BOARD OF STUDIES
NEW SOUTH WALES

2004

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Use integration by parts to find $\int xe^{3x}\,dx$. **2**

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\,dx$. **3**

(c) By completing the square, find $\int \frac{dx}{\sqrt{5+4x-x^2}}$. **2**

(d) (i) Find real numbers a and b such that **2**

$$\frac{x^2-7x+4}{(x+1)(x-1)^2} \equiv \frac{a}{x+1} + \frac{b}{x-1} - \frac{1}{(x-1)^2}.$$

(ii) Hence find $\int \frac{x^2-7x+4}{(x+1)(x-1)^2}\,dx$. **2**

(e) Use the substitution $x = 2\sin\theta$ to find $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$. **4**

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = 1 + 2i$ and $w = 3 - i$.

Find, in the form $x + iy$,

(i) zw **1**

(ii) $\overline{\left(\frac{10}{z}\right)}$. **1**

(b) Let $\alpha = 1 + i\sqrt{3}$ and $\beta = 1 + i$.

(i) Find $\frac{\alpha}{\beta}$, in the form $x + iy$. **1**

(ii) Express α in modulus-argument form. **2**

(iii) Given that β has the modulus-argument form **1**

$$\beta = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$$

find the modulus-argument form of $\frac{\alpha}{\beta}$.

(iv) Hence find the exact value of $\sin\frac{\pi}{12}$. **1**

(c) Sketch the region in the complex plane where the inequalities **3**

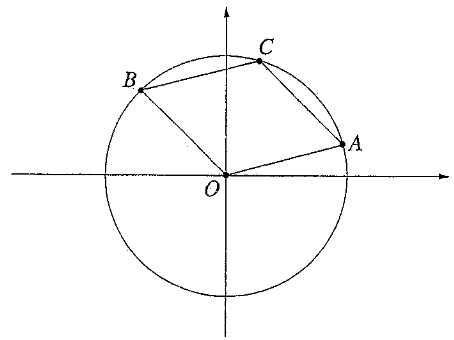
$$|z + \bar{z}| \le 1 \text{ and } |z - i| \le 1$$

hold simultaneously.

Question 2 continues on page 4

Question 2 (continued) **Marks**

(d) The diagram shows two distinct points A and B that represent the complex numbers z and w respectively. The points A and B lie on the circle of radius r centred at O. The point C representing the complex number $z + w$ also lies on this circle.



Copy the diagram into your writing booklet.

(i) Using the fact that C lies on the circle, show geometrically that **2**

$$\angle AOB = \frac{2\pi}{3}.$$

(ii) Hence show that $z^3 = w^3$. **2**

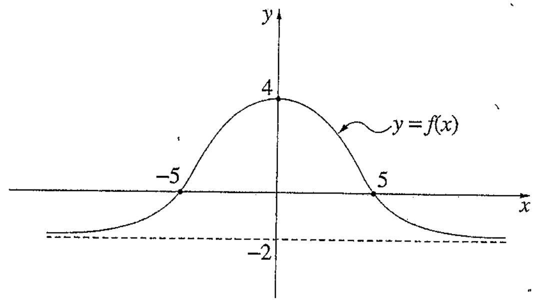
(iii) Show that $z^2 + w^2 + zw = 0$. **1**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Sketch the curve $y = \dfrac{4x^2}{x^2 - 9}$ showing all asymptotes. **3**

(b) The diagram shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = |f(x)|$ **2**

(ii) $y = (f(x))^2$ **2**

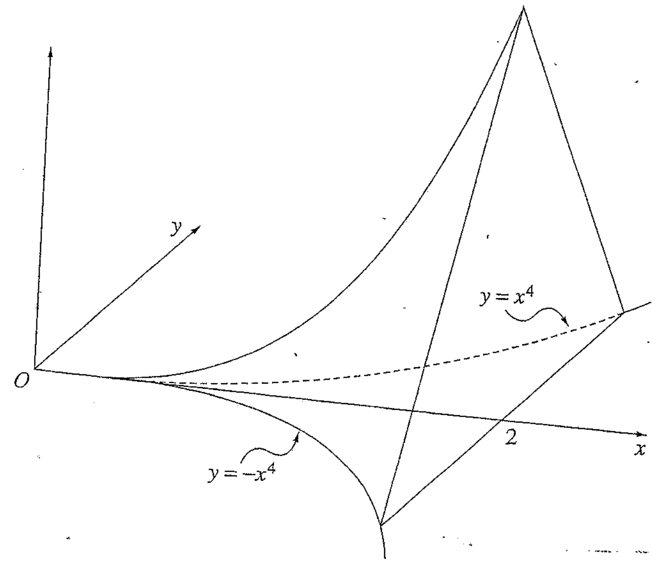
(iii) $y = \dfrac{1}{\sqrt{f(x)}}$. **2**

(c) Find the equation of the tangent to the curve defined by $x^2 - xy + y^3 = 5$ at the point $(2, -1)$. **3**

Question 3 continues on page 6

Question 3 (continued) **Marks**

(d) The base of a solid is the region in the xy plane enclosed by the curves $y = x^4$, $y = -x^4$ and the line $x = 2$. Each cross-section perpendicular to the x-axis is an equilateral triangle.



(i) Show that the area of the triangular cross-section at $x = h$ is $\sqrt{3}\, h^8$. **1**

(ii) Hence find the volume of the solid. **2**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let α, β, and γ be the zeros of the polynomial $p(x) = 3x^3 + 7x^2 + 11x + 51$.

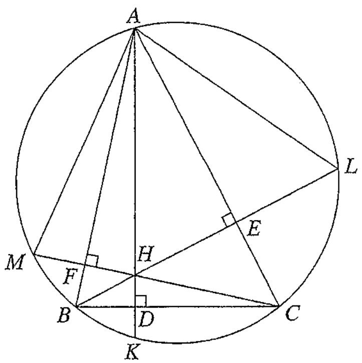
(i) Find $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. **1**

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$. **2**

(iii) Using part (ii), or otherwise, determine how many of the zeros of $p(x)$ are real. Justify your answer. **1**

(b) The vertices of an acute-angled triangle ABC lie on a circle. The perpendiculars from A, B and C meet BC, AC and AB at D, E and F respectively. These perpendiculars meet at H.

The perpendiculars AD, BE and CF are produced to meet the circle at K, L and M respectively.



(i) Prove that $\angle AHE = \angle DCE$. **2**

(ii) Deduce that $AH = AL$. **1**

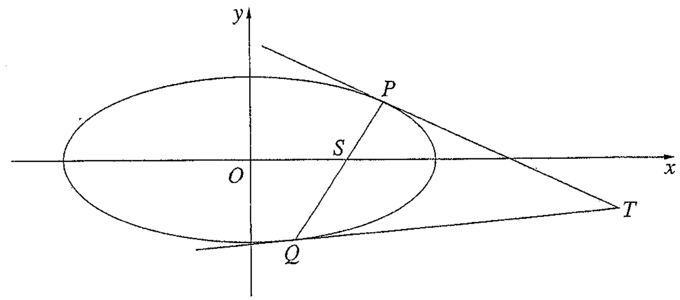
(iii) State a similar result for triangle AMH. **1**

(iv) Show that the length of the arc BKC is half the length of the arc MKL. **2**

Question 4 continues on page 8

Question 4 (continued) **Marks**

(c)



The point P lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The chord through P and the focus $S(ae, 0)$ meets the ellipse at Q. The tangents to the ellipse at P and Q meet at the point $T(x_0, y_0)$, so the equation of PQ is $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$. (Do NOT prove this.)

(i) Using the equation of PQ, show that T lies on the directrix. **1**

The point P is now chosen so that T also lies on the x-axis.

(ii) What is the value of the ratio $\frac{PS}{ST}$? **2**

(iii) Show that $\angle PTQ$ is less than a right angle. **1**

(iv) Show that the area of triangle PQT is $b^2\left(\frac{1}{e} - e\right)$. **1**

End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Let $a > 0$. Find the points where the line $y = ax$ and the curve $y = x(x - a)$ intersect. **1**

(ii) Let R be the region in the plane for which $x(x-a) \le y \le ax$. Sketch R. **1**

(iii) A solid is formed by rotating the region R about the line $x = -2a$. Use the method of cylindrical shells to find the volume of the solid. **4**

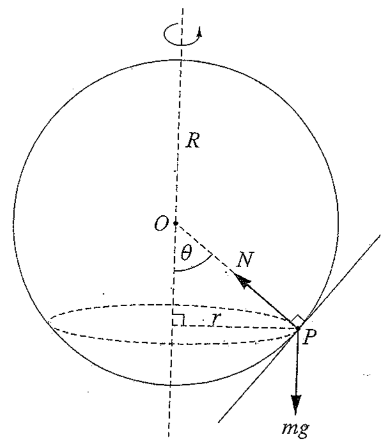
(b) (i) In how many ways can n students be placed in two distinct rooms so that neither room is empty? **1**

(ii) In how many ways can five students be placed in three distinct rooms so that no room is empty? **2**

Question 5 continues on page 10

Marks

(c) A smooth sphere with centre O and radius R is rotating about its vertical diameter at a uniform angular velocity, ω radians per second. A marble is free to roll around the inside of the sphere.



Assume that the marble can be considered as a point P which is acted upon by gravity and the normal reaction force N from the sphere. The marble describes a horizontal circle of radius r with the same uniform angular velocity, ω radians per second. Let the angle between OP and the vertical diameter be θ.

(i) Explain why $mr\omega^2 = N\sin\theta$ and $mg = N\cos\theta$. **2**

(ii) Show that either $\cos\theta = \dfrac{g}{R\omega^2}$ or $\theta = 0$. **3**

(iii) Hence, or otherwise, show that if $\theta \ne 0$ then $\omega > \sqrt{\dfrac{g}{R}}$. **1**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that **2**

$$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = \frac{\pi}{2}.$$

(ii) By making the substitution $x = \pi - u$, find **3**

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx.$$

(b) A particle is released from the origin O with an initial velocity of $A\ \mathrm{m\,s^{-1}}$ directed vertically downward. The particle is subject to a constant gravitational force and a resistance which is proportional to the velocity, $v\ \mathrm{m\,s^{-1}}$, of the particle.

Let x be the displacement in metres of the particle below O at time t seconds after the release of the particle, so that the equation of motion is

$$\ddot{x} = g - kv,$$

where $g\ \mathrm{m\,s^{-2}}$ is the acceleration due to gravity.

(i) The terminal velocity of the particle is $B\ \mathrm{m\,s^{-1}}$. Show that $k = \frac{g}{B}$. **1**

(ii) Verify that v satisfies the equation $\frac{d}{dt}\left(ve^{kt}\right) = ge^{kt}$. **2**

(iii) Hence show that the velocity of the particle is given by **2**

$$v = B - (B - A)e^{-\frac{gt}{B}}.$$

(iv) Deduce that $x = Bt - \frac{B}{g}(B - A)\left(1 - e^{-\frac{gt}{B}}\right)$. **2**

Question 6 continues on page 12

 Marks

At the same time as the particle is released from O, an identical particle is released from the point P which is h metres below O. The second particle has an initial velocity of $A\ \mathrm{m\,s^{-1}}$ directed vertically upward.

Its displacement below O is given by $x = h + Bt - \frac{B}{g}(B + A)\left(1 - e^{-\frac{gt}{B}}\right)$.
(Do NOT prove this.)

(v) Suppose that the two particles meet after T seconds. Show that **2**

$$T = \frac{B}{g}\log_e\left(\frac{2AB}{2AB - gh}\right).$$

(vi) The value of A can be varied. What condition must A satisfy so that the two particles can meet? **1**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Let a be a positive real number. Show that $a+\frac{1}{a}\geq 2$. **2**

(ii) Let n be a positive integer and $a_1, a_2, \ldots, a_n$ be n positive real numbers. Prove by induction that $(a_1+a_2+\ldots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)\geq n^2$. **4**

(iii) Hence show that $\operatorname{cosec}^2\theta+\sec^2\theta+\cot^2\theta\geq 9\cos^2\theta$. **1**

(b) Let α be a real number and suppose that z is a complex number such that

$$z+\frac{1}{z}=2\cos\alpha.$$

(i) By reducing the above equation to a quadratic equation in z, solve for z and use de Moivre's theorem to show that **3**

$$z^n+\frac{1}{z^n}=2\cos n\alpha.$$

(ii) Let $w=z+\frac{1}{z}$. Prove that **2**

$$w^3+w^2-2w-2=\left(z+\frac{1}{z}\right)+\left(z^2+\frac{1}{z^2}\right)+\left(z^3+\frac{1}{z^3}\right).$$

(iii) Hence, or otherwise, find all solutions of **3**

$$\cos\alpha+\cos 2\alpha+\cos 3\alpha=0,$$

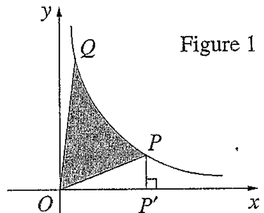
in the range $0\leq\alpha\leq 2\pi$.

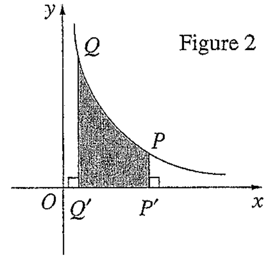
REPLACEMENT PAGE 14

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Let $P\left(p,\frac{1}{p}\right)$ and $Q\left(q,\frac{1}{q}\right)$ be points on the hyperbola $y=\frac{1}{x}$ with $p>q>0$. Let P' be the point $(p, 0)$ and Q' be the point $(q, 0)$. The shaded region OPQ in Figure 1 is bounded by the lines OP, OQ and the hyperbola. The shaded region $Q'QPP'$ in Figure 2 is bounded by the lines QQ', PP', $P'Q'$ and the hyperbola.

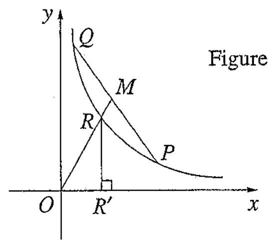




(i) Find the area of triangle OPP'. **1**

(ii) Prove that the area of the shaded region OPQ is equal to the area of the shaded region $Q'QPP'$. **1**

Let M be the midpoint of the chord PQ and $R\left(r,\frac{1}{r}\right)$ be the intersection of the line OM with the hyperbola. Let R' be the point $(r, 0)$, as shown in Figure 3.



(iii) By using similar triangles, or otherwise, prove that $r^2=pq$. **2**

(iv) By using integration, or otherwise, show that the line RR' divides the shaded region $Q'QPP'$ into two pieces of equal area. **2**

(v) Deduce that the line OR divides the shaded region OPQ into two pieces of equal area. **1**

Question 8 continues on page 15

Marks

Question 8 (continued)

(b) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$ and let $J_n = (-1)^n I_{2n}$ for $n = 0, 1, 2, \ldots$

(i) Show that $I_n + I_{n+2} = \frac{1}{n+1}$. 2

(ii) Deduce that $J_n - J_{n-1} = \frac{(-1)^n}{2n-1}$ for $n \geq 1$. 1

(iii) Show that $J_m = \frac{\pi}{4} + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}$. 2

(iv) Use the substitution $u = \tan x$ to show that $I_n = \int_0^1 \frac{u^n}{1+u^2} \, du$. 1

(v) Deduce that $0 \leq I_n \leq \frac{1}{n+1}$ and conclude that $J_n \to 0$ as $n \to \infty$. 2

End of paper

2004 HIGHER SCHOOL CERTIFICATE SOLUTIONS

MATHEMATICS EXTENSION 2

QUESTION 1

(a) $\displaystyle\int xe^{3x}\,dx = \frac{1}{3}\int x\,d\left(e^{3x}\right)$

$\displaystyle = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx$

$\displaystyle = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

$\displaystyle = \frac{e^{3x}}{9}(3x-1) + C.$

(b) This problem can be done using many different substitutions, or equivalently using the function of a function rule in reverse. Here we show one example of each type of method.

METHOD 1 (Substitution)

Let $u = \cos x$, $du = -\sin x\,dx$.

When $x = 0$, $u = 1$.

When $x = \frac{\pi}{4}$, $u = \frac{1}{\sqrt{2}}$.

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\,dx = \int_1^{\frac{1}{\sqrt{2}}} -u^{-3}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{u^2}\right]_1^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{2}(2-1)$$

$$= \frac{1}{2}.$$

Other possible substitutions include $u = \cos^2 x$, $u = \cos^3 x$, $u = \sin x$ and $u = \dfrac{1}{\cos^2 x}\left(= \sec^2 x\right)$.

METHOD 2

(Function of a function in reverse)

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\,dx = \int_0^{\frac{\pi}{4}} \tan x \sec^2 x\,dx$$

$$= \frac{1}{2}\Big[\tan^2 x\Big]_0^{\frac{\pi}{4}} \left(\text{or } \frac{1}{2}\Big[\sec^2 x\Big]_0^{\frac{\pi}{4}}\right)$$

$$= \frac{1}{2}(1-0)$$

$$= \frac{1}{2}.$$

(c) $\displaystyle\int \frac{dx}{\sqrt{5+4x-x^2}} = \int \frac{dx}{\sqrt{9-(x-2)^2}}$

$\displaystyle = \sin^{-1}\left(\frac{x-2}{3}\right) + C.$

(d) (i) $\displaystyle\frac{x^2-7x+4}{(x+1)(x-1)^2} \equiv \frac{a}{x+1} + \frac{b}{x-1} - \frac{1}{(x-1)^2}$

$\therefore\ a(x-1)^2 + b(x+1)(x-1) - (x+1) \equiv x^2 - 7x + 4.$

Let $x = -1$: $4a = 1 + 7 + 4$

$a = 3.$

Coefficient of x^2: $a + b = 1$

$b = -2.$

Therefore $a = 3$ and $b = -2$.

(ii) $\displaystyle\int \frac{x^2-7x+4}{(x+1)(x-1)^2}\,dx$

$\displaystyle = \int\left(\frac{3}{x+1} - \frac{2}{x-1} - \frac{1}{(x-1)^2}\right)dx$

$\displaystyle = 3\ln|x+1| - 2\ln|x-1| + \frac{1}{x-1} + C$

$\displaystyle = \ln\frac{|x+1|^3}{(x-1)^2} + \frac{1}{x-1} + C.$

(e) $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta.$

When $x = 1$, $\theta = \frac{\pi}{6}$.

When $x = 0$, $\theta = 0$.

$$\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{2\cos\theta} \times 2\cos\theta\,d\theta,$$

$\left(\text{since } \cos\theta > 0 \text{ for } 0 \le \theta \le \frac{\pi}{6}\right)$

$$= 4\int_0^{\frac{\pi}{6}} \sin^2\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{6}} (1-\cos 2\theta)\,d\theta$$

$$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= 2\left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right) - 0$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2}, \text{ or } \frac{1}{6}\left(2\pi - 3\sqrt{3}\right).$$

QUESTION 2

(a) (i) $zw = (1+2i)(3-i)$

$= (3 - i + 6i + 2)$

$= 5 + 5i.$

(ii) *METHOD* 1

$\displaystyle\frac{10}{z} = \frac{10}{1+2i}$

$\displaystyle = \frac{10}{1+2i} \cdot \frac{1-2i}{1-2i}$

$\displaystyle = \frac{10(1-2i)}{1+4}$

$= 2 - 4i.$

$\therefore\ \overline{\left(\dfrac{10}{z}\right)} = 2 + 4i.$

METHOD 2

$\displaystyle\overline{\left(\frac{10}{z}\right)} = \frac{10}{\bar z}$

$\displaystyle = \frac{10z}{\bar z z}$

$\displaystyle = \frac{10(1+2i)}{1^2+2^2}$

$= 2 + 4i.$

(b) (i) $\displaystyle\frac{\alpha}{\beta} = \frac{1+i\sqrt{3}}{1+i}$

$\displaystyle = \frac{1+i\sqrt{3}}{1+i} \times \frac{1-i}{1-i}$

$\displaystyle = \frac{1-i+i\sqrt{3}+\sqrt{3}}{2}$

$\displaystyle = \frac{\left(1+\sqrt{3}\right)+i\left(\sqrt{3}-1\right)}{2}$

$\displaystyle = \left(\frac{1+\sqrt{3}}{2}\right) + i\left(\frac{\sqrt{3}-1}{2}\right).$

(ii) $|\alpha| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$

$\therefore\ \alpha = 2\left(\dfrac{1}{2} + i\cdot\dfrac{\sqrt{3}}{2}\right)$

$= 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right).$

(iii) $|\beta| = \sqrt{2}$

$\displaystyle\left|\frac{\alpha}{\beta}\right| = \frac{|\alpha|}{|\beta|} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\displaystyle\arg\left(\frac{\alpha}{\beta}\right) = \arg\alpha - \arg\beta$

$\displaystyle = \frac{\pi}{3} - \frac{\pi}{4}$

$\displaystyle = \frac{\pi}{12}.$

$\displaystyle\therefore\ \frac{\alpha}{\beta} = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right).$

(iv) Comparing (i) and (iii), and equating imaginary parts:

$\displaystyle\sqrt{2}\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2}$

$\displaystyle\therefore\ \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$

(c) Let $z = x + iy$.

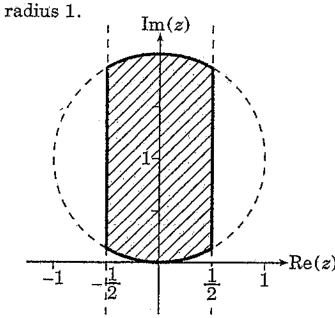
Then $z + \bar z = 2x$

$\Rightarrow\ |z + \bar z| \le 1$

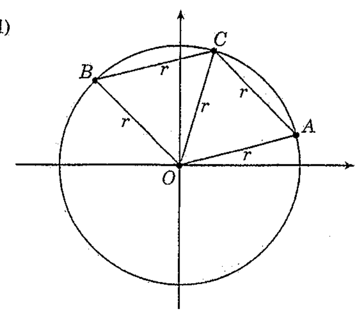
$\Rightarrow\ |2x| \le 1$

$|x| \le \frac{1}{2}.$

$|z - i| \le 1$ represents the interior and circumference of a circle centre $(0, 1)$, radius 1.



(d)



(i) Since C represents $z+w$, $OACB$ is a parallelogram.

$\therefore$ $OB = AC$ and $BC = OA$.

Also, $OA = OB = OC$ (radii of circle).

$\therefore$ ΔOBC and ΔOCA are equilateral.

$\therefore \angle BOC = \angle COA = \dfrac{\pi}{3}$.

$$\therefore \angle AOB = \angle AOC + \angle COA = \frac{2\pi}{3}.$$

(ii) *METHOD 1*

Let $z = r\operatorname{cis}\alpha$

$$w = r\operatorname{cis}\left(\alpha + \frac{2\pi}{3}\right), \text{ from (i)}$$

$$\therefore\ w^3 = r^3\operatorname{cis}\left[3\left(\alpha + \frac{2\pi}{3}\right)\right],$$ using de Moivre's rule

$$= r^3\operatorname{cis}(3\alpha + 2\pi)$$
$$= r^3\operatorname{cis}3\alpha$$
$$= r^3(\operatorname{cis}\alpha)^3, \text{ using de Moivre's rule}$$
$$= z^3.$$

METHOD 2

$$w = z\operatorname{cis}\frac{2\pi}{3}, \text{ from (i)}$$

$$w^3 = z^3\left(\operatorname{cis}\frac{2\pi}{3}\right)^3$$
$$= z^3\operatorname{cis}2\pi, \text{ using de Moivre's rule}$$
$$= z^3 \times 1$$
$$= z^3.$$

(iii) From (ii), $z^3 - w^3 = 0$

ie. $(z-w)(z^2 + w^2 + zw) = 0$.

$\therefore$ $z^2 + w^2 + zw = 0$, since $z \neq w$.

QUESTION 3

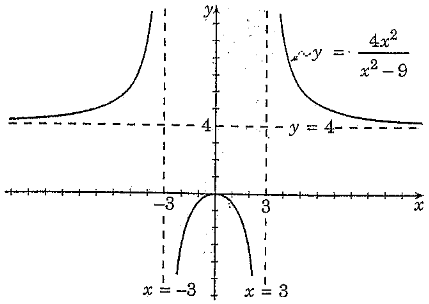
(a)
$$y = \frac{4x^2}{x^2-9} \quad \text{—①}$$
$$= \frac{4x^2}{(x-3)(x+3)} \quad \text{—②}$$
$$= \frac{4}{1-\dfrac{9}{x^2}} \quad \text{—③}$$
$$= 4 + \frac{36}{x^2-9}. \quad \text{—④}$$

From ① we note that the function is an even function, and so the curve is symmetric about the y-axis. It goes through the origin with a gradient of zero.

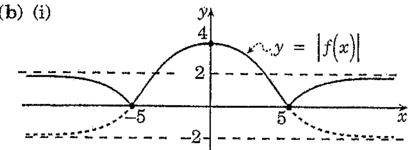
From ② we note vertical asymptotes at $x = \pm 3$.

From ③ and ④ we note that $y \to 4$ as $x \to \pm\infty$, and so $y = 4$ is a horizontal asymptote.

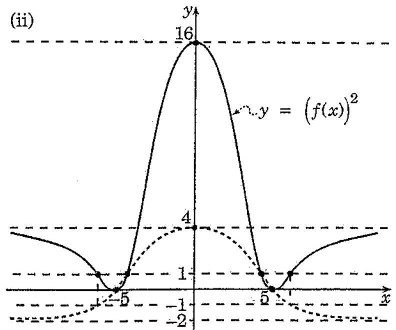
From ① we note that $y \le 0$ for $|x| < 3$ and from ④ we note that $y > 4$ for $|x| > 3$.



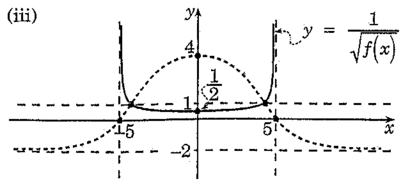
(b) (i)



(ii)



(iii)



(c) $x^2 - xy + y^3 = 5$

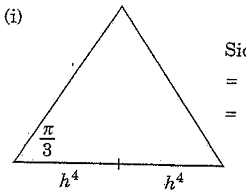
Differentiate both sides with respect to x.

$$2x - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{y-2x}{3y^2-x}.$$

At (2, −1),
$$\frac{dy}{dx} = \frac{-1-4}{3-2} = -5.$$

The tangent is
$$y+1 = -5(x-2)$$
$$y = -5x+9$$
or
$$5x + y - 9 = 0.$$

(d) (i)



Side length
$$= h^4 - (-h^4)$$
$$= 2h^4.$$

The cross-sectional area is
$$A = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2}(2h^4)^2\sin\frac{\pi}{3}$$
$$= \frac{1}{2}\times 4h^8 \times \frac{\sqrt{3}}{2}$$
$$= \sqrt{3}h^8.$$

(ii)
$$\text{Volume} = \int_0^2 A\,dx$$
$$= \int_0^2 \sqrt{3}x^8\,dx$$
$$= \frac{\sqrt{3}}{9}\Big[x^9\Big]_0^2$$
$$= \frac{512\sqrt{3}}{9} \text{ cubic units.}$$

QUESTION 4

(a) $p(x) = 3x^3 + 7x^2 + 11x + 51$.

(i)
$$\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 = \alpha\beta\gamma(\alpha+\beta+\gamma)$$
$$= -\frac{51}{3}\times\left(-\frac{7}{3}\right)$$
$$= \frac{119}{3}$$
$$= 39\tfrac{2}{3}.$$

(ii) *METHOD 1*

$$\alpha^2 + \beta^2 + \gamma^2$$
$$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$$
$$= \left(-\frac{7}{3}\right)^2 - 2\times\frac{11}{3}$$
$$= -\frac{17}{9}$$
$$= -1\tfrac{8}{9}.$$

METHOD 2

Form a new polynomial whose roots are α^2, β^2 and γ^2.

Let $u = x^2$, so that $x = \pm\sqrt{u}$, and substitute into $p(x) = 0$.

$$\pm 3\sqrt{u}\,u + 7u^2 \pm 11\sqrt{u} + 51 = 0$$
$$\pm\sqrt{u}\,(3u+11) = -(7u+51)$$
$$u(3u+11)^2 = (7u+51)^2$$
$$9u^3 + 66u^2 + \cdots = 49u^2 + \cdots$$

(*Note*: Only need first two terms.)

$$9u^3 + 17u^2 + \cdots = 0.$$

$$\therefore\ \alpha^2 + \beta^2 + \gamma^2 = -\frac{17}{9} = -1\tfrac{8}{9}.$$

(iii) *METHOD 1*

As $\alpha^2 + \beta^2 + \gamma^2 < 0$, from (ii), there is at least one unreal zero. As the coefficients of $p(x)$ are real, the unreal zeros occur in conjugate pairs, so there are at least 2 unreal zeros.

As deg. $(p(x)) = 3$, there is only one more zero which must be real. Therefore there is exactly one real zero of $p(x)$.

METHOD 2

$$p'(x) = 9x^2 + 14x + 11$$
$$\Delta = 14^2 - 4\times 9\times 11$$
$$< 0.$$

Therefore $p'(x)$ is positive definite, so $p(x)$ is monotonic increasing and so has exactly one real zero.

METHOD 3

Possible rational zeros are ± 1, ± 3, ± 17, ± 51 only. Testing these leads to $p(-3) = 0$, and then to

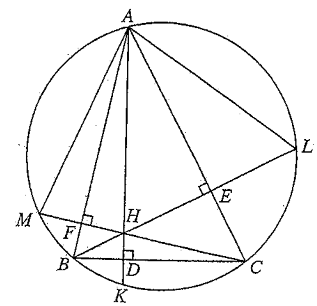
$$p(x) = (x+3)(3x^2 - 2x + 17).$$

Now for $3x^2 - 2x + 17$

$$\Delta = 2^2 - 4\times 3\times 17$$
$$< 0.$$

Therefore there are two unreal zeros and exactly one real zero.

(b)



(i) $\angle AHE + \angle AEH + \angle HAE$ ($\angle$ sum, ΔAEH
$= \angle DCE + \angle ADC + \angle CAD$ and ΔADC)
Now $\angle HAE = \angle CAD$ (common $\angle$)
and $\angle AEH = \angle ACD$ (both right $\angle$s)
$\therefore \quad \angle AHE = \angle DCE.$

This can also be done many other ways, such as by showing $\Delta AEH \parallel\!\!\mid \Delta ADC$, or by showing $\Delta HBD \parallel\!\!\mid \Delta CBE$, or by showing that quadrilateral $EHDC$ is cyclic.

(ii) $\angle ALB = \angle ACB$ ($\angle$s in same segment standing on same arc AB)
that is, $\angle ALE = \angle ECD$ (same $\angle$s)
but $\angle AHE = \angle DCE$, from (i)
$\therefore \quad \angle AHE = \angle ALE$
$\therefore \quad AH = AL$ (sides opposite equal $\angle$s are equal)

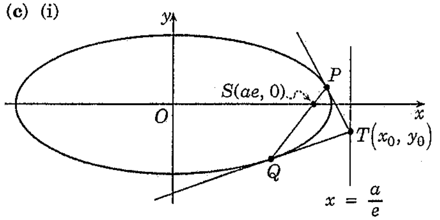
Again, other methods are possible, such as showing $\Delta AEH \equiv \Delta AEL$.

(iii) Similarly, $AH = AM$ (diagram is symmetric)

(iv) $\angle AHE = \angle ALE$, from (ii)
$\therefore \quad \angle HAE = \angle LAE$ ($\angle$ sum of Δ)
$\therefore$ arc KC = arc CL (equal $\angle$s stand on equal arcs)

Similarly from (iii),
arc KB = arc BM.
$\therefore$ arc MKL
= arc MB + arc BK + arc KC + arc CL
= 2 arc BK + 2 arc KC
= 2(arc BK + arc KC)
= 2 arc BKC.
$\therefore$ arc $BKC = \frac{1}{2}$ arc MKL.

(c) (i)

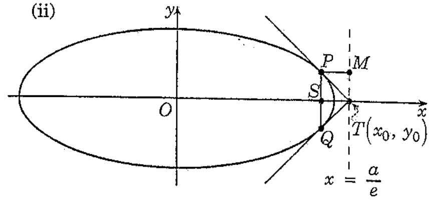


Since $S(ae, 0)$ lies on PQ, its coordinates will satisfy the equation

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$$
$$\therefore \quad \frac{aex_0}{a^2} + 0 = 1$$
$$x_0 = \frac{a}{e}.$$

But the directrix has equation $x = \frac{a}{e}$.
Therefore $T(x_0, y_0)$ lies on the directrix.

(ii)



If T lies on the x-axis then the symmetry of the ellipse means that P and Q will be vertically aligned so that PSQ will be a vertical line.

This can also be shown as follows. Since T lies on the x-axis its coordinates will be $(x_0, y_0) = \left(\frac{a}{e}, 0\right)$, and so the equation of PQ will be

$$\frac{x \times \frac{a}{e}}{a^2} + \frac{y \times 0}{b^2} = 1$$
$$\frac{x}{ae} = 1$$
$$x = ae,$$

which is a vertical line through S.

Let M be the closest point to P on the directrix as shown.

$$\frac{PS}{ST} = \frac{PS}{PM}, \text{ since } PM = ST$$
$$= e. \quad \text{(focus–directrix definition)}$$

(iii) $\tan \angle PTS = \frac{PS}{ST}$
$= e$, from (ii)
< 1
$\therefore \quad \angle PTS < \frac{\pi}{4}.$

Similarly, $\angle QTS = \angle PTS$
$< \frac{\pi}{4}.$
$\therefore \quad \angle PTQ < \frac{\pi}{2}.$

(iv) At P, $x = ae$

$$\therefore \quad \frac{a^2e^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\therefore \quad \frac{y^2}{b^2} = 1 - e^2$$
$$y^2 = b^2\left(1 - e^2\right).$$

That is, $PS^2 = b^2\left(1 - e^2\right).$

$$\text{Area } \Delta PQT = 2 \text{ area } \Delta PTS$$
$$= PS \times ST$$
$$= PS \times \frac{PS}{e}$$
$$= \frac{b^2\left(1 - e^2\right)}{e}$$
$$= b^2\left(\frac{1}{e} - e\right).$$

QUESTION 5

(a) (i)
$$y = ax \quad \text{—①}$$
$$y = x(x - a) \quad \text{—②}$$

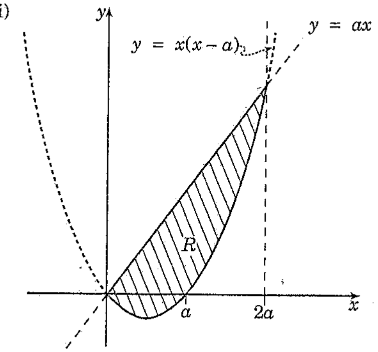
For the intersection, substitute ① into ②:

$$ax = x(x - a)$$
$$0 = x^2 - 2ax$$
$$x(x - 2a) = 0$$
$$\therefore \quad x = 0 \text{ or } 2a.$$

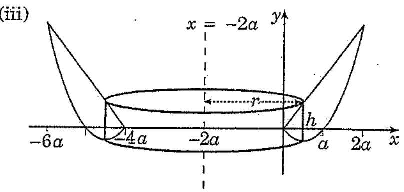
Substitute these values in ①:
$y = 0$ or $2a^2$.

Therefore the points of intersection are $(0, 0)$ and $\left(2a, 2a^2\right)$.

(ii)



(iii)



Consider the solid as being made up of cylindrical shells of height h and radius r as shown.

From the diagram, $2a \le r \le 4a$.

$$\therefore \quad V = \int_{2a}^{4a} 2\pi rh \; dr.$$

Now $h = ax - x(x - a)$
$= x(2a - x)$
and $r = x + 2a$
so $dr = dx$.
When $r = 2a$, $x = 0$.
When $r = 4a$, $x = 2a$.

$$\therefore \quad V = \int_0^{2a} 2\pi(x + 2a)\,x(2a - x) \; dx$$
$$= 2\pi \int_0^{2a} x\left(4a^2 - x^2\right) dx$$
$$= 2\pi \int_0^{2a} 4a^2x - x^3 \; dx$$
$$= 2\pi \left[2a^2x^2 - \frac{x^4}{4}\right]_0^{2a}$$
$$= 2\pi \left(2a^2 . 4a^2 - \frac{16a^4}{4}\right)$$
$$= 2\pi \left(8a^4 - 4a^4\right)$$
$$= 8\pi a^4 \text{ units}^3.$$

(b) (i) *METHOD 1*

Each student can be placed in one of the two rooms, so that there are 2^n ways altogether. However, in 2 of these ways one of the rooms is empty.

$\therefore$ Number of ways $= 2^n - 2$.

METHOD 2

Possible configurations are $1, (n - 1)$
$2, (n - 2)$
$\vdots$
$(n - 1), 1.$

$\therefore$ Number of ways

$$= \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n-1}.$$

Now $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$
$= 2^n$,
and $\binom{n}{0} = \binom{n}{n} = 1.$

$\therefore$ Number of ways $= 2^n - 2$.

(ii) *METHOD 1*

Possible configurations are 1, 1, 3 and 1, 2, 2.

For 1, 1, 3, number of ways

$$= \binom{5}{1} \times \binom{4}{1} \times \binom{3}{3}$$
$$= 20.$$

For 1, 2, 2, number of ways

$$= \binom{5}{1} \times \binom{4}{2} \times \binom{2}{2}$$
$$= 30.$$

However, each of these configurations can be arranged 3 ways in the 3 rooms.

$\therefore$ Number of ways $= 3 \times (20 + 30)$
$= 150.$

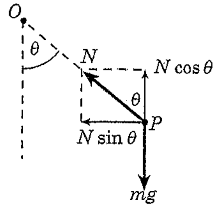
METHOD 2

If empty rooms are allowed, the number of ways $= 3^5$. The number of ways with one room empty = number of ways with all in two rooms $= 3 \times \left(2^5 - 2\right)$, using (i).

Number of ways with 2 rooms empty = 3.

$\therefore$ Number of ways $= 3^5 - 3\left(2^5 - 2\right) - 3$
$= 3\left(81 - 30 - 1\right)$
$= 150.$

(c) (i)



The only forces acting on P are N and mg. The particle is travelling in a horizontal circle of radius r, with uniform angular velocity ω.

$\therefore$ Net vertical force $= 0$. —①

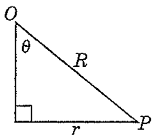
Net inward radial force $= mr\omega^2$. —②

① becomes $0 = N\cos\theta - mg$

$\therefore \quad mg = N\cos\theta$ —③

and ② becomes $mr\omega^2 = N\sin\theta$. —④

(ii)



From the diagram, $\sin\theta = \dfrac{r}{R}$.

Substitute into ④: $\dfrac{Nr}{R} = mr\omega^2$.

If the marble is at the bottom of the sphere, then $\theta = 0$ and $r = 0$, and this equation simply says $0 = 0$.

Otherwise $r \neq 0$, so divide both sides by r:

$$\frac{N}{R} = m\omega^2$$
$$N = mR\omega^2.$$

Substitute into ③:

$$mR\omega^2\cos\theta = mg$$
$$\cos\theta = \frac{g}{R\omega^2}.$$

(iii)
$$\theta > 0$$
$$\therefore \cos\theta < 1$$
$$\therefore \frac{g}{R\omega^2} < 1$$
$$\omega^2 > \frac{g}{R}$$
$$\omega > \sqrt{\frac{g}{R}}, \text{ since } \omega > 0.$$

QUESTION 6

(a) (i) $I = \displaystyle\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx.$

Let $u = \cos x$, so $du = -\sin x\,dx$

$x = 0, \quad u = \cos 0 = 1$

$x = \pi, \quad u = \cos\pi = -1.$

$$\therefore \quad I = \int_1^{-1} \frac{-du}{1+u^2}$$
$$= \int_{-1}^{1} \frac{du}{1+u^2}$$
$$= \Big[\tan^{-1} u\Big]_{-1}^{1}$$
$$= \tan^{-1}(1) - \tan^{-1}(-1)$$
$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$$
$$= \frac{\pi}{2}.$$

(ii) $I = \displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

Let $x = \pi - u$, so $dx = -du$

$x = 0, \quad u = \pi$

$x = \pi, \quad u = 0.$

$$\therefore \ I = \int_{\pi}^{0} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}\,(-du)$$
$$= \int_0^{\pi} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}\,du.$$

Now $\sin(\pi - u) = \sin u$

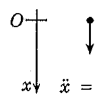
and $\cos(\pi - u) = -\cos u$.

$$\therefore \ I = \int_0^{\pi} \frac{(\pi-u)\sin u}{1+\cos^2 u}\,du$$
$$= \int_0^{\pi} \frac{\pi\sin u}{1+\cos^2 u}\,du - \int_0^{\pi} \frac{u\sin u}{1+\cos^2 u}\,du$$
$$= \pi\int_0^{\pi} \frac{\sin u}{1+\cos^2 u}\,du - I.$$

Now, from (i), $\displaystyle\int_0^{\pi} \frac{\sin u}{1+\cos^2 u}\,du = \frac{\pi}{2}$

$$\therefore \quad 2I = \pi \times \frac{\pi}{2}$$
$$\therefore \quad I = \frac{\pi^2}{4}.$$

(b)



$v_0 = A \text{ m s}^{-1}$

$\ddot{x} = g - kv$

(i) At terminal velocity, acceleration is zero.

$\therefore$ When $v = B$, $\quad \ddot{x} = 0,$

$\therefore \quad 0 = g - kB$

$\therefore \quad k = \dfrac{g}{B}.$

(ii) $\dfrac{d}{dt}\left(ve^{kt}\right)$
$$= e^{kt}\frac{dv}{dt} + v \cdot ke^{kt}$$
$$= e^{kt}\left(g - kv\right) + kve^{kt}, \quad \text{since } \frac{dv}{dt} = \ddot{x},$$
$$= ge^{kt}.$$

(iii) Now, $\dfrac{d}{dt}\left(ve^{kt}\right) = ge^{kt}.$

Integrating with respect to t,
$$ve^{kt} = \frac{g}{k}e^{kt} + C_1$$
$$= Be^{kt} + C_1, \quad \text{from (i).}$$

When $t = 0$, $\quad v = A$

$\therefore \quad A = B + C_1$

$C_1 = A - B$

$\therefore \quad ve^{kt} = Be^{kt} + A - B$

$v = B + \left(A - B\right)e^{-kt}.$

ie. $\quad v = B - \left(B - A\right)e^{-\frac{gt}{B}}$, using (i).

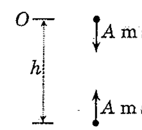
(iv) From (iii), $\dfrac{dx}{dt} = B - \left(B - A\right)e^{-\frac{gt}{B}}.$

$$\therefore \quad x = Bt - \left(B - A\right)e^{-\frac{gt}{B}} \cdot \left(-\frac{B}{g}\right) + C_2$$
$$= Bt + \frac{B}{g}\left(B - A\right)e^{-\frac{gt}{B}} + C_2.$$

When $t = 0$, $\quad x = 0$.

$$\therefore \quad 0 = 0 + \frac{B}{g}\left(B - A\right) + C_2$$
$$C_2 = -\frac{B}{g}\left(B - A\right).$$
$$\therefore \quad x = Bt + \frac{B}{g}\left(B - A\right)\left(e^{-\frac{gt}{B}} - 1\right)$$
$$= Bt - \frac{B}{g}\left(B - A\right)\left(1 - e^{-\frac{gt}{B}}\right).$$

(v)



When the two particles meet, their displacements below O are equal, ie. putting $t = T$ in the two displacement equations:

$$BT - \frac{B}{g}\left(B - A\right)\left(1 - e^{-\frac{gT}{B}}\right)$$
$$= h + BT - \frac{B}{g}\left(B + A\right)\left(1 - e^{-\frac{gT}{B}}\right).$$
$$-B\left(B - A\right)\left(1 - e^{-\frac{gT}{B}}\right)$$
$$= gh - B\left(B + A\right)\left(1 - e^{-\frac{gT}{B}}\right).$$
$$0 = gh + B\left(1 - e^{-\frac{gT}{B}}\right)\left[\left(B - A\right) - \left(B + A\right)\right]$$
$$= gh - 2AB\left(1 - e^{-\frac{gT}{B}}\right).$$
$$\therefore \ 1 - e^{-\frac{gT}{B}} = \frac{gh}{2AB}$$
$$e^{-\frac{gT}{B}} = 1 - \frac{gh}{2AB}$$
$$\therefore \quad -\frac{gT}{B} = \log_e\left(1 - \frac{gh}{2AB}\right)$$
$$T = -\frac{B}{g}\log_e\left(\frac{2AB - gh}{2AB}\right)$$
$$= \frac{B}{g}\log_e\left(\frac{2AB}{2AB - gh}\right).$$

(vi) From (v), $T \to \infty$ as $2AB - gh \to 0$.

$\therefore$ In order to meet, $2AB - gh > 0$

$\therefore \quad A > \dfrac{gh}{2B}.$

Note: This is a necessary condition and it is the one the examiners were looking for. However, if A is close to $\dfrac{gh}{2B}$, the particles may meet after a very long time, and therefore may have travelled a very long distance.

In practice, there will be a finite time at which the second particle hits the ground, and A must be sufficiently large for the particles to meet before this occurs.

Students may find it worthwhile to examine this question in more depth, and determine a sufficient condition for the particles to meet. It will be necessary to consider how far above the ground P is.

QUESTION 7

(a) (i) *METHOD 1*

$$\left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \geq 0, \quad \text{for all } a > 0$$
$$a - 2 + \frac{1}{a} \geq 0$$
$$a + \frac{1}{a} \geq 2.$$

METHOD 2

Let $f(x) = x + \dfrac{1}{x}, \quad x > 0.$

$f'(x) = 1 - \dfrac{1}{x^2} = 0$ when $x = 1$.

$f''(x) = \dfrac{2}{x^3} > 0$ for $x > 0$,

so $f(x)$ is concave up and the stationary point is an absolute minimum.

$\therefore \quad f(x) \ge f(1) = 2$

$\therefore \quad x + \frac{1}{x} \ge 2$ for $x > 0$,

that is, $a + \frac{1}{a} \ge 2$ for $a > 0$.

METHOD 3

Consider values of k for which $a + \frac{1}{a} = k$ has solutions for $a > 0$.

$$a^2 - ka + 1 = 0.$$

To have solutions, we need $\Delta \ge 0$.

$$k^2 - 4 \ge 0$$

$$k \ge 2, \text{ since } k > 0 \text{ if } a > 0.$$

$$\therefore \quad a + \frac{1}{a} \ge 2 \text{ for } a > 0.$$

(ii) Prove $(a_1 + a_2 + \cdots + a_n) \times \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \ge n^2$.

Test for $n = 1$: LHS $= a_1 \times \frac{1}{a_1} = 1$.

RHS $= 1^2 = 1$.

Since LHS = RHS, the result is true when $n = 1$. Assume the result is true for $n = k$. That is, assume

$$(a_1 + a_2 + \cdots + a_k) \times \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}\right) \ge k^2.$$

When $n = k+1$,

$$\begin{aligned}
\text{LHS} &= (a_1 + a_2 + \cdots + a_k + a_{k+1}) \times \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} + \frac{1}{a_{k+1}}\right) \\
&= (a_1 + a_2 + \cdots + a_k) \times \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}\right) \\
&\quad + (a_1 + a_2 + \cdots + a_k) \times \frac{1}{a_{k+1}} \\
&\quad + a_{k+1}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}\right) \\
&\quad + a_{k+1} \times \frac{1}{a_{k+1}} \\
&\ge k^2 + \left(\frac{a_1}{a_{k+1}} + \frac{a_{k+1}}{a_1}\right) + \left(\frac{a_2}{a_{k+1}} + \frac{a_{k+1}}{a_2}\right) + \cdots \\
&\quad + \left(\frac{a_k}{a_{k+1}} + \frac{a_{k+1}}{a_k}\right) + 1, \quad \text{using the assumption, and rearranging terms,} \\
&\ge k^2 + \underbrace{2 + 2 + \cdots + 2}_{k \text{ terms}} + 1, \quad \text{using (i) } k \text{ times.} \\
&= k^2 + 2k + 1 \\
&= (k+1)^2.
\end{aligned}$$

Therefore if the result is true for $n = k$, it is also true for $n = k+1$. Since it is true for $n = 1$ it is true for all integers $n \ge 1$.

(iii) $(\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta) \times \left(\frac{1}{\operatorname{cosec}^2\theta} + \frac{1}{\sec^2\theta} + \frac{1}{\cot^2\theta}\right) \ge 3^2$, from (ii),

$$(\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta) \times (\sin^2\theta + \cos^2\theta + \tan^2\theta) \ge 9$$

$$(\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta)(1 + \tan^2\theta) \ge 9$$

$$(\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta)\sec^2\theta \ge 9$$

$$\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta \ge 9\cos^2\theta.$$

(b) (i) $z + \frac{1}{z} = 2\cos\alpha$.

$$\begin{aligned}
z^2 - 2\cos\alpha\, z + 1 &= 0 \\
(z - \cos\alpha)^2 &= \cos^2\alpha - 1 \\
&= -\sin^2\alpha. \\
z - \cos\alpha &= \pm i\sin\alpha \\
z &= \cos\alpha \pm i\sin\alpha \\
&= \operatorname{cis}\alpha \text{ or } \operatorname{cis}(-\alpha).
\end{aligned}$$

If $z = \operatorname{cis}\alpha$ then, from de Moivre's rule, $z^n = \operatorname{cis} n\alpha$ and $z^{-n} = \operatorname{cis}(-n\alpha)$.

If $z = \operatorname{cis}(-\alpha)$, then $z^n = \operatorname{cis}(-n\alpha)$ and $z^{-n} = \operatorname{cis} n\alpha$.

In either case,

$$\begin{aligned}
z^n + \frac{1}{z^n} &= \operatorname{cis} n\alpha + \operatorname{cis}(-n\alpha) \\
&= (\cos n\alpha + i\sin n\alpha) + (\cos n\alpha - i\sin n\alpha) \\
&= 2\cos n\alpha.
\end{aligned}$$

(ii) $w = z + \frac{1}{z}$.

METHOD 1

$$\begin{aligned}
w^3 - 2w &= z^3 + 3z + 3 \times \frac{1}{z} + \frac{1}{z^3} - 2z - 2 \times \frac{1}{z} \\
&= \left(z^3 + \frac{1}{z^3}\right) + \left(z + \frac{1}{z}\right). \\
w^2 - 2 &= z^2 + 2 + \frac{1}{z^2} - 2 \\
&= z^2 + \frac{1}{z^2}.
\end{aligned}$$

$\therefore \quad w^3 + w^2 - 2w - 2$

$$= \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right).$$

METHOD 2

$$\begin{aligned}
&w^3 + w^2 - 2w - 2 \\
&= w^2(w+1) - 2(w+1) \\
&= (w+1)(w^2 - 2) \\
&= \left(z + \frac{1}{z} + 1\right)\left(z^2 + \frac{1}{z^2}\right) \\
&= z^3 + \frac{1}{z} + z + \frac{1}{z^3} + z^2 + \frac{1}{z^2} \\
&= \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right).
\end{aligned}$$

(iii)

$$\cos\alpha + \cos 2\alpha + \cos 3\alpha = 0$$

$$2\cos\alpha + 2\cos 2\alpha + 2\cos 3\alpha = 0$$

$$\left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right) = 0,$$

where $z + \frac{1}{z} = 2\cos\alpha$ and using (i) with $n = 2, 3$, $w^3 + w^2 - 2w - 2 = 0$,

where $w = z + \frac{1}{z}$, using (ii),

$$(w+1)(w^2 - 2) = 0.$$

$$\therefore \quad w = -1, \sqrt{2} \text{ or } -\sqrt{2}.$$

$$2\cos\alpha = -1, \sqrt{2} \text{ or } -\sqrt{2},$$

since $w = z + \frac{1}{z} = 2\cos\alpha$,

$$\cos\alpha = -\frac{1}{2}, \frac{1}{\sqrt{2}} \text{ or } -\frac{1}{\sqrt{2}}.$$

$\cos\alpha = -\frac{1}{2}$: $\quad \alpha = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

$\cos\alpha = \frac{1}{\sqrt{2}}$: $\quad \alpha = \frac{\pi}{4}$ or $\frac{7\pi}{4}$

$\cos\alpha = -\frac{1}{\sqrt{2}}$: $\quad \alpha = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$.

The last 4 solutions can also be obtained from $w^2 - 2 = 0$

$\cos 2\alpha = 0$, since $w^2 - 2 = z^2 + \frac{1}{z^2} = 2\cos 2\alpha$.

$$2\alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

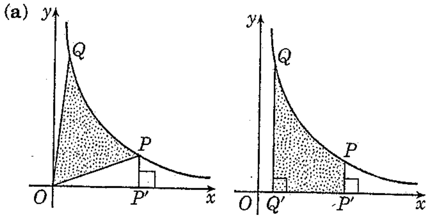
$$\alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

Thus there are six solutions:

$$\alpha = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{4\pi}{3} \text{ and } \frac{7\pi}{4}.$$

QUESTION 8

(a)

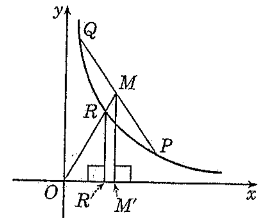


(i) Area $\Delta OPP' = \frac{1}{2} \times p \times \frac{1}{p} = \frac{1}{2}$.

(ii) Similarly, area $\Delta OQQ' = \frac{1}{2}$.

$$\begin{aligned}
\text{Area } OPQ &= \text{area } Q'QPP' + \text{area } \Delta OQQ' - \text{area } \Delta OPP' \\
&= \text{area } Q'QPP' + \frac{1}{2} - \frac{1}{2} \\
&= \text{area } Q'QPP'.
\end{aligned}$$

(iii)



Let M' be the point on the x-axis below M.

M has coordinates $\left(\frac{p+q}{2}, \frac{p+q}{2pq}\right)$ and

M' has coordinates $\left(\frac{p+q}{2}, 0\right)$.

$\Delta ORR' \,|||\, \Delta OMM'$ (equal $\angle$s)

$\therefore \quad \frac{OR'}{RR'} = \frac{OM'}{MM'}$ (equivalent ratios in similar Δs)

$$r \div \frac{1}{r} = \frac{p+q}{2} \div \frac{p+q}{2pq}$$

$$r^2 = pq.$$

(iv) Area $Q'QRR' = \int_q^r \frac{1}{x}\,dx$

$$= \ln\left(\frac{r}{q}\right).$$

Area $R'RPP' = \int_r^p \frac{1}{x}\,dx$

$$= \ln\left(\frac{p}{r}\right).$$

Now $r^2 = pq$ implies $\frac{r}{q} = \frac{p}{r}$.

Therefore the two areas are equal.

(v) Area OQR = area $Q'QRR'$, (from (ii), with R replacing P)
= area $R'RPP'$, (from (iv))
= area ORP, (from (ii), with R replacing Q).

Therefore OR divides region OPQ into two pieces of equal area.

(b) (i)
$$I_n + I_{n+2} = \int_0^{\frac{\pi}{4}} \left(\tan^n x + \tan^{n+2} x\right) dx$$
$$= \int_0^{\frac{\pi}{4}} \tan^n x\left(1 + \tan^2 x\right) dx$$
$$= \int_0^{\frac{\pi}{4}} \tan^n x \sec^2 x \ dx$$
$$= \left[\frac{\tan^{n+1} x}{n+1}\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{n+1}.$$

(ii)
$$J_n - J_{n-1} = (-1)^n I_{2n} - (-1)^{n-1} I_{2n-2}$$
$$= (-1)^n \left(I_{2n} + I_{2n-2}\right)$$
$$= (-1)^n \times \frac{1}{(2n-2)+1}, \text{ from (i)},$$
$$= \frac{(-1)^n}{2n-1}.$$

Note: $n \geq 1$ is required so that J_{n-1}, I_{2n-2} are defined.

(iii)
$$\sum_{n=1}^{m} \frac{(-1)^n}{2n-1}$$
$$= \sum_{n=1}^{m} \left(J_n - J_{n-1}\right)$$
$$= \left(J_1 - J_0\right) + \left(J_2 - J_1\right) + \cdots + \left(J_m - J_{m-1}\right)$$
$$= J_m - J_0.$$
$$J_0 = I_0 = \int_0^{\frac{\pi}{4}} dx = \frac{\pi}{4}.$$
$$\therefore\ J_m = \frac{\pi}{4} + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}.$$

(iv)
$$u = \tan x$$
$$x = \tan^{-1} u$$
$$dx = \frac{du}{1+u^2}.$$

When $x = \frac{\pi}{4}$, $u = 1$.

When $x = 0$, $u = 0$.
$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \ dx$$
$$= \int_0^1 u^n \cdot \frac{du}{1+u^2}$$
$$= \int_0^1 \frac{u^n}{1+u^2} \ du.$$

(v) $\dfrac{u^n}{1+u^2} \geq 0$ for $0 \leq u \leq 1$.
$$\therefore\ I_n = \int_0^1 \frac{u^n}{1+u^2} \ du \geq 0$$
$$I_n = \frac{1}{n+1} - I_{n+2} \text{ from (i)},$$
$$\leq \frac{1}{n+1}, \text{ since } I_{n+2} \geq 0.$$
$$\therefore\ 0 \leq I_n \leq \frac{1}{n+1}$$
$$\therefore\ I_n \to 0 \text{ as } n \to \infty.$$
$$\therefore\ J_n = (-1)^n I_{2n} \to 0 \text{ as } n \to \infty.$$

Note that this result produces an infinite series for $\frac{\pi}{4}$.
$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$
$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

END OF MATHEMATICS EXTENSION 2 SOLUTIONS