BOARD OF STUDIES
NEW SOUTH WALES

2011
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

3380

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int x \ln x \, dx$. **2**

(b) Evaluate $\int_0^3 x\sqrt{x+1}\, dx$. **3**

(c) (i) Find real numbers a, b and c such that **2**

$$\frac{1}{x^2(x-1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1}.$$

(ii) Hence, find $\int \frac{1}{x^2(x-1)}\, dx$. **2**

(d) Find $\int \cos^3\theta \, d\theta$. **3**

(e) Evaluate $\int_{-1}^{1} \frac{1}{5-2t+t^2}\, dt$. **3**

Question 2 (15 marks) Use a SEPARATE writing booklet.

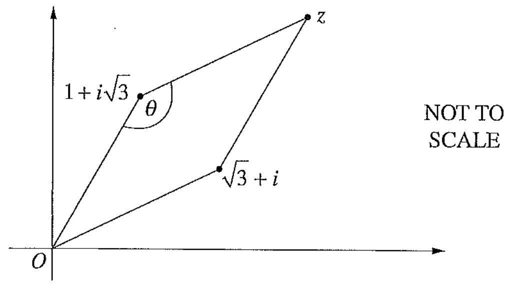
(a) Let $w = 2-3i$ and $z = 3+4i$.

(i) Find $\overline{w} + z$. **1**

(ii) Find $|w|$. **1**

(iii) Express $\frac{w}{z}$ in the form $a+ib$, where a and b are real numbers. **2**

(b) On the Argand diagram, the complex numbers 0, $1+i\sqrt{3}$, $\sqrt{3}+i$ and z form a rhombus.



(i) Find z in the form $a+ib$, where a and b are real numbers. **1**

(ii) An interior angle, θ, of the rhombus is marked on the diagram. **2**

Find the value of θ.

(c) Find, in modulus-argument form, all solutions of $z^3 = 8$. **2**

(d) (i) Use the binomial theorem to expand $(\cos\theta + i\sin\theta)^3$. **1**

(ii) Use de Moivre's theorem and your result from part (i) to prove that **3**

$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta.$$

(iii) Hence, or otherwise, find the smallest positive solution of **2**

$$4\cos^3\theta - 3\cos\theta = 1.$$

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Draw a one-third page sketch of the graph $y = \sin\frac{\pi}{2}x$ for $0 < x < 4$. **1**

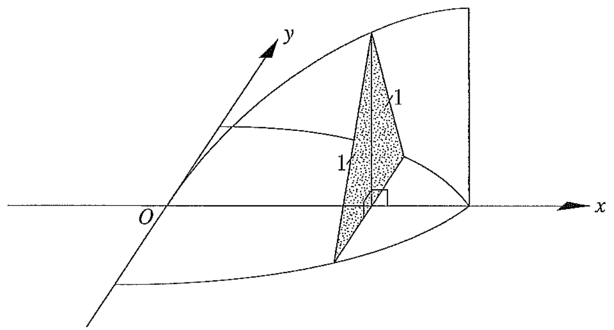
(ii) Find $\lim_{x \to 0} \frac{x}{\sin\frac{\pi}{2}x}$. **1**

(iii) Draw a one-third page sketch of the graph $y = \frac{x}{\sin\frac{\pi}{2}x}$ for $0 < x < 4$. **2**

(Do NOT calculate the coordinates of any turning points.)

(b) The base of a solid is formed by the area bounded by $y = \cos x$ and $y = -\cos x$ for $0 \le x \le \frac{\pi}{2}$. **3**

Vertical cross-sections of the solid taken parallel to the y-axis are in the shape of isosceles triangles with the equal sides of length 1 unit as shown in the diagram.



Find the volume of the solid.

Question 3 continues on page 5

(c) Use mathematical induction to prove that $(2n)! \ge 2^n (n!)^2$ for all positive integers n. **3**

(d) The equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$ represents a hyperbola.

(i) Find the eccentricity e. **1**

(ii) Find the coordinates of the foci. **1**

(iii) State the equations of the asymptotes. **1**

(iv) Sketch the hyperbola. **1**

(v) For the general hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, describe the effect on the hyperbola as $e \to \infty$. **1**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet.

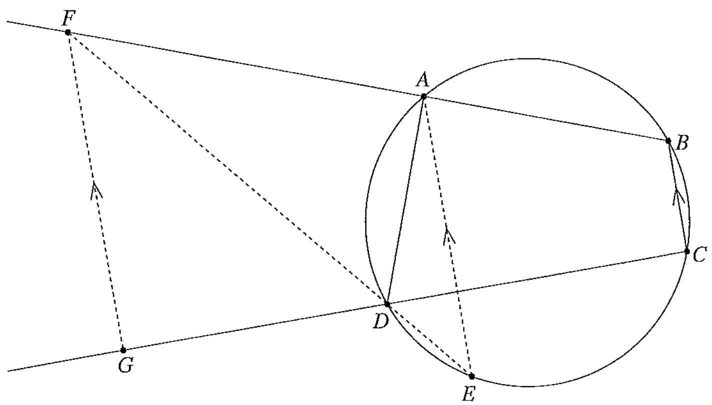
(a) Let a and b be real numbers with $a \neq b$. Let $z = x + iy$ be a complex number such that

$$|z-a|^2 - |z-b|^2 = 1.$$

(i) Prove that $x = \dfrac{a+b}{2} + \dfrac{1}{2(b-a)}$. **2**

(ii) Hence, describe the locus of all complex numbers z such that $|z-a|^2 - |z-b|^2 = 1$. **1**

(b) In the diagram, $ABCD$ is a cyclic quadrilateral. The point E lies on the circle through the points A, B, C and D such that $AE \parallel BC$. The line ED meets the line BA at the point F. The point G lies on the line CD such that $FG \parallel BC$.



Copy or trace the diagram into your writing booklet.

(i) Prove that $FADG$ is a cyclic quadrilateral. **2**

(ii) Explain why $\angle GFD = \angle AED$. **1**

(iii) Prove that GA is a tangent to the circle through the points A, B, C and D. **2**

Question 4 continues on page 7

(c) A mass is attached to a spring and moves in a resistive medium. The motion of the mass satisfies the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0,$$

where y is the displacement of the mass at time t.

(i) Show that, if $y = f(t)$ and $y = g(t)$ are both solutions to the differential equation and A and B are constants, then **2**

$$y = Af(t) + Bg(t)$$

is also a solution.

(ii) A solution of the differential equation is given by $y = e^{kt}$ for some values of k, where k is a constant. **2**

Show that the only possible values of k are $k = -1$ and $k = -2$.

(iii) A solution of the differential equation is **3**

$$y = Ae^{-2t} + Be^{-t}.$$

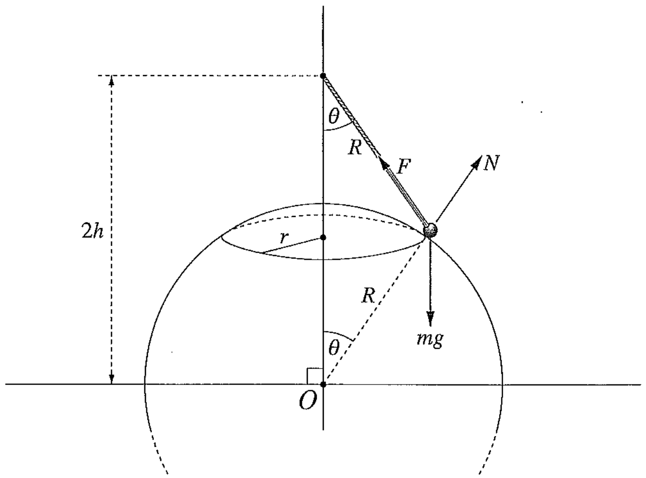
When $t = 0$, it is given that $y = 0$ and $\dfrac{dy}{dt} = 1$.

Find the values of A and B.

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) A small bead of mass m is attached to one end of a light string of length R. The other end of the string is fixed at height $2h$ above the centre of a sphere of radius R, as shown in the diagram. The bead moves in a circle of radius r on the surface of the sphere and has constant angular velocity $\omega > 0$. The string makes an angle of θ with the vertical.



Three forces act on the bead: the tension force F of the string, the normal reaction force N to the surface of the sphere, and the gravitational force mg.

(i) By resolving the forces horizontally and vertically on a diagram, show that **2**

$$F\sin\theta - N\sin\theta = m\omega^2 r$$

and

$$F\cos\theta + N\cos\theta = mg.$$

(ii) Show that **2**

$$N = \frac{1}{2}mg\sec\theta - \frac{1}{2}m\omega^2 r\operatorname{cosec}\theta.$$

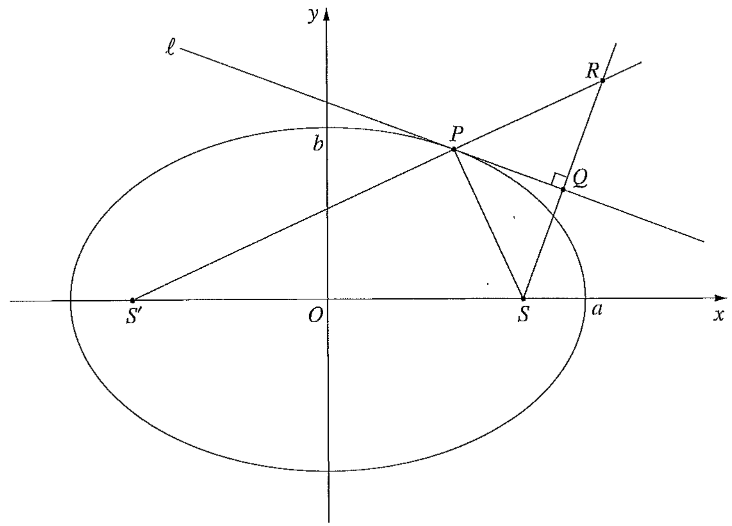
(iii) Show that the bead remains in contact with the sphere if $\omega \le \sqrt{\frac{g}{h}}$. **2**

Question 5 continues on page 9

(b) If p, q and r are positive real numbers and $p + q \ge r$, prove that **3**

$$\frac{p}{1+p} + \frac{q}{1+q} - \frac{r}{1+r} \ge 0.$$

(c) The diagram shows the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$. The line ℓ is the tangent to the ellipse at the point P. The foci of the ellipse are S and S'. The perpendicular to ℓ through S meets ℓ at the point Q. The lines SQ and $S'P$ meet at the point R.



Copy or trace the diagram into your writing booklet.

(i) Use the reflection property of the ellipse at P to prove that $SQ = RQ$. **2**

(ii) Explain why $S'R = 2a$. **1**

(iii) Hence, or otherwise, prove that Q lies on the circle $x^2 + y^2 = a^2$. **3**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) Jac jumps out of an aeroplane and falls vertically. His velocity at time t after his parachute is opened is given by $v(t)$, where $v(0) = v_0$ and $v(t)$ is positive in the downwards direction. The magnitude of the resistive force provided by the parachute is kv^2, where k is a positive constant. Let m be Jac's mass and g the acceleration due to gravity. Jac's terminal velocity with the parachute open is v_T.

Jac's equation of motion with the parachute open is

$$m\frac{dv}{dt} = mg - kv^2. \quad \text{(Do NOT prove this.)}$$

(i) Explain why Jac's terminal velocity v_T is given by $\sqrt{\frac{mg}{k}}$. **1**

(ii) By integrating the equation of motion, show that t and v are related by the equation **3**

$$t = \frac{v_T}{2g}\ln\left[\frac{(v_T + v)(v_T - v_0)}{(v_T - v)(v_T + v_0)}\right].$$

(iii) Jac's friend Gil also jumps out of the aeroplane and falls vertically. Jac and Gil have the same mass and identical parachutes. **3**

Jac opens his parachute when his speed is $\frac{1}{3}v_T$. Gil opens her parachute when her speed is $3v_T$. Jac's speed increases and Gil's speed decreases, both towards v_T.

Show that in the time taken for Jac's speed to double, Gil's speed has halved.

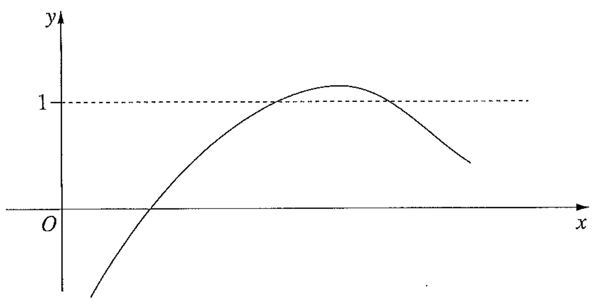
Question 6 continues on page 11

(b) Let $f(x)$ be a function with a continuous derivative.

(i) Prove that $y = (f(x))^3$ has a stationary point at $x = a$ if $f(a) = 0$ or $f'(a) = 0$. **2**

(ii) Without finding $f''(x)$, explain why $y = (f(x))^3$ has a horizontal point of inflexion at $x = a$ if $f(a) = 0$ and $f'(a) \neq 0$. **1**

(iii) The diagram shows the graph $y = f(x)$. **3**



Copy or trace the diagram into your writing booklet.

On the diagram in your writing booklet, sketch the graph $y = (f(x))^3$, clearly distinguishing it from the graph $y = f(x)$.

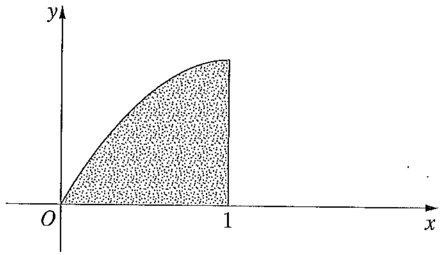
(c) On an Argand diagram, sketch the region described by the inequality **2**

$$\left|1 + \frac{1}{z}\right| \leq 1.$$

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of $f(x) = \dfrac{x}{1+x^2}$ for $0 \le x \le 1$. **4**



The area bounded by $y = f(x)$, the line $x = 1$ and the x-axis is rotated about the line $x = 1$ to form a solid.

Use the method of cylindrical shells to find the volume of the solid.

(b) Let $I = \displaystyle\int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx$.

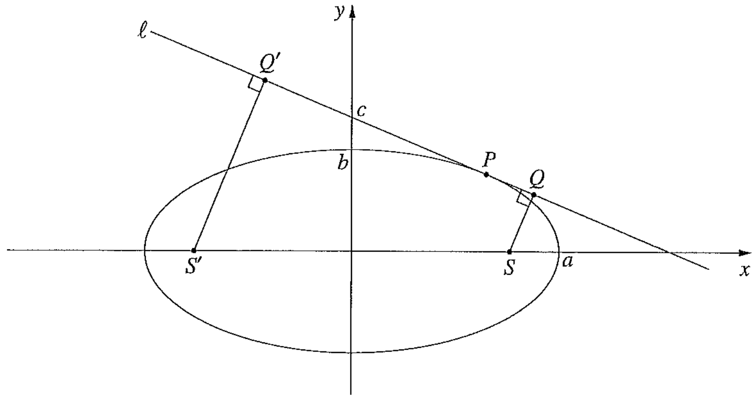
(i) Use the substitution $u = 4 - x$ to show that **2**

$$I = \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du.$$

(ii) Hence, find the value of I. **3**

Question 7 continues on page 13

(c) The diagram shows the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b$. Let e be the eccentricity of the ellipse.



The line ℓ is the tangent to the ellipse at the point P. The line ℓ has equation $y = mx + c$, where m is the slope and c is the y-intercept.

The points S and S' are the focal points of the ellipse, where S is on the positive x-axis. The perpendiculars to ℓ through S and S' intersect ℓ at Q and Q' respectively.

(i) By substituting the equation for ℓ into the equation for the ellipse, show that **3**

$$a^2m^2 + b^2 = c^2.$$

(ii) Show that the perpendicular distance from S to ℓ is given by **1**

$$QS = \frac{|mae + c|}{\sqrt{1+m^2}}.$$

(iii) It is given that $Q'S' = \dfrac{|mae - c|}{\sqrt{1+m^2}}$. **2**

Hence, prove that $QS \times Q'S' = b^2$.

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) For every integer $m \geq 0$ let **3**

$$I_m = \int_0^1 x^m \left(x^2 - 1\right)^5 dx.$$

Prove that for $m \geq 2$

$$I_m = \frac{m-1}{m+11} I_{m-2}.$$

(b) A bag contains seven balls numbered from 1 to 7. A ball is chosen at random and its number is noted. The ball is then returned to the bag. This is done a total of seven times.

(i) What is the probability that each ball is selected exactly once? **1**

(ii) What is the probability that at least one ball is not selected? **1**

(iii) What is the probability that exactly one of the balls is not selected? **2**

Question 8 continues on page 15

(c) Let β be a root of the complex monic polynomial

$$P(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0.$$

Let M be the maximum value of $\left|a_{n-1}\right|, \left|a_{n-2}\right|, \ldots, \left|a_0\right|$.

(i) Show that $\left|\beta\right|^n \leq M\left(\left|\beta\right|^{n-1} + \left|\beta\right|^{n-2} + \cdots + \left|\beta\right| + 1\right)$. **2**

(ii) Hence, show that for any root β of $P(z)$ **3**

$$\left|\beta\right| < 1 + M.$$

(d) Let $S(x) = \sum_{k=0}^{n} c_k \left(x + \frac{1}{x}\right)^k$, where the real numbers c_k satisfy $\left|c_k\right| \leq \left|c_n\right|$ for all $k < n$, and $c_n \neq 0$. **3**

Using part (c), or otherwise, show that $S(x) = 0$ has no real solutions.

End of paper

2011 Higher School Certificate Solutions Mathematics Extension 2

Question 1

(a) Let $u = \ln x$ and $\dfrac{dv}{dx} = x$

$\dfrac{du}{dx} = \dfrac{1}{x}, \qquad v = \dfrac{x^2}{2}$

$$\int x \ln x \, dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}.\frac{1}{x}\,dx$$
$$= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$$
$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c.$$

(b) Let $u = x+1$ when $x=0,\ u=1$

$\dfrac{du}{dx} = 1$ when $x=3,\ u=4$

$du = dx$

$$\int_0^3 x\sqrt{x+1}\,dx = \int_1^4 (u-1)\sqrt{u}\,du$$
$$= \int_1^4 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$
$$= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^4$$
$$= \left(\frac{2}{5}\times 32 - \frac{2}{3}\times 8\right) - \left(\frac{2}{5} - \frac{2}{3}\right)$$
$$= \frac{116}{15}.$$

(c) (i) $\dfrac{1}{x^2(x-1)} = \dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{c}{x-1}$

$1 = ax(x-1) + b(x-1) + cx^2$

when $x=1,\ 1=c \quad \therefore c=1$

when $x=0,\ 1=-b \quad \therefore b=-1$

Coefficients of x^2, $0 = a+c \quad \therefore a=-1$

$\therefore a=-1,\ b=-1,\ c=1.$

(ii)
$$\int \frac{1}{x^2(x-1)}\,dx = \int\left(-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\right)dx$$
$$= -\ln x + \frac{1}{x} + \ln(x-1) + c$$
$$= \ln\left(\frac{x-1}{x}\right) + \frac{1}{x} + c.$$

(d)
$$\int \cos^3\theta\,d\theta = \int \cos^2\theta.\cos\theta\,d\theta$$
$$= \int\left(1-\sin^2\theta\right)\cos\theta\,d\theta$$
$$= \int\left(\cos\theta - \sin^2\theta.\cos\theta\right)d\theta$$
$$= \sin\theta - \frac{1}{3}\sin^3\theta + c.$$

(e)
$$\int_{-1}^{1}\frac{1}{5-2t+t^2}\,dt = \int_{-1}^{1}\frac{1}{4+1-2t+t^2}\,dt$$
$$= \int_{-1}^{1}\frac{1}{4+(t-1)^2}\,dt$$
$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{t-1}{2}\right)\right]_{-1}^{1}$$
$$= \frac{1}{2}\left(\tan^{-1}(0) - \tan^{-1}(-1)\right)$$
$$= \frac{1}{2}\left(0 - \left(-\frac{\pi}{4}\right)\right)$$
$$= \frac{\pi}{8}.$$

Question 2

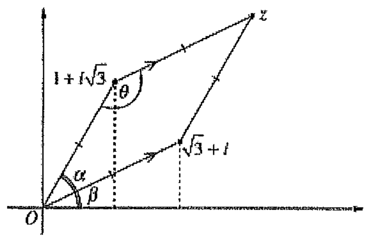
(a) (i) $\overline{w} + z = 2+3i+3+4i$
$= 5+7i.$

(ii) $|w| = \sqrt{2^2+(-3)^2}$
$= \sqrt{13}.$

(iii)
$$\frac{w}{z} = \frac{2-3i}{3+4i}\times\frac{3-4i}{3-4i}$$
$$= \frac{6-8i-9i-12}{9+16}$$
$$= \frac{-6-17i}{25}$$
$$= -\frac{6}{25} + i\left(-\frac{17}{25}\right).$$

(b) (i) $z = \sqrt{3}+i+1+i\sqrt{3}$
$= \sqrt{3}+1+i(\sqrt{3}+1).$

(ii)



Method 1:

$\tan\alpha = \dfrac{\sqrt{3}}{1} \Rightarrow \alpha = \dfrac{\pi}{3}$

$\tan\beta = \dfrac{1}{\sqrt{3}} \Rightarrow \beta = \dfrac{\pi}{6}$

$\therefore \alpha - \beta = \dfrac{\pi}{3} - \dfrac{\pi}{6} = \dfrac{\pi}{6}$

$\therefore \theta = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}.$ (Co-interior angles)

Method 2:

Let $w = \sqrt{3}+i$

$\arg z = \tan^{-1}\left(\dfrac{\sqrt{3}+1}{\sqrt{3}+1}\right) = \dfrac{\pi}{4}$

$\arg(\sqrt{3}+i) = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

Difference $= \dfrac{\pi}{4} - \dfrac{\pi}{6} = \dfrac{\pi}{12}$

Thus $\theta = \pi - 2\times\dfrac{\pi}{12}$ ($\angle$ sum of Δ)

$= \dfrac{5\pi}{6}.$

Method 3:

$$|z| = \sqrt{\left(1+\sqrt{3}\right)^2 + \left(\sqrt{3}+1\right)^2}$$
$$= \sqrt{2\left(1+\sqrt{3}\right)^2}$$
$$= \sqrt{2}\left(1+\sqrt{3}\right)$$

and $\left|\sqrt{3}+i\right| = \left|1+i\sqrt{3}\right| = 2$

By the cosine rule:

$$\cos\theta = \frac{2^2+2^2-\left[\sqrt{2}\left(1+\sqrt{3}\right)\right]^2}{2\times 2\times 2}$$
$$= \frac{8-2\left(1+2\sqrt{3}+3\right)}{8}$$
$$= \frac{-4\sqrt{3}}{8}$$
$$= -\frac{\sqrt{3}}{2}$$
$$\theta = \pi - \frac{\pi}{6}$$
$$= \frac{5\pi}{6}.$$

(c) *Method 1:*

$z^3 = 8$

$z^3 = 8[\cos(2k\pi) + i\sin(2k\pi)]$ $\quad k$ integer

$z = 2\left[\cos\left(\dfrac{2k\pi}{3}\right) + i\sin\left(\dfrac{2k\pi}{3}\right)\right]$ (de Moivre)

$k=0, \quad z = 2[\cos 0 + i\sin 0]$

$k=1, \quad z = 2\left[\cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right)\right]$

$k=-1, \quad z = 2\left[\cos\left(\dfrac{-2\pi}{3}\right) + i\sin\left(\dfrac{-2\pi}{3}\right)\right].$

Method 2:

$$z^3 = 8$$
$$z^3 - 2^3 = 0$$
$$(z-2)(z^2+2z+4) = 0$$
$$z = 2 \text{ or } z^2+2z+4=0$$
$$z = \frac{-2 \pm \sqrt{2^2 - 4\times1\times4}}{2\times1}$$
$$z = \frac{-2\pm\sqrt{-12}}{2}$$
$$z = -1 \pm \sqrt{-3}$$
$$= -1 \pm i\sqrt{3}$$
$$z_1 = 2 = 2cis0$$
$$z_2 = -1 + i\sqrt{3}$$
$$= 2cis\left(\frac{2\pi}{3}\right)$$
$$z_3 = -1 - i\sqrt{3}$$
$$= 2cis\left(\frac{-2\pi}{3}\right).$$

(d) (i)

$$(\cos\theta + i\sin\theta)^3 = \sum_{k=0}^{3} {}^3C_k(\cos\theta)^{3-k}(i\sin\theta)^k$$
$$= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$$
$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$

(ii) By de Moivre's theorem,

$$(\cos\theta + i\sin\theta)^3 = \cos3\theta + i\sin3\theta$$

From (i), equating real parts:

$$\cos^3\theta - 3\cos\theta\sin^2\theta = \cos3\theta$$
$$\cos^3\theta - 3\cos\theta(1-\cos^2\theta) = \cos3\theta$$
$$4\cos^3\theta - 3\cos\theta = \cos3\theta$$
$$\therefore \cos^3\theta = \frac{1}{4}\cos3\theta + \frac{3}{4}\cos\theta.$$

(iii)

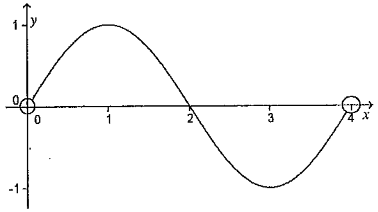
$$4\cos^3\theta - 3\cos\theta = 1$$
$$4\left(\frac{1}{4}\cos3\theta + \frac{3}{4}\cos\theta\right) - 3\cos\theta = 1$$
$$\cos3\theta + 3\cos\theta - 3\cos\theta = 1$$
$$\cos3\theta = 1$$

$$3\theta = 2k\pi \quad k \text{ integer}$$
$$\theta = \frac{2k\pi}{3}$$

The smallest positive solution is when $k=1$ giving a value of $\frac{2\pi}{3}$.

Question 3

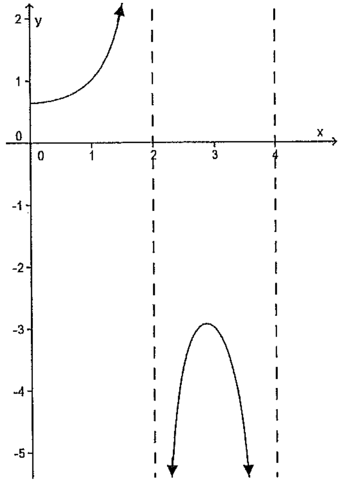
(a) (i)



(ii)

$$\lim_{x\to0}\frac{x}{\sin\frac{\pi}{2}x} = \frac{2}{\pi}\times\lim_{x\to0}\frac{\frac{\pi}{2}x}{\sin\frac{\pi}{2}x}$$
$$= \frac{2}{\pi}\times1$$
$$= \frac{2}{\pi}$$

(iii)



(b) Base cross-section $= 2\cos x$
Height $= \sin x$
The volume of the triangular strip is given by:

$$V = \lim_{\delta x\to0}\sum_{x=0}^{\frac{\pi}{2}}\frac{1}{2}\times2\cos x\times\sin x.\,\delta x$$
$$= \int_0^{\frac{\pi}{2}}\sin x\cos x\,dx$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin2x\,dx$$
$$= \left[-\frac{1}{4}\cos2x\right]_0^{\frac{\pi}{2}}$$
$$= -\frac{1}{4}(-1-1)$$
$$= \frac{1}{2}$$

$\therefore$ the volume is $\frac{1}{2}$ units^3.

(c) Need to prove $(2n)! \geq 2^n(n!)^2$

For $n=1$:
LHS $= 2! = 2$
RHS $= 2^1(1!)^2 = 2$
$\therefore$ true for n= 1
Assume that it is true for $n=k$.
i.e. $(2k)! \geq 2^k(k!)^2$ (α)
For $n=k+1$: $(2(k+1))! \geq 2^{k+1}((k+1)!)^2$

$$LHS = (2(k+1))!$$
$$= (2k+2)!$$
$$= (2k)!(2k+1)(2k+2)$$
$$\geq 2^k(k!)^2(2k+1)(2k+2) \quad \text{from } (\alpha)$$
$$\geq 2^{k+1}(k!)^2(2k+1)(k+1)$$
$$\geq 2^{k+1}(k!)^2(k+1)^2 \quad \text{since } 2k+1 > k+1$$
$$\geq 2^{k+1}((k+1)!)^2$$
$$= RHS$$

$\therefore$ statement is true for $n=k+1$.
Thus it is true for $n=1$ and is true for $n=k+1$ assuming it is true for $n=k$.
Therefore by Mathematical Induction it is true for all positive integers.

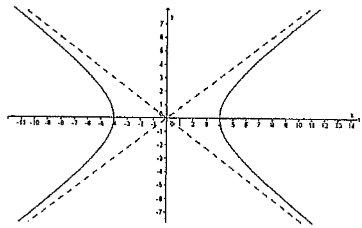
(d) (i)

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$
$$\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$$
$$a=4, \quad b=3$$
$$b^2 = a^2(e^2-1)$$
$$3^2 = 4^2(e^2-1)$$
$$e^2 - 1 = \frac{9}{16}$$
$$e^2 = \frac{25}{16}$$
$$e = \frac{5}{4}.$$

(ii) Foci $(\pm ae, 0)$
i.e. $(5,0)$ and $(-5,0)$.

(iii) Asymptotes are $y = \pm\frac{b}{a}x$
i.e. $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$.

(iv)



(v) As $e\to\infty$, $b\to\infty$. This means that the gradient of the asymptotes $\to\infty$.
Thus the asymptotes (and the hyperbola) tends to the y-axis from both sides.

Question 4

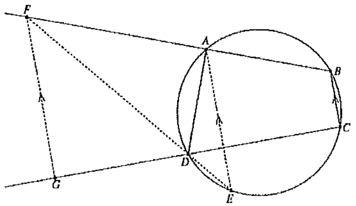
(a) (i)

$$|z-a|^2 - |z-b|^2 = 1$$
$$|(x+iy)-a|^2 - |(x+iy)-b|^2 = 1$$
$$|x-a+iy|^2 - |x-b+iy|^2 = 1$$
$$\left(\sqrt{(x-a)^2+y^2}\right)^2 - \left(\sqrt{(x-b)^2+y^2}\right)^2 = 1$$
$$(x-a)^2 + y^2 - \left[(x-b)^2+y^2\right] = 1$$

$$(x-a)^2-(x-b)^2=1$$
$$x^2-2ax+a^2-x^2+2bx-b^2=1$$
$$-2ax+a^2+2bx-b^2=1$$
$$2x(b-a)=b^2-a^2+1$$
$$\therefore x=\frac{b+a}{2}+\frac{1}{2(b-a)} \text{ as required.}$$

(ii) The locus is a vertical line with equation $x=\dfrac{b+a}{2}+\dfrac{1}{2(b-a)}$.

(b) (i)



Let $\angle BCD = x$
$\angle FAD = x$ (exterior $\angle$ of cyclic quad)
$\angle FGC = \pi - x$ (co-interior $\angle$'s, with $FG \parallel BC$)
$\angle FAD$ and $\angle FGD$ are supplementary, thus $FADG$ is a cyclic quadrilateral.

(ii) $\angle GFD = \angle AED$ (alternate $\angle$'s, with $FG \parallel BC$).

(iii) $\angle GFD = \angle GAD$ ($\angle$'s on circumference)
$\angle GFD = \angle AED$ (alternate $\angle$'s, FG||AE)
$\therefore \angle AED = \angle GAD$ (both equal $\angle GFD$)
Since $\angle AED = \angle GAD$, this satisfies the condition for the angle in the alternate segment, hence GA is a tangent to the circle $ABCD$.

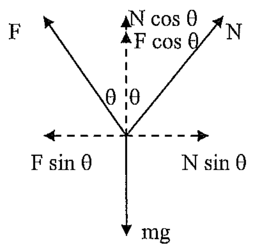
(c) (i) If $y=f(t)$ an dy $=g(t)$ are solutions of:
$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=0,$$
then $f''(t)+3f'(t)+2f(t)=0$
and $g''(t)+3g'(t)+2g(t)=0$
If $y=Af(t)+Bg(t)$ then
$$\frac{dy}{dt}=Af'(t)+Bg'(t) \quad \text{and}$$
$$\frac{d^2y}{dt^2}=Af''(t)+Bg''(t)$$
$$\begin{aligned}\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y &= Af''(t)+Bg''(t)\\ &\quad+3\left(Af'(t)+Bg'(t)\right)\\ &\quad+2A\left(f(t)+Bg(t)\right)\\ &= A\left(f''(t)+3f'(t)+2f(t)\right)\\ &\quad+B\left(g''(t)+3g'(t)+2g(t)\right)\\ &= A(0)+B(0)\\ &=0\end{aligned}$$
Hence $y=Af(t)+Bg(t)$ is a solution.

(ii) If $y=e^{kt}$ is a solution then
$$k^2e^{kt}+3ke^{kt}+2e^{kt}=0$$
$$e^{kt}\left(k^2+3k+2\right)=0, \quad e^{kt}\neq 0$$
$$(k+2)(k+1)=0$$
$$k=-1,-2$$
$\therefore k=-1,-2$ are the only solutions.

(iii) $y=Ae^{-2t}+Be^{-t}$
$$\frac{dy}{dx}=-2Ae^{-2t}-Be^{-t}$$
when $t=0,\ y=0 \quad \rightarrow \quad A+B=0$
when $t=0,\ \dfrac{dy}{dx}=1 \quad \rightarrow -2A-B=1$
On inspection $\quad A=-1, \quad B=1$
$\therefore y=-e^{-2t}+e^{2t}$.

Question 5

(a) (i)



Horizontally, net force into the centre of the circle,
$$F\sin\theta-N\sin\theta=mr\omega^2 \quad ①$$

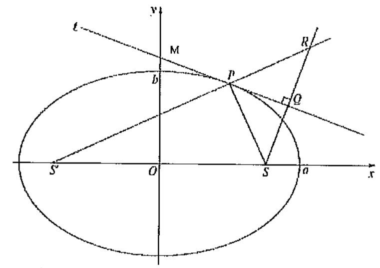
Vertically, equilibrium
$$F\cos\theta+N\cos\theta=mg \quad ②$$

(ii) Multiply ① by $\cos\theta$ and ② by $\sin\theta$:
$$F\sin\theta\cos\theta-N\sin\theta\cos\theta=mr\omega^2\cos\theta$$
$$F\sin\theta\cos\theta+N\sin\theta\cos\theta=mg\sin\theta$$
Subtracting gives
$$2N\sin\theta\cos\theta=mg\sin\theta-mr\omega^2\cos\theta$$
$$N=\frac{mg}{2\cos\theta}-\frac{mr\omega^2}{2\sin\theta}$$
$$N=\frac{1}{2}mg\sec\theta-\frac{1}{2}mr\omega^2\operatorname{cosec}\theta.$$

(iii) When $N=0$, the bead is at the point of losing contact with the sphere;
i.e. the bead is in contact when $N\geq 0$
$$\frac{1}{2}mg\sec\theta-\frac{1}{2}mr\omega^2\operatorname{cosec}\theta\geq 0$$
$$\frac{1}{2}mg\sec\theta\geq\frac{1}{2}mr\omega^2\operatorname{cosec}\theta$$
m, r, and $\operatorname{cosec}\theta$ are all positive
$$\omega^2\leq\frac{g\sin\theta}{r\cos\theta}$$
$$\omega^2\leq\frac{g}{r}\tan\theta \qquad \text{but } \tan\theta=\frac{r}{h}$$
$$\omega^2\leq\frac{g}{r}\times\frac{r}{h}$$
$$\omega^2\leq\frac{g}{h}$$
$$\therefore\ \omega\leq\sqrt{\frac{g}{h}} \text{ (since } g>0 \text{ and } h>0\text{).}$$

(b)
$$\frac{p}{1+p}+\frac{q}{1+q}-\frac{r}{1+r}$$
$$=\frac{p(1+q)(1+r)+q(1+p)(1+r)-r(1+p)(1+q)}{(1+p)(1+q)(1+r)}$$
$$=\frac{p(1+q+r+qr)+q(1+p+r+pr)}{(1+p)(1+q)(1+r)}+\frac{-r(1+p+q+pq)}{(1+p)(1+q)(1+r)}$$
$$=\frac{p+pq+pr+pqr+q+pq+qr+pqr}{(1+p)(1+q)(1+r)}+\frac{-r-pr-qr-pqr}{(1+p)(1+q)(1+r)}$$
$$=\frac{p+q+2pq+pqr-r}{(1+p)(1+q)(1+r)}$$
$$\geq\frac{r+2pq+pqr-r}{(1+p)(1+q)(1+r)} \quad \text{since } p+q\geq r$$
$$\geq\frac{2pq+pqr}{(1+p)(1+q)(1+r)}$$
$$\geq 0 \quad \text{since } p,\ q,\ r\geq 0.$$

(c) (i)



M is the intersection of ℓ with y-axis.
$\angle MPS' = \angle QPS$ (Property of ellipse)
$\angle MPS' = \angle QPR$ (Vert. opp. $\angle$s)
In ΔSPQ and ΔRPQ:
$\angle QPR = \angle QPS$ (Proven above)
$\angle RQP = \angle SQP = 90°$ ($SQ \perp \ell$)
PQ is common.
$\Delta SPQ \equiv \Delta RPQ$ (ASA)
$\therefore SQ = RQ$ (corresponding sides of congruent triangles).

(ii) $S'P+PS=2a$ (Property of an ellipse)
$S'P+PR=2a$ (PR=PS corresponding sides of congruent triangles from (i))
$\therefore S'R=2a$ ($S'P+PR=S'R$).

(iii) Consider $S'R$ as a radius of a circle with centre $S'(-ae,0)$ with radius $2a$ from (ii) above.
Let R have coordinates (x_R,y_R), where
$$\left(x_R--ae\right)^2+\left(y_R-0\right)^2=(2a)^2$$
$$(x_R+ae)^2+{y_R}^2=4a^2$$
Q is the midpoint of RS, Since $RQ=SQ$ (corresponding sides of congruent triangles are equal) with $S(ae,0)$.
Thus Q has coordinates $\left(\dfrac{ae+x_R}{2},\dfrac{y_R}{2}\right)$.

At Q:

$$x^2+y^2=\frac{(ae+x_R)^2}{4}+\frac{y_R^{\ 2}}{4}$$
$$=\frac{1}{4}\left[(x_R+ae)^2+y_R^{\ 2}\right]$$
$$=\frac{1}{4}\times 4a^2$$
$$=a^2$$

$\therefore$ Q lies on the circle $x^2+y^2=a^2$ as required.

Question 6

(a) (i) $m\dfrac{dv}{dt}=mg-kv^2$

Terminal velocity occurs when $\dfrac{dv}{dt}\to 0$

$$mg-kv^2=0$$
$$mg=kv^2$$
$$v^2=\frac{mg}{k}$$
$$v_T=\sqrt{\frac{mg}{k}}.$$

(ii)

$$m\frac{dv}{dt}=mg-kv^2$$
$$\int\frac{m\,dv}{mg-kv^2}=\int dt$$
$$\frac{m}{k}\int\frac{dv}{\frac{mg}{k}-v^2}=\int dt$$
$$\frac{m}{k}\int\frac{dv}{\frac{mg}{k}-v^2}=\int dt$$
$$\int\frac{dv}{v_T^{\ 2}-v^2}=\frac{k}{m}\int dt$$
$$\frac{1}{2v_T}\int_0^v\frac{1}{v_T-v}+\frac{1}{v_T+v}\,dv=\frac{k}{m}\int_0^t dt$$
$$\frac{1}{2v_T}\left[-\ln(v_T-v)+\ln(v_T+v)\right]_0^v=\frac{k}{m}\left[t\right]_0^t$$
$$\frac{1}{2v_T}\ln\left(\frac{v_T+v}{v_T-v}\right)=\frac{kt}{m}$$

At $t=0, v=v_0$ $\quad\therefore c=\dfrac{1}{2v_T}\ln\left(\dfrac{v_T+v_0}{v_T-v_0}\right)$

Hence:

$$\frac{1}{2v_T}\ln\left(\frac{v_T+v}{v_T-v}\right)=\frac{kt}{m}+\frac{1}{2v_T}\ln\left(\frac{v_T+v_0}{v_T-v_0}\right)$$

and $\dfrac{kt}{m}=\dfrac{1}{2v_T}\ln\left(\dfrac{(v_T+v)(v_T-v_0)}{(v_T-v)(v_T+v_0)}\right)$

$$\therefore t=\frac{m}{2kv_T}\ln\left(\frac{(v_T+v)(v_T-v_0)}{(v_T-v)(v_T+v_0)}\right)$$

Noting $\dfrac{m}{k}=\dfrac{v_T^{\ 2}}{g}$ (from terminal velocity)

$$t=\frac{v_T}{2g}\ln\left(\frac{(v_T+v)(v_T-v_0)}{(v_T-v)(v_T+v_0)}\right).$$

(iii) Jac opens his parachute when $v_0=\dfrac{v_T}{3}$.

Let t_{Jac} be the time for Jac to double his speed (at which point $v=\dfrac{2}{3}v_T$).

$$t_{Jac}=\frac{v_T}{2g}\ln\left(\frac{\left(v_T+\frac{2}{3}v_T\right)\left(v_T-\frac{v_T}{3}\right)}{\left(v_T-\frac{2}{3}v_T\right)\left(v_T+\frac{v_T}{3}\right)}\right)$$
$$=\frac{v_T}{2g}\ln\left(\frac{\left(\frac{5}{3}v_T\right)\left(\frac{2v_T}{3}\right)}{\left(\frac{1}{3}v_T\right)\left(\frac{4v_T}{3}\right)}\right).$$
$$=\frac{v_T}{2g}\ln\left(\frac{\frac{10}{9}v_T^{\ 2}}{\frac{4}{9}v_T^{\ 2}}\right)$$
$$=\frac{v_T}{2g}\ln\left(\frac{5}{2}\right)$$

Gil opens her parachute when $v_0=3v_T$.

Let t_{Gil} be the time for Gil to halve her speed (at which point $v=\dfrac{3}{2}v_T$).

$$t_{Gil}=\frac{v_T}{2g}\ln\left(\frac{\left(v_T+\frac{3}{2}v_T\right)(v_T-3v_T)}{\left(v_T-\frac{3}{2}v_T\right)(v_T+3v_T)}\right)$$

$$t_{Gil}=\frac{v_T}{2g}\ln\left(\frac{\left(\frac{5}{2}v_T\right)(-2v_T)}{\left(-\frac{1}{2}v_T\right)(4v_T)}\right)$$
$$=\frac{v_T}{2g}\ln\left(\frac{-5v_T^{\ 2}}{-2v_T^{\ 2}}\right)$$
$$=\frac{v_T}{2g}\ln\left(\frac{5}{2}\right)$$
$$\therefore t_{Jac}=t_{Gil}=\frac{v_T}{2g}\ln\left(\frac{5}{2}\right)$$

The time taken for Jac's speed to double is the same as the time taken for Gil's speed to halve, as required.

(b) (i) If $y=(f(x))^3$, using the chain rule

$$\frac{dy}{dx}=3(f(x))^2f'(x)$$

If $\dfrac{dy}{dx}=0$ at $x=a$,

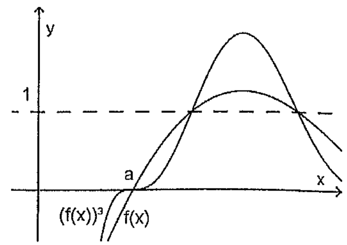
$$3(f(a))^2f'(a)=0$$
$$\therefore f'(a)=0 \text{ or } f(a)=0.$$

(ii) $y=(f(x))^3$ has a root of multiplicity 3. If $f(a)=0$, then $f'(a)=0$ and $f''(a)=0$ so there is a possible horizontal point of inflexion.

$\dfrac{dy}{dx}=3(f(x))^2f'(x)$, at points $x=a\pm\varepsilon$ (where ε is small):

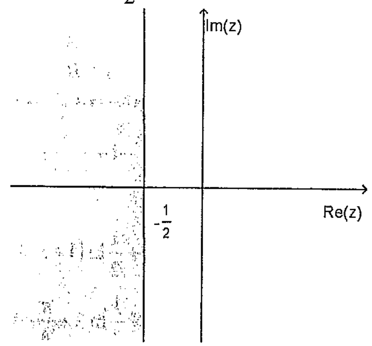
$3(f(a\pm\varepsilon))^2\geq 0$ and so the sign of $\dfrac{dy}{dx}$ is the same sign as $f'(x)$. Since $f'(a)\neq 0$, and the gradient has the same sign about a stationary point, the point is a horizontal point of inflexion.

(iii)



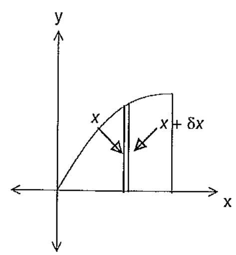
(c)

$$\left|1+\frac{1}{z}\right|\leq 1$$
$$\left|\frac{z+1}{z}\right|\leq 1$$
$$\frac{|z+1|}{|z|}\leq 1$$
$$|z-(-1)|\leq|z|$$
$$|x+1+iy|\leq|x+1+iy|$$
$$\sqrt{(x+1)^2+y^2}\leq\sqrt{x^2+y^2}$$
$$(x+1)^2+y^2\leq x^2+y^2$$
$$2x+1\leq 0$$
$$x\leq-\frac{1}{2}$$
$$\therefore \operatorname{Re}(z)\leq-\frac{1}{2}$$



Question 7

(a)



Volume of the shell (δV) is the large cylinder minus the small cylinder.

$\delta V=\pi R^2H-\pi r^2h$ where:

$R = 1 - x,$

$r = 1 - x - \delta x,$

$H = h = y$

$\delta V = \pi y\left\{(1-x)^2 - (1-x-\delta x)^2\right\}$

$= \pi y(2 - 2x - \delta x)\delta x$

$\approx 2\pi y(1-x)\delta x \ \left(\delta x^2 \text{ is negligible}\right)$

$V_{solid} = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi y(1-x)\delta x$

$= 2\pi \int_0^1 \frac{x(1-x)}{1+x^2}\,dx$

$= 2\pi \int_0^1 \frac{x - x^2}{1+x^2}\,dx$

Let $\frac{x-x^2}{1+x^2} = A + \frac{Bx+C}{1+x^2}$

$x - x^2 = A\left(1+x^2\right) + Bx + C$

Equate x^2: $\quad -1 = A$

Equate x: $\quad 1 = B$

Sub $x = 0$: $\quad 0 = A + C \to C = 1$

$V_{solid} = 2\pi \int_0^1 \left(-1 + \frac{x}{1+x^2} + \frac{1}{1+x^2}\right) dx$

$V_{solid} = 2\pi \left[-x + \frac{1}{2}\ln\left(1+x^2\right) + \tan^{-1} x\right]_0^1$

$= 2\pi \left[-1 + \frac{1}{2}\ln 2 + \frac{\pi}{4} - 0\right]$

$\therefore$ Volume $= \pi\left(\ln 2 + \frac{\pi}{2} - 2\right)$ units^3.

(b) (i) Let $u = 4 - x$ when $x = 1, u = 3$

$\frac{du}{dx} = -1$ when $x = 3, u = 1$

$du = -dx$

$\int_1^3 \frac{\cos^2 \frac{\pi x}{8}}{x(4-x)}\,dx = \int_3^1 \frac{\cos^2\left(\frac{\pi}{8}(4-u)\right)}{(4-u)u}(-du)$

$= -\int_3^1 \frac{\cos^2\left(\frac{\pi}{2} - \frac{\pi u}{8}\right)}{(4-u)u}\,du$

$= \int_1^3 \frac{\sin^2 \frac{\pi u}{8}}{(4-u)u}\,du.$

(ii) $\therefore \int_1^3 \frac{\cos^2 \frac{\pi x}{8}}{x(4-x)}\,dx = \int_1^3 \frac{\sin^2 \frac{\pi x}{8}}{(4-x)x}\,dx$

Add the equal integrals to get:

$2I = \int_1^3 \frac{\cos^2 \frac{\pi x}{8}}{x(4-x)}\,dx + \int_1^3 \frac{\sin^2 \frac{\pi x}{8}}{x(4-x)}\,dx$

$= \int_1^3 \frac{1}{x(4-x)}\,dx$

$I = \frac{1}{2}\int_1^3 \frac{1}{x(4-x)}\,dx$

$\frac{1}{x(4-x)} = \frac{A}{x} + \frac{B}{4-x}$

$1 = A(4-x) + Bx$

when $x = 4$, $B = \frac{1}{4}$

when $x = 0$, $A = \frac{1}{4}$

$I = \frac{1}{8}\int_1^3 \left(\frac{1}{x} + \frac{1}{4-x}\right) dx$

$= \frac{1}{8}\left[\ln x - \ln(4-x)\right]_1^3$

$= \frac{1}{8}\left(\ln 3 - \ln 1 - \ln 1 + \ln 3\right)$

$= \frac{1}{4}\ln 3.$

(c) (i)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$

$b^2x^2 + a^2\left(m^2x^2 + 2cmx + c^2\right) = a^2b^2$

$\left(b^2 + a^2m^2\right)x^2 + 2a^2cmx + a^2c^2 - a^2b^2 = 0$

But this is a tangent $\therefore \Delta = 0$:

[Note- $a \neq 0$ and $b \neq 0$]

$4a^4c^2m^2 - 4\left(b^2 + a^2m^2\right)a^2\left(c^2 - b^2\right) = 0$

$a^2c^2m^2 - \left(b^2 + a^2m^2\right)\left(c^2 - b^2\right) = 0$

$a^2c^2m^2 - b^2c^2 + b^4 - a^2m^2c^2 + a^2b^2m^2 = 0$

$b^4 - b^2c^2 + a^2b^2m^2 = 0$

$b^2 + a^2m^2 - c^2 = 0$

$a^2m^2 + b^2 = c^2.$

(ii) S is $(ae, 0)$, ℓ is $mx - y + c = 0$

$QS = \left|\frac{ax_o + by_o + c}{\sqrt{a^2+b^2}}\right|$

$= \left|\frac{m(ae) - 1(0) + c}{\sqrt{m^2 + (-1)^2}}\right|$

$= \left|\frac{mae + c}{\sqrt{1+m^2}}\right|.$

(iii) $QS \times Q'S' = \left|\frac{mae+c}{\sqrt{m^2+1}}\right| \times \left|\frac{mae-c}{\sqrt{m^2+1}}\right|$

$= \left|\frac{m^2a^2e^2 - c^2}{m^2+1}\right|$

$= \left|\frac{m^2a^2e^2 - \left(a^2m^2 + b^2\right)}{m^2+1}\right|$ from (i)

$= \left|\frac{m^2a^2\left(e^2-1\right) - b^2}{m^2+1}\right|$

$= \left|\frac{-m^2b^2 - b^2}{m^2+1}\right|$

$= \left|\frac{-b^2\left(m^2+1\right)}{m^2+1}\right|$

$= b^2.$

Question 8

(a) $I_m = \int_0^1 x^m\left(x^2-1\right)^5 dx$

$= \int_0^1 x^{m-1}\, x\left(x^2-1\right)^5 dx$

Let $u = x^{m-1}$ $\qquad dv = x\left(x^2-1\right)^5 dx$

$du = (m-1)x^{m-2}dx \qquad v = \frac{\left(x^2-1\right)^6}{12}$

$I_m = \left[x^{m-1}\frac{\left(x^2-1\right)^6}{12}\right]_0^1 - (m-1)\int_0^1 x^{m-2}\frac{\left(x^2-1\right)^6}{12}\,dx$

$= 0 - \frac{(m-1)}{12}\int_0^1 x^{m-2}\left(x^2-1\right)^5\left(x^2-1\right) dx$

$= -\frac{(m-1)}{12}\int_0^1 x^m\left(x^2-1\right)^5 dx$

$+ \frac{(m-1)}{12}\int_0^1 x^{m-2}\left(x^2-1\right)^5 dx$

$= -\frac{(m-1)}{12}I_m + \frac{(m-1)}{12}I_{m-2}$

$I_m\left(1 + \frac{(m-1)}{12}\right) = \frac{(m-1)}{12}I_{m-2}$

$I_m\left(\frac{12+m-1}{12}\right) = \frac{(m-1)}{12}I_{m-2}$

$I_m\left(\frac{11+m}{12}\right) = \frac{(m-1)}{12}I_{m-2}$

$I_m = \frac{(m-1)}{12} \times \frac{12}{11+m}I_{m-2}$

$I_m = \frac{m-1}{m+11}I_{m-2}.$

(b) (i) $P(E) = \frac{7}{7} \times \frac{6}{7} \times \frac{5}{7} \times \ldots \times \frac{1}{7} = \frac{7!}{7^7}$

$= \frac{6!}{7^6} = \frac{720}{117649}.$

(ii) $P(E) = 1 - \frac{6!}{7^6} = \frac{116929}{117649}.$

(iii) If one ball, say the 7 is left out then 1,1,2,3,4,5,6 is a possibility.

There are $\frac{7!}{2!}$ of selecting these numbers.

Given there can be double 2's, 3's all the way to double 6's, there are $6 \times \frac{7!}{2!}$ permutations of leaving out the 7.
As each of the other numbers can be left out, the total permutations are

$$7\times 6\times\frac{7!}{2!}=21\times 7!$$

$$P(E)=\frac{21\times 7!}{7^7}$$
$$=\frac{3\times 6!}{7^5}$$
$$=\frac{2160}{16807}.$$

(c) (i) $P(z)=z^n+a_{n-1}z^{n-1}+...+a_1z+a_o$

If $P(\beta)=0$:

$$0=\beta^n+a_{n-1}\beta^{n-1}+...+a_1\beta+a_o$$
$$\beta^n=-\left(a_{n-1}\beta^{n-1}+...+a_1\beta+a_o\right)$$
$$\left|\beta^n\right|=\left|a_{n-1}\beta^{n-1}+...+a_1\beta+a_o\right|$$
$$\le\left|a_{n-1}\beta^{n-1}\right|+...+\left|a_1\beta\right|+\left|a_o\right|$$
$$\left|\beta\right|^n\le\left|a_{n-1}\right|\left|\beta\right|^{n-1}+...+\left|a_1\right|\left|\beta\right|+\left|a_o\right|$$
$$\le M\left(\left|\beta\right|^{n-1}+...+\left|\beta\right|+1\right).$$

(ii) $\left|\beta\right|^{n-1}+...+\left|\beta\right|+1$ is a GP with $a=1$, $r=\left|\beta\right|$ with n terms

$$\left|\beta\right|^n\le M\left(\left|\beta\right|^{n-1}+...+\left|\beta\right|+1\right)$$
$$\left|\beta\right|^n\le M\left(\frac{1\left(\left|\beta\right|^n-1\right)}{\left|\beta\right|-1}\right)$$
$$\left|\beta\right|^n\left(\left|\beta\right|-1\right)\le M\left(\left|\beta\right|^n-1\right)\quad\text{if }\left|\beta\right|>1$$
$$\left|\beta\right|-1\le M\frac{\left|\beta\right|^n-1}{\left|\beta\right|^n}$$
$$\left|\beta\right|-1\le M\left(1-\frac{1}{\left|\beta\right|^n}\right)$$
$$\left|\beta\right|-1\le M$$
$$\left|\beta\right|\le 1+M.$$

If $\left|\beta\right|<1$ then $\left|\beta\right|\le 1+M$.

Thus $\left|\beta\right|\le 1+M$.

(d) $$S(x)=\sum_{k=0}^{n}c_k\left(x+\frac{1}{x}\right)^k\quad\text{where }\left|c_k\right|\le\left|c_n\right|$$
$$=c_n\left(x+\frac{1}{x}\right)^n+c_{n-1}\left(x+\frac{1}{x}\right)^{n-1}+$$
$$...+c_1\left(x+\frac{1}{x}\right)+c_0$$

Hence:

$$\frac{S(x)}{c_n}=\left(x+\frac{1}{x}\right)^n+\frac{c_{n-1}}{c_n}\left(x+\frac{1}{x}\right)^{n-1}+$$
$$...+\frac{c_1}{c_n}\left(x+\frac{1}{x}\right)+\frac{c_0}{c_n}$$

which is a monic polynomial in terms of $x+\frac{1}{x}$. Let $z=x+\frac{1}{x}$ to get

$P(z)=z^n+a_{n-1}z^{n-1}+...+a_1z+a_o$

(as in part c)

$\left|a_k\right|=\frac{\left|c_k\right|}{\left|c_n\right|}\le 1$ as $\left|c_k\right|\le\left|c_n\right|$

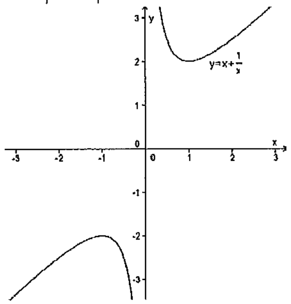
Let β be a real root of $S(x)$,

i.e. $\beta=x+\frac{1}{x}$

From (c)(ii) $\left|\beta\right|<1+M$ so

$\left|\beta\right|<2$ since $M\le 1$

But $\left|x+\frac{1}{x}\right|\ge 2$ [see graph]



This would mean that $\left|\beta\right|\ge 2$ but this is a contradiction as earlier it was shown that $\left|\beta\right|<2$. This contraction means that the assumption that β is a real root of $S(x)$ is incorrect. Thus $S(x)$ has no real solutions.

End of Mathematics solutions