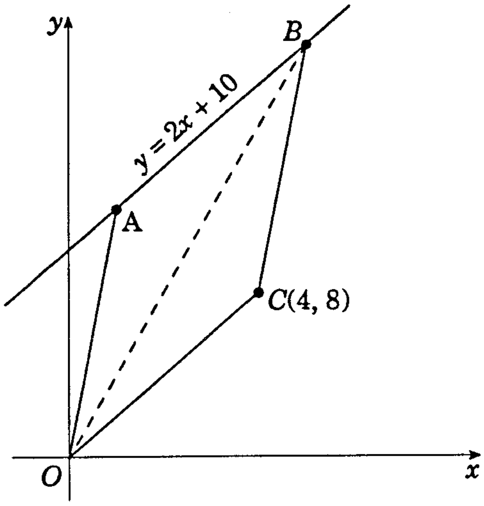
OR Since the mass halves in 100 years, it will halve ten times in 1000 years.

$\therefore\quad \text{Mass} = \frac{10}{2^{10}} = 0{\cdot}009\,766$ kg.

(iv) $t = ?,\ M = 8$
$$8 = 10e^{-kt}$$
$$0{\cdot}8 = e^{-kt}$$
$$\ln 0{\cdot}8 = \ln e^{-kt}$$
$$\ln 0{\cdot}8 = -kt$$
$$t = \frac{\ln 0{\cdot}8}{-k}$$
$$= \frac{\ln 0{\cdot}8\times 100}{\ln 0{\cdot}5}$$
$$= 32{\cdot}192\,809\,49$$
$$= 32{\cdot}2 \text{ years (1 dec. pl.).}$$

That is, it will take 32·2 years to reduce to 8 kg.

(b)



(i) Equation of AB is $y = 2x + 10$,
$\therefore$ gradient = 2.

$$\text{Gradient } OC = \frac{8-0}{4-0} = 2 = \text{gradient } AB.$$

$\therefore OC \parallel AB$.

(ii) $\angle ABO = \angle BOC$ because they are alternate angles between parallel sides AB and OC.

(iii) Assume that in the congruent triangles ABO matches COB. With this assumption any of the following three arguments complete the proof.

$AO = BC$ (corresponding sides of congruent Δs)

$AB = CO$ (corresponding sides of congruent Δs)

$\therefore OABC$ is a parallelogram (both pairs of opposite sides equal).

OR $\angle AOB = \angle CBO$ (corresponding angles of congruent Δs)

$OA \parallel CD$ (alternate angles equal)

$\therefore OABC$ is a parallelogram (both pairs of opposite sides parallel).

OR $AB = CO$ (corresponding sides of congruent Δs)

$\therefore OABC$ is a parallelogram (one pair of sides, AB and OC, are both parallel and equal).

There is a logic problem with this answer. The question does not actually say that ABO matches COB. Since there are no other restrictions on the points A and B, it is possible that the match could be done differently.

There are 5 other possibilities to consider. Two of them (ABO matches BCO and ABO matches OBC) are impossible, since the triangles would have to be equilateral.

The other three *are* possible. However, they all lead to isosceles triangles, from which one can deduce that a congruence with ABO matching COB is *also* possible. One can then complete the proof as above.

QUESTION 7

(a) (i) $\$A_1 = \$P \times 1{\cdot}08 - \$3000$

(ii)
$$\begin{aligned}\$A_2 &= (\$P \times 1{\cdot}08 - \$3000)1{\cdot}08 - 3000 \\ &= \$P \times 1{\cdot}08^2 - \$3000 \times 1{\cdot}08 - 3000 \\ &= \$P \times 1{\cdot}08^2 - \$3000(1 + 1{\cdot}08).\end{aligned}$$

(iii)
$$\$A_6 = \$P \times 1{\cdot}08^6 - \$3000\left(1 + 1{\cdot}08 + \cdots + 1{\cdot}08^5\right) = 0.$$

$$\therefore \quad \$P \times 1{\cdot}08^6 = \$3000\left(\frac{1{\cdot}08^6 - 1}{1{\cdot}08 - 1}\right)$$

$$\$P = \frac{\$3000\left(1{\cdot}08^6 - 1\right)}{1{\cdot}08^6 \times 0{\cdot}08} = \$13\,868{\cdot}64.$$

OR Investment $= P\left(1{\cdot}08^6\right)$.

Withdrawal
$$\begin{aligned}&= 3000 + 3000 \times 1{\cdot}08 + 3000 \times 1{\cdot}08^2 + \cdots + 3000 \times 1{\cdot}08^5 \\ &= 3000\left(1 + 1{\cdot}08 + 1{\cdot}08^2 + \cdots + 1{\cdot}08^5\right).\end{aligned}$$

That is, GP where $a = 1$, $r = 1{\cdot}08$, $n = 6$,

$$S_6 = \frac{3000\left(1{\cdot}08^6 - 1\right)}{0{\cdot}08} = 22\,007{\cdot}79.$$

Amount invested will be enough when investment = withdrawals.

That is, $P = \dfrac{2207{\cdot}79}{1{\cdot}08^6}$

$\therefore \quad \$P = \$13\,868{\cdot}64$.

(b) $x = 2\sin t - t, \quad t \geq 0$.

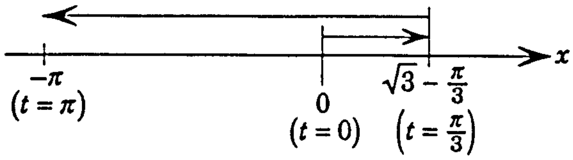
(i) $v = \dfrac{dx}{dt} = 2\cos t - 1$.

(ii) At $t = 0$, $v = 2 - 1 = 1$,
$\therefore$ the particle is moving in a positive direction.

(iii)
$$\begin{aligned}2\cos t - 1 &= 0 \\ \cos t &= \frac{1}{2} \\ t &= \frac{\pi}{3} \text{ seconds.}\end{aligned}$$

(iv) $a = \dfrac{dv}{dt} = -2\sin t$.

Acceleration is negative when $0 < t < \pi$.



$$\text{Total distance} = \left(\sqrt{3} - \frac{\pi}{3}\right) + \left(\pi + \sqrt{3} - \frac{\pi}{3}\right)$$
$$= 2\sqrt{3} + \frac{\pi}{3}.$$

QUESTION 8

(a) (i) $y = e^{x^2}$

$x^2 = \log_e y$

Rotating about the y axis,

$$V = \pi \int_1^7 x^2 \, dy$$
$$= \pi \int_1^7 \log_e y \, dy.$$

(ii)

y	1	4	7
$\log_e y$	0	1·386	1·946

(iii) $V = \pi \int_1^7 \log_e y \, dy$

$$\doteqdot \pi \times \frac{3}{2}\left[(0 + 1{\cdot}386) + (1{\cdot}386 + 1{\cdot}946)\right].$$

$\therefore$ Volume $\doteqdot$ 22·2 cubic units.

(b) (i) $\text{P(5 then 3)} = \text{P}(5) \times \text{P}(3)$

$$= \frac{2}{5} \times \frac{2}{4}$$
$$= \frac{1}{5}.$$

(ii) P(sum at least 8)

$= \text{P(sum 8)} + \text{P(sum 10)}$

$= \text{P(5, 3 or 3, 5)} + \text{P(5, 5)}$

$$= \frac{1}{5} + \frac{1}{5} + \frac{2}{5} \times \frac{1}{4}$$
$$= \frac{1}{2}.$$

(iii) The probability that the second card is a 3 is the same as the probability that the first card is a 3, that is, $\frac{2}{5}$.

OR $\text{P}(x, 3) = \text{P}(0, 3) + \text{P}(3, 3) + \text{P}(5, 3)$

$$= \frac{1}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{1}{4} + \frac{2}{5} \times \frac{2}{4}$$
$$= \frac{2}{5}.$$

(c) Stationary points occur when $f'(x) = 0$, that is, at $x = 1$ and at $x = 5$.

To find the minimum, use the first derivative or second derivative test.

First derivative test:

	$x < 1$	$x = 1$	$1 < x < 5$	$x = 5$	$x > 5$
$f'(x)$	> 0	0	< 0	0	> 0
$f(x)$	/	—	\	—	/

There is a local minimum at $x = 5$.

Second derivative test:

At $x = 1$, $f'(x)$ is decreasing so the curve is concave down (that is, $f''(x) < 0$). This is a local maximum.

At $x = 5$, $f'(x)$ is increasing so the curve is concave up (that is, $f''(x) > 0$). This is a local minimum.

QUESTION 9

(a) $A = \int_0^{\frac{\pi}{4}} \sec^2 x \, dx - A\Delta$

$$= \left[\tan x\right]_0^{\frac{\pi}{4}} - \frac{1}{2} \times \left(\frac{\pi}{4}\right)^2$$
$$= \tan\frac{\pi}{4} - \tan 0 - \frac{\pi^2}{32}$$
$$= 1 - 0 - \frac{\pi^2}{32}$$
$$= 1 - \frac{\pi^2}{32}$$
$$= 0{\cdot}692 \text{ (3 sig. figs).}$$

(b) (i)
1. $\angle STR = \angle PTQ$ (vert. opp.)
2. $\angle SRT = \angle TPQ$ (alt. $\angle$s, $SR \parallel PQ$)

$\therefore \Delta SRT \,|||\, \Delta QPT$ (equal $\angle$s)

$\therefore \dfrac{SR}{PQ} = \dfrac{VT}{UT}$ (ratios of corresp. lengths in similar Δs. N.B. VT and UT are corresp. altitudes).

(ii)
$$\frac{SR}{4} = \frac{3-y}{y}$$
$$SR = \frac{12-4y}{y}$$
$$\therefore SR = \frac{12}{y} - 4.$$

(iii) $A(\Delta PTQ + \Delta RTS)$

$$= \frac{1}{2}\left[SR \times (3-y) + 4y\right]$$
$$= \frac{1}{2}\left[\left(\frac{12}{y} - 4\right)(3-y) + 4y\right]$$
$$= \left(\frac{6}{y} - 2\right)(3-y) + 2y$$
$$= \frac{18}{y} - 6 - 6 + 2y + 2y$$
$$= 4y - 12 + \frac{18}{y}.$$

(iv) Minimum value of A occurs when $\frac{dA}{dy} = 0$ and $\frac{d^2A}{dy^2} > 0$.

$A = 4y - 12 + 18y^{-1}$

$\therefore \quad \frac{dA}{dy} = 4 - 18y^{-2}$

$\frac{d^2A}{dy^2} = 36y^{-3}$.

$\frac{dA}{dy} = 0, \quad \therefore \quad 4 - \frac{18}{y^2} = 0$

$4y^2 - 18 = 0$

$y^2 = \frac{18}{4}$

$y = \pm\frac{3\sqrt{2}}{2}$.

But $y > 0$, so $y = \frac{3\sqrt{2}}{2}$.

When $y = \frac{3\sqrt{2}}{2}$, $\frac{d^2A}{dy^2} = \frac{36}{\left(\frac{3\sqrt{2}}{2}\right)^3} > 0$.

$\therefore$ The value of y that minimises A is $\frac{3\sqrt{2}}{2}$ metres.

QUESTION 10

(a) (i) LHS: $\sin x = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

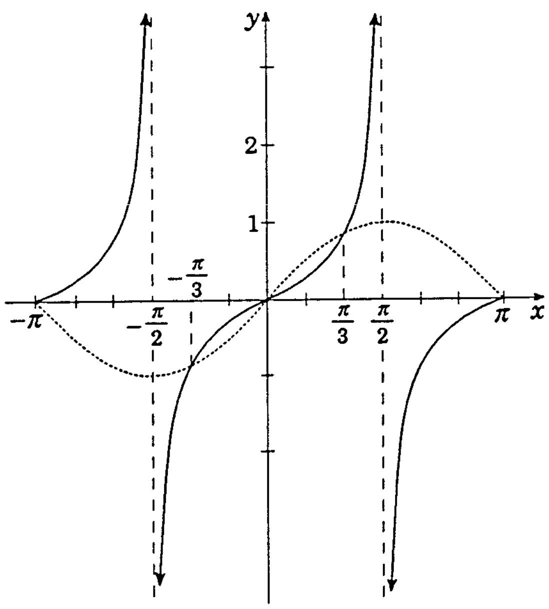
RHS: $\frac{1}{2}\tan x = \frac{1}{2}\tan\frac{\pi}{3}$

$= \frac{1}{2}\sqrt{3}$

$= \frac{\sqrt{3}}{2} = \text{LHS}$.

That is, $\sin x = \frac{1}{2}\tan x$ when $x = \frac{\pi}{3}$.

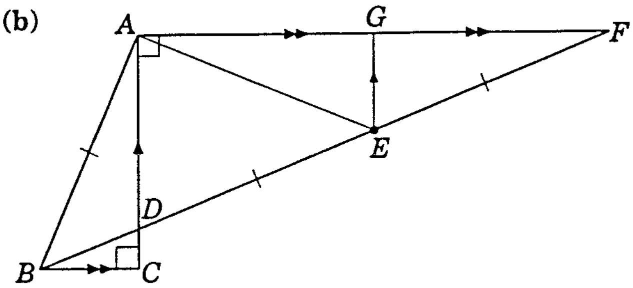
(ii)



(iii) Solutions are $x = 0, \frac{\pi}{3}, -\frac{\pi}{3}$.

(iv) $\sin x \le \frac{1}{2}\tan x$ for $\frac{\pi}{3} < x \le \frac{\pi}{2}$ or $-\frac{\pi}{3} \le x \le 0$.

(b)



(i) $\angle BCD = \angle DAF$ (both 90°)

$\angle BDC = \angle ADF$ (vertically opposite $\angle$s)

$\therefore \ \Delta DBC \parallel\!\mid \Delta DFA$ (equal $\angle$s)

$\therefore \ \angle DBC = \angle DFA$ (corresponding $\angle$s in similar Δs).

OR $\angle BCD = \angle DAF$ (both 90°)

$\therefore \ AF \| BC$ (alternate $\angle$s equal)

$\therefore \ \angle DBC = \angle DFA$ (alternate $\angle$s, $AF \| BC$).

(ii) $\angle AGE = 90°$ (coint. with $\angle GAD$, $AD \| GE$)

$\angle EGF = 90°$ ($\angle$ on a straight line)

$\therefore \ \angle AGE = \angle EGF$ (both 90°).

Since E is the midpoint of DF, G must be be the midpoint of AF (by equal intercept theorem, given that $GE \| AD$).

$\therefore \ AG = GF$

$GE = GE$ (common)

$\therefore \ \Delta AGE \equiv \Delta FGE$ (SAS).

(iii) Let $\angle DBC = \alpha$

$\angle AFE = \alpha$ (from (i))

$\angle EAF = \angle AFE$ (corresp. $\angle$s, $\Delta AGE \equiv \Delta FGE$)

$= \alpha$,

$\therefore \ \angle AED = 2\alpha$ (ext. $\angle$ of Δ equals sum of int. opp. $\angle$s).

Now $AE = EF$ (corresp. sides, $\Delta AGE \equiv \Delta FGE$)

$\therefore \ AB = AE$ (both equal to EF)

$\therefore \ \angle ABD = \angle AED$ (base $\angle$s of isos. Δ)

$= 2\alpha$

$= 2\angle DBC$.

END OF 2/3 UNIT (COMMON) MATHEMATICS SOLUTIONS