BOARD OF STUDIES

NEW SOUTH WALES

2006

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–10
- All questions are of equal value

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Attempt Questions 1–10
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

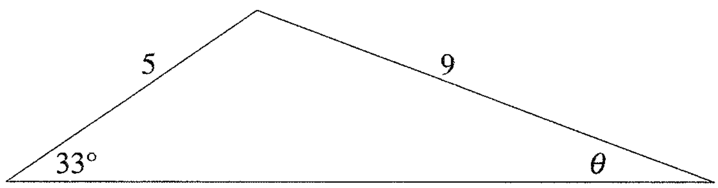
Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $e^{-0.5}$ correct to three decimal places. **2**

(b) Factorise $2x^2 + 5x - 3$. **2**

(c) Sketch the graph of $y = |x + 4|$. **2**

(d) **2**



NOT TO SCALE

Find the value of θ in the diagram. Give your answer to the nearest degree.

(e) Solve $3 - 5x \le 2$. **2**

(f) Find the limiting sum of the geometric series $\frac{13}{5} + \frac{13}{25} + \frac{13}{125} + \cdots$. **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Differentiate with respect to x:

(i) $x \tan x$ **2**

(ii) $\dfrac{\sin x}{x+1}$. **2**

(b) (i) Find $\int 1 + e^{7x}\, dx$. **2**

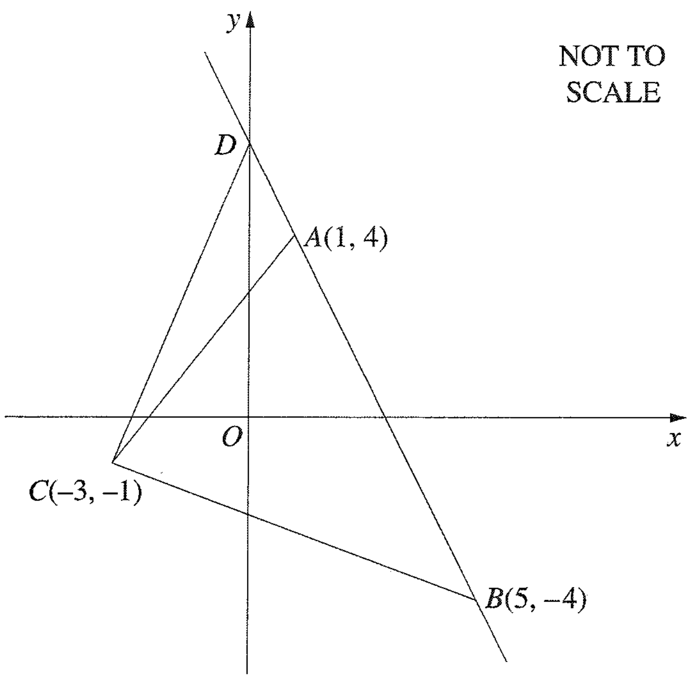
(ii) Evaluate $\displaystyle\int_0^3 \frac{8x}{1+x^2}\, dx$. **3**

(c) Find the equation of the tangent to the curve $y = \cos 2x$ at the point whose x-coordinate is $\dfrac{\pi}{6}$. **3**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a)



In the diagram, A, B and C are the points $(1, 4)$, $(5, -4)$ and $(-3, -1)$ respectively. The line AB meets the y-axis at D.

(i) Show that the equation of the line AB is $2x + y - 6 = 0$. **2**

(ii) Find the coordinates of the point D. **1**

(iii) Find the perpendicular distance of the point C from the line AB. **1**

(iv) Hence, or otherwise, find the area of the triangle ADC. **2**

Question 3 continues on page 5

Question 3 (continued)

(b) Evaluate $\sum_{r=2}^{4} \frac{1}{r}$. **1**

(c) On the first day of the harvest, an orchard produces 560 kg of fruit. On the next day, the orchard produces 543 kg, and the amount produced continues to decrease by the same amount each day.

(i) How much fruit is produced on the fourteenth day of the harvest? **2**

(ii) What is the total amount of fruit that is produced in the first 14 days of the harvest? **1**

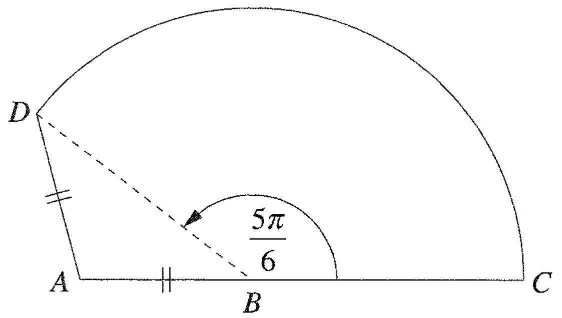
(iii) On what day does the daily production first fall below 60 kg? **2**

End of Question 3

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a)



NOT TO SCALE

In the diagram, $ABCD$ represents a garden. The sector BCD has centre B and $\angle DBC = \frac{5\pi}{6}$.

The points A, B and C lie on a straight line and $AB = AD = 3$ metres.

Copy or trace the diagram into your writing booklet.

(i) Show that $\angle DAB = \frac{2\pi}{3}$. **1**

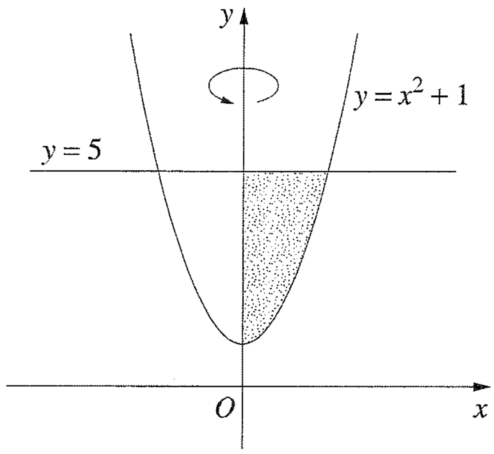
(ii) Find the length of BD. **2**

(iii) Find the area of the garden $ABCD$. **2**

Question 4 continues on page 7

Question 4 (continued)

(b) **3**



In the diagram, the shaded region is bounded by the parabola $y = x^2 + 1$, the y-axis and the line $y = 5$.

Find the volume of the solid formed when the shaded region is rotated about the y-axis.

(c) A chessboard has 32 black squares and 32 white squares. Tanya chooses three different squares at random.

(i) What is the probability that Tanya chooses three white squares? **2**

(ii) What is the probability that the three squares Tanya chooses are the same colour? **1**

(iii) What is the probability that the three squares Tanya chooses are not the same colour? **1**

End of Question 4

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) A function $f(x)$ is defined by $f(x) = 2x^2(3 - x)$.

(i) Find the coordinates of the turning points of $y = f(x)$ and determine their nature. **3**

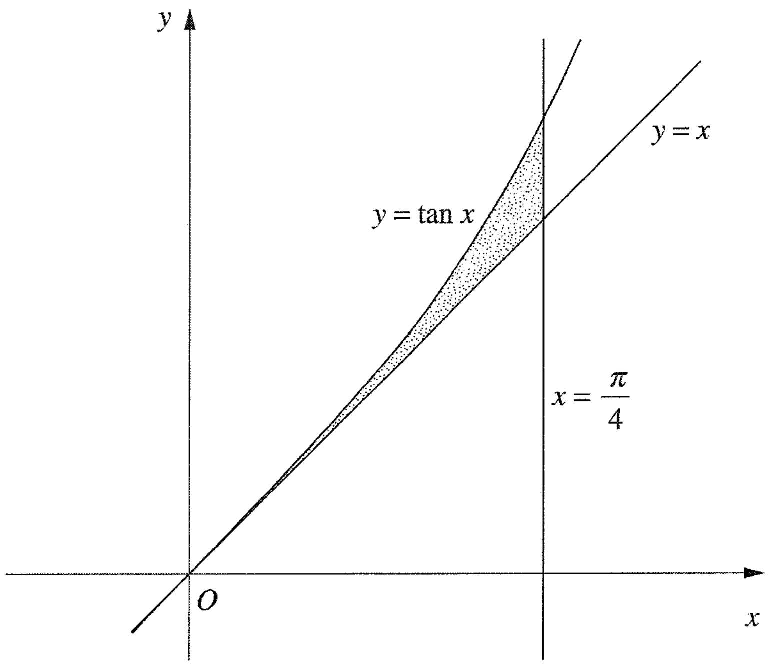
(ii) Find the coordinates of the point of inflexion. **1**

(iii) Hence sketch the graph of $y = f(x)$, showing the turning points, the point of inflexion and the points where the curve meets the x-axis. **3**

(iv) What is the minimum value of $f(x)$ for $-1 \le x \le 4$? **1**

(b) (i) Show that $\frac{d}{dx}\log_e(\cos x) = -\tan x$. **1**

(ii) **3**



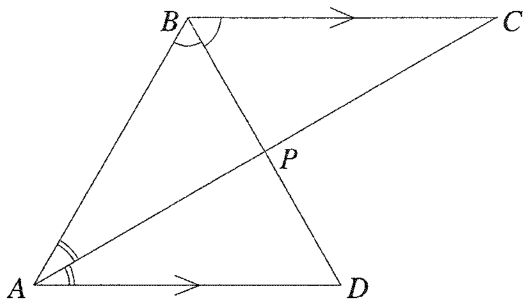
The shaded region in the diagram is bounded by the curve $y = \tan x$ and the lines $y = x$ and $x = \frac{\pi}{4}$.

Using the result of part (i), or otherwise, find the area of the shaded region.

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a)



In the diagram, AD is parallel to BC, AC bisects $\angle BAD$ and BD bisects $\angle ABC$. The lines AC and BD intersect at P.

Copy or trace the diagram into your writing booklet.

(i) Prove that $\angle BAC = \angle BCA$. **1**

(ii) Prove that $\Delta ABP \equiv \Delta CBP$. **2**

(iii) Prove that $ABCD$ is a rhombus. **3**

(b) A rare species of bird lives only on a remote island. A mathematical model predicts that the bird population, P, is given by

$$P = 150 + 300e^{-0.05t}$$

where t is the number of years after observations began.

(i) According to the model, how many birds were there when observations began? **1**

(ii) According to the model, what will be the rate of change in the bird population ten years after observations began? **2**

(iii) What does the model predict will be the limiting value of the bird population? **1**

(iv) The species will become eligible for inclusion in the endangered species list when the population falls below 200. When does the model predict that this will occur? **2**

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Let α and β be the solutions of $x^2 - 3x + 1 = 0$.

(i) Find $\alpha\beta$. **1**

(ii) Hence find $\alpha + \dfrac{1}{\alpha}$. **1**

(b) A function $f(x)$ is defined by $f(x) = 1 + 2\cos x$.

(i) Show that the graph of $y = f(x)$ cuts the x-axis at $x = \dfrac{2\pi}{3}$. **1**

(ii) Sketch the graph of $y = f(x)$ for $-\pi \le x \le \pi$ showing where the graph cuts each of the axes. **3**

(iii) Find the area under the curve $y = f(x)$ between $x = -\dfrac{\pi}{2}$ and $x = \dfrac{2\pi}{3}$. **3**

(c) (i) Write down the discriminant of $2x^2 + (k-2)x + 8$, where k is a constant. **1**

(ii) Hence, or otherwise, find the values of k for which the parabola $y = 2x^2 + kx + 9$ does not intersect the line $y = 2x + 1$. **2**

Marks

Question 8 (12 marks) Use a SEPARATE writing booklet.

(a) A particle is moving in a straight line. Its displacement, x metres, from the origin, O, at time t seconds, where $t \geq 0$, is given by $x = 1 - \dfrac{7}{t+4}$.

(i) Find the initial displacement of the particle. **1**

(ii) Find the velocity of the particle as it passes through the origin. **3**

(iii) Show that the acceleration of the particle is always negative. **1**

(iv) Sketch the graph of the displacement of the particle as a function of time. **2**

(b) Joe borrows \$200 000 which is to be repaid in equal monthly instalments. The interest rate is 7.2% per annum reducible, calculated monthly.

It can be shown that the amount, $\$A_n$, owing after the nth repayment is given by the formula:

$$A_n = 200\,000r^n - M(1 + r + r^2 + \cdots + r^{n-1}),$$

where $r = 1.006$ and $\$M$ is the monthly repayment. (Do NOT show this.)

(i) The minimum monthly repayment is the amount required to repay the loan in 300 instalments. **3**

Find the minimum monthly repayment.

(ii) Joe decides to make repayments of \$2800 each month from the start of the loan. **2**

How many months will it take for Joe to repay the loan?

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Marks

Question 9 (12 marks) Use a SEPARATE writing booklet.

(a) Find the coordinates of the focus of the parabola $12y = x^2 - 6x - 3$. **2**

(b) During a storm, water flows into a 7000-litre tank at a rate of $\frac{dV}{dt}$ litres per minute, where $\frac{dV}{dt} = 120 + 26t - t^2$ and t is the time in minutes since the storm began.

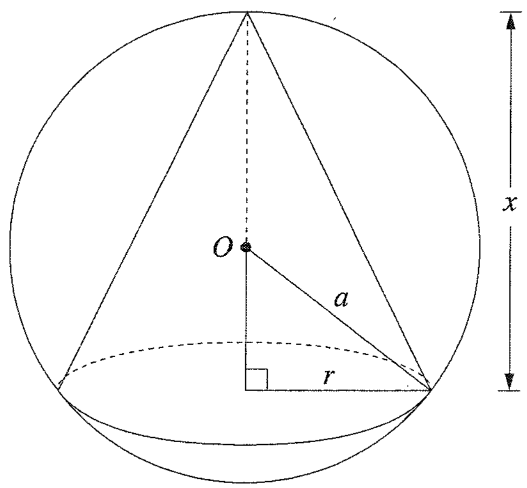
(i) At what times is the tank filling at twice the initial rate? **2**

(ii) Find the volume of water that has flowed into the tank since the start of the storm as a function of t. **1**

(iii) Initially, the tank contains 1500 litres of water. When the storm finishes, 30 minutes after it began, the tank is overflowing. **2**

How many litres of water have been lost?

(c)



A cone is inscribed in a sphere of radius a, centred at O. The height of the cone is x and the radius of the base is r, as shown in the diagram.

(i) Show that the volume, V, of the cone is given by $V = \frac{1}{3}\pi\left(2ax^2 - x^3\right)$. **2**

(ii) Find the value of x for which the volume of the cone is a maximum. You must give reasons why your value of x gives the maximum volume. **3**

Marks

Question 10 (12 marks) Use a SEPARATE writing booklet.

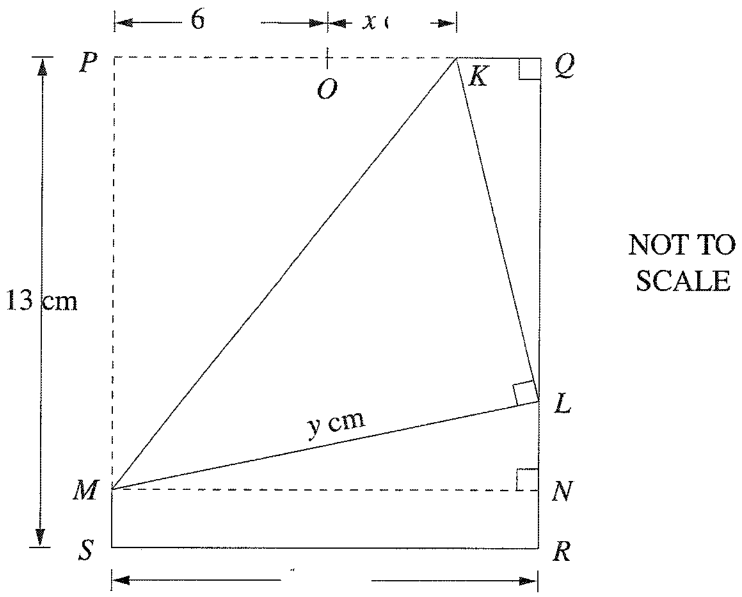
(a) Use Simpson's rule with three function values to find an approximation to the value of $\displaystyle\int_{0.5}^{1.5} \left(\log_e x\right)^3 dx$. **2**

Give your answer correct to three decimal places.

Question 10 continues on page 15

Question 10 (continued)

(b)



A rectangular piece of paper $PQRS$ has sides $PQ = 12$ cm and $PS = 13$ cm. The point O is the midpoint of PQ. The points K and M are to be chosen on OQ and PS respectively, so that when the paper is folded along KM, the corner that was at P lands on the edge QR at L. Let $OK = x$ cm and $LM = y$ cm.

Copy or trace the diagram into your writing booklet.

(i) Show that $QL^2 = 24x$. **1**

(ii) Let N be the point on QR for which MN is perpendicular to QR. **3**

By showing that $\Delta QKL \parallel\!| \Delta NLM$, deduce that $y = \dfrac{\sqrt{6}(6+x)}{\sqrt{x}}$.

(iii) Show that the area, A, of ΔKLM is given by $A = \dfrac{\sqrt{6}(6+x)^2}{2\sqrt{x}}$. **1**

(iv) Use the fact that $12 \le y \le 13$ to find the possible values of x. **2**

(v) Find the minimum possible area of ΔKLM. **3**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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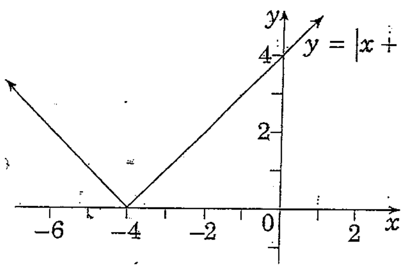
MATHEMATICS

QUESTION 1

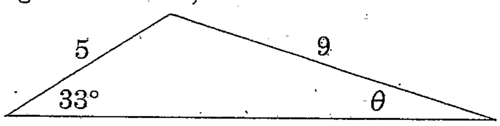
(a) $e^{-0.5} = 0.6065\ldots$
$= 0.607$ (3 decimal places).

(b) $2x^2 + 5x - 3$ $\quad AB = 2 \times -3 \quad A + B = 5$
$\quad -1, 6$
$= 2x^2 - x + 6x - 3$
$= x(2x-1) + 3(2x-1)$
$= (2x-1)(x+3)$.

(c)



(d) Using the sine rule,



$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin\theta}{5} = \frac{\sin 33°}{9}$$
$$\sin\theta = \frac{5\sin 33°}{9}$$
$$\therefore \quad \theta = 17.6\ldots°$$
$$= 18° \text{ (to the nearest degree)}.$$

(e)
$$3 - 5x \le 2$$
$$-5x \le -1$$
$$\frac{-5x}{-5} \ge \frac{-1}{-5}$$
$$x \ge \frac{1}{5}.$$

(f) Using the limiting sum formula:
$$S = \frac{a}{1-r}, \quad |r| < 1$$
$$a = \frac{13}{5}, \quad r = \frac{13}{25} \div \frac{13}{5} = \frac{1}{5}$$
$$S = \frac{\frac{13}{5}}{1 - \frac{1}{5}} = \frac{13}{4}.$$

QUESTION 2

(a) (i) Product rule: $\frac{d}{dx}(uv) = uv' + vu'$
$$\therefore \quad \frac{d}{dx}(x\tan x) = x\sec^2 x + \tan x \times 1$$
$$= x\sec^2 x + \tan x.$$

(ii) Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
$$\therefore \quad \frac{d}{dx}\left(\frac{\sin x}{x+1}\right) = \frac{(x+1)\cos x - \sin x \times 1}{(x+1)^2}$$
$$= \frac{(x+1)\cos x - \sin x}{(x+1)^2}.$$

(b) (i) $\int (1 + e^{7x})\, dx = x + \frac{e^{7x}}{7} + c.$

(ii)
$$\int_0^3 \frac{8x}{1+x^2}\, dx = 4\int_0^3 \frac{2x}{1+x^2}\, dx$$
$$= 4\left[\log_e(1+x^2)\right]_0^3$$
$$= 4\left[\log_e 10 - \log_e 1\right]$$
$$= 4\log_e 10 \quad (\log_e 1 = 0).$$

(c) $y = \cos 2x$.

At $x = \frac{\pi}{6}$: $\quad y = \cos\left(2 \times \frac{\pi}{6}\right)$
$$= \cos\frac{\pi}{3}$$
$$= \frac{1}{2}.$$

Point on the curve is $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

Gradient: $\frac{dy}{dx} = -2\sin 2x$

At $x = \frac{\pi}{6}$: $\quad \frac{dy}{dx} = -2\sin\left(2 \times \frac{\pi}{6}\right)$
$$= -2\sin\frac{\pi}{3}$$
$$= -2 \times \frac{\sqrt{3}}{2}$$
$$= -\sqrt{3}.$$

Equation of tangent: $y - y_1 = m(x - x_1)$

$$y - \frac{1}{2} = -\sqrt{3}\left(x - \frac{\pi}{6}\right)$$

$$y - \frac{1}{2} = -\sqrt{3}x + \frac{\sqrt{3}\pi}{6}$$

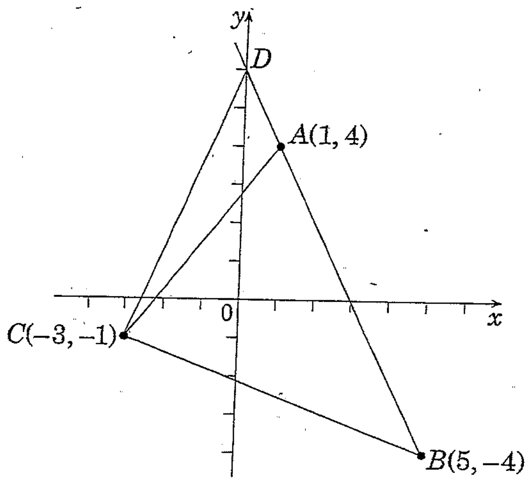
$$y = -\sqrt{3}x + \frac{\sqrt{3}\pi}{6} + \frac{1}{2},$$

or General Form: $6y = -6\sqrt{3}x + \sqrt{3}\pi + 3$

$$6\sqrt{3}x + 6y - \sqrt{3}\pi - 3 = 0.$$

QUESTION 3

(a)



(i) Gradient of $AB = \frac{-4-4}{5-1}$

$$= \frac{-8}{4}$$

$$= -2.$$

Using $y - y_1 = m(x - x_1)$ and $A(1, 4)$, $m = -2$:

$$y - 4 = -2(x - 1)$$

$$y - 4 = -2x + 2$$

$$\therefore \quad 2x + y - 6 = 0.$$

(ii) y-intercept occurs when $x = 0$,

$$\therefore \quad 2(0) + y - 6 = 0$$

$$y = 6.$$

$\therefore$ The coordinates of D are $(0, 6)$.

(iii) Perpendicular distance

$$= \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$$= \left|\frac{2(-3) + 1(-1) - 6}{\sqrt{2^2 + 1^2}}\right|$$

$$= \left|\frac{-13}{\sqrt{5}}\right|$$

$$= \frac{13}{\sqrt{5}}.$$

$\therefore$ The perpendicular distance is $\frac{13}{\sqrt{5}}$ units.

(iv) Area of $\Delta ADC = \frac{1}{2} \times$ length of AD $\times$ perpendicular height.

Using the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AD = \sqrt{(1-0)^2 + (4-6)^2}$$

$$= \sqrt{5}.$$

$$\therefore \text{ Area of } \Delta ADC = \frac{1}{2} \times \sqrt{5} \times \frac{13}{\sqrt{5}}$$

$$= 6{\cdot}5$$

$\therefore$ the area is $6{\cdot}5$ unit^2.

(b) $\displaystyle\sum_{r=2}^{4} \frac{1}{r} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}.$

(c) (i) Fruit produced: 560, 543, 526, ...

This is an arithmetic sequence, with $a = 560$ and $d = 543 - 560 = -17$.

$$T_n = a + (n-1)d$$

$$\therefore \quad T_{14} = 560 + (14 - 1) \times -17$$

$$= 339.$$

$\therefore$ Fruit produced on the 14th day is 339 kg.

(ii) Fruit produced $= 560 + 543 + 526 + \cdots + 339$.

METHOD 1

Using the formula $S_n = \frac{n}{2}[a + \ell]$:

$$S_{14} = \frac{14}{2}[560 + 339]$$

$$= 6293$$

$\therefore$ total fruit produced is 6293 kg.

METHOD 2

Using $S_n = \frac{n}{2}[2a + (n-1)d]$:

$$S_{14} = \frac{14}{2}[2 \times 560 + (14 - 1) \times (-17)]$$

$$= 6293.$$

(iii) Need first term of arithmetic series to be less than 60.

$$T_n < 60$$

$$a + (n-1)d < 60$$

$$560 + (n-1) \times (-17) < 60$$

$$560 - 17n + 17 < 60$$

$$-17n < -517$$

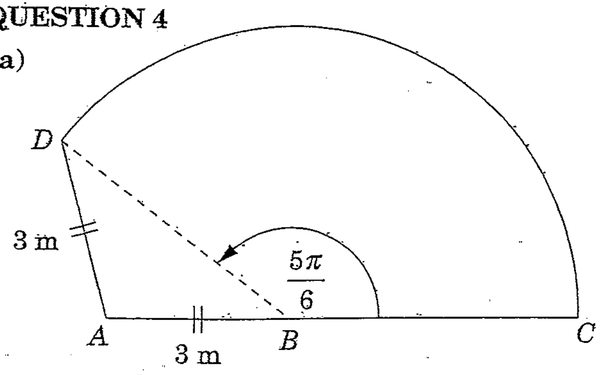
$$n > \frac{-517}{-17}$$

$$n > 30{\cdot}411\,76\ldots$$

$\therefore$ on the 31st day production falls below 60 kg.

QUESTION 4

(a)



(i) $\angle DBA = \pi - \frac{5\pi}{6}$ ($\angle ABC$ is a straight $\angle$)

$= \frac{\pi}{6}$.

$\angle BDA = \frac{\pi}{6}$ (equal $\angle$s opp. equal sides of isosceles ΔBDA)

$\therefore \angle DAB = \pi - \left(\frac{\pi}{6} + \frac{\pi}{6}\right)$ ($\angle$ sum of ΔBDA is π)

$= \pi - \frac{\pi}{3}$

$\angle DAB = \frac{2\pi}{3}$.

(ii) Using the cosine rule

$a^2 = b^2 + c^2 - 2bc\cos A$

$BD^2 = 3^2 + 3^2 - 2\times 3\times 3\times \cos\frac{2\pi}{3}$

$= 9 + 9 - 18\times\left(-\frac{1}{2}\right)$

$= 18 + 9$

$\therefore BD^2 = 27$.

$BD = \sqrt{27}$ m or $3\sqrt{3}$ m or $5{\cdot}196\ldots$ m.

(iii) Using $A = \frac{1}{2}ab\sin C$,

$\text{area}\,\Delta ABD = \frac{1}{2}\times 3\times 3\times \sin\frac{2\pi}{3}$

$= \frac{1}{2}\times 3\times 3\times\frac{\sqrt{3}}{2}$

$= \frac{9\sqrt{3}}{4}$.

Using $A = \frac{1}{2}r^2\theta$, $r = \sqrt{27}$,

$\text{area of sector } DBC = \frac{1}{2}\times\left(\sqrt{27}\right)^2\times\frac{5\pi}{6}$

$= \frac{1}{2}\times \overset{9}{\cancel{27}}\times\frac{5\pi}{\cancel{6}_2}$

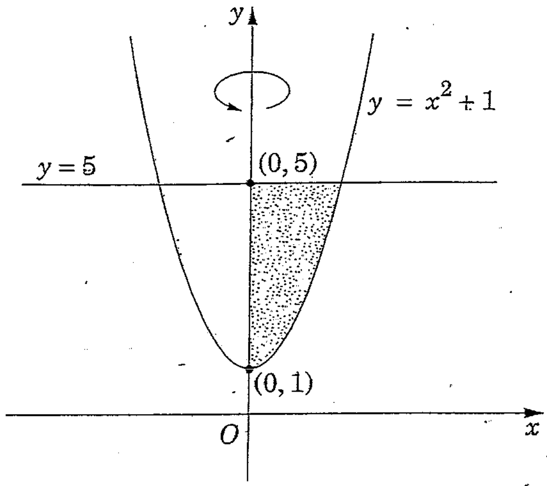
$= \frac{45\pi}{4}$.

$\text{Garden area} = \frac{9\sqrt{3}}{4} + \frac{45\pi}{4}\ \text{m}^2$

$= \frac{9\sqrt{3} + 45\pi}{4}\ \text{m}^2$

or $39{\cdot}240\ldots\ \text{m}^2$.

(b) At $x = 0$, $y = 1$.



$V = \pi\int_1^5 x^2\,dy$ $\qquad y = x^2 + 1$, $y - 1 = x^2$

$= \pi\int_1^5 (y-1)\,dy$

$= \pi\left[\frac{y^2}{2} - y\right]_1^5$

$= \pi\left[\left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} - 1\right)\right]$

$= \pi\left(\frac{15}{2} + \frac{1}{2}\right)$

$= 8\pi$.

$\therefore$ The volume is 8π unit^3.

(c) (i) $P(\text{white, white, white}) = \frac{32}{64}\times\frac{31}{63}\times\frac{30}{62}$

$= \frac{5}{42}$ or $0{\cdot}119\ldots$

(ii) $P(\text{same colour}) = P(\text{white, white, white}) + P(\text{black, black, black})$

$= \frac{32}{64}\times\frac{31}{63}\times\frac{30}{62} + \frac{32}{64}\times\frac{31}{63}\times\frac{30}{62}$

$= \frac{5}{42} + \frac{5}{42}$

$= \frac{5}{21}$.

(iii) $P(\text{not the same colour})$

$= 1 - P(\text{same colour})$

$= 1 - \frac{5}{21}$

$= \frac{16}{21}$.

QUESTION 5

(a) (i)
$$f(x) = 2x^2(3-x) = 6x^2 - 2x^3$$
$$f'(x) = 12x - 6x^2$$
$$f''(x) = 12 - 12x$$

Stationary points occur when $f'(x) = 0$

$\therefore \quad 12x - 6x^2 = 0$

$\quad 6x(2-x) = 0$

$\therefore \ x = 0,\ x = 2.$

To determine the nature of turning points, the second derivative may be used.

When $x = 0$, $f(0) = 6(0)^2 - 2(0)^3 = 0$

and $f''(x) = 12 - 12(0) = 12 > 0 \Rightarrow$ concave up

$\therefore$ Minimum turning point at (0, 0).

When $x = 2$, $f(2) = 6(2)^2 - 2(2)^3 = 8$

and $f''(2) = 12 - 12(2) = -12 < 0 \Rightarrow$ concave down

$\therefore$ Maximum turning point at (2, 8).

(ii) Point of inflexion occurs when $f''(x) = 0$ and concavity changes.

$$12 - 12x = 0$$
$$12 = 12x$$
$$x = 1,$$
$$f(1) = 6(1)^2 - 2(1)^3 = 4.$$

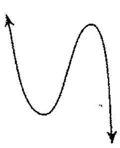
Considering concavity:

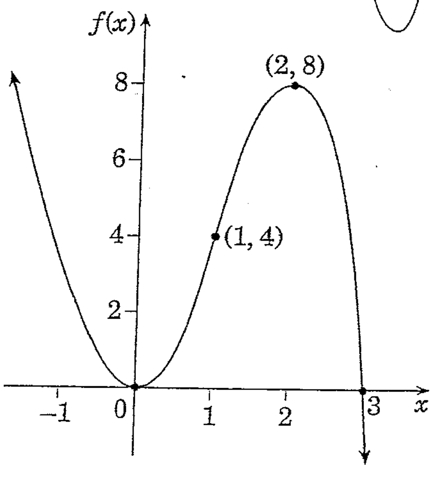
x	$\frac{1}{2}$	1	$1\frac{1}{2}$
$f''(x)$	+6	0	−6

$\therefore$ Concavity changes from positive to negative at $x = 1$.

$\therefore$ Point of inflexion at $(1, 4)$.

(iii) Note that the function has the shape of a negative cubic.





x-intercepts occur when $f(x) = 0$

$\therefore \quad 2x^2(3-x) = 0.$

$\therefore \ x = 0,\ x = 3$ are x-intercepts.

(iv) When restricting the domain to $-1 \le x \le 4$, the graph will have a minimum value at $x = 4$.

$\therefore$ Minimum value $= f(4) = 6(4)^2 - 2(4)^3 = -32.$

(b) (i) Using $\frac{d}{dx}\left[\log_e f(x)\right] = \frac{f'(x)}{f(x)}$

$$\frac{d}{dx}\left[\log_e(\cos x)\right] = \frac{-\sin x}{\cos x} = -\tan x.$$

(ii) Shaded area = area under $(y = \tan x)$ − area under $(y = x.)$

$\therefore$ Shaded area

$$= \int_0^{\frac{\pi}{4}} \tan x \ dx - \int_0^{\frac{\pi}{4}} x \ dx$$

$$= \left[-\log_e(\cos x) - \frac{x^2}{2}\right]_0^{\frac{\pi}{4}} \quad \text{using (i)}$$

$$= -\log_e\left(\cos\frac{\pi}{4}\right) - \frac{\left(\frac{\pi}{4}\right)^2}{2} - \left[-\log_e(\cos 0) - \frac{0^2}{2}\right]$$

$$= -\log_e\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi^2}{32} + \log_e 1$$

$$= \log_e\left(\frac{1}{\sqrt{2}}\right)^{-1} - \frac{\pi^2}{32} + 0$$

$$= \log_e\sqrt{2} - \frac{\pi^2}{32}$$

$$= \frac{1}{2}\log_e 2 - \frac{\pi^2}{32} \quad \text{or } 0{\cdot}0381\ldots$$

$\therefore$ The area is $\left(\frac{1}{2}\log_e 2 - \frac{\pi^2}{32}\right)$ unit^2.

Note: $\int_0^{\frac{\pi}{4}} x \ dx$ represents the area of a triangle of base $\frac{\pi}{4}$ units and height $\frac{\pi}{4}$ units.

$$\therefore \int_0^{\frac{\pi}{4}} x \ dx = \frac{1}{2} \times \frac{\pi}{4} \times \frac{\pi}{4} = \frac{\pi^2}{32}.$$

The particle passes through the origin when $x = 0$.

$$\therefore \quad 0 = 1 - \frac{7}{t+4}$$
$$\frac{7}{t+4} = 1$$
$$7 = t+4$$
$$3 = t.$$

$\therefore$ Velocity at $t = 3$,

$$\frac{dx}{dt} = \frac{7}{(3+4)^2} = \frac{1}{7}.$$

$\therefore$ The particle passes through the origin at $\frac{1}{7}$ ms^{-1}.

(iii) Acceleration is $\frac{d^2x}{dt^2}$.

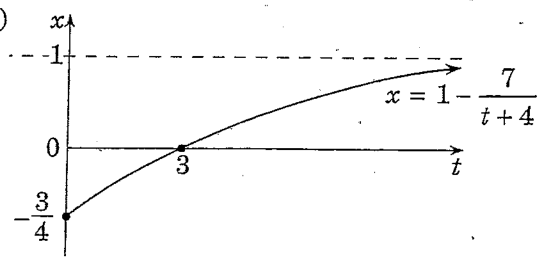
$$\frac{dx}{dt} = 7(t+4)^{-2}$$
$$\frac{d^2x}{dt^2} = -14(t+4)^{-3} \times 1 = \frac{-14}{(t+4)^3}.$$

As $t > 0$, then $(t+4)^3 > 0$

$$\therefore \quad \frac{-14}{(t+4)^3} < 0 \text{ for all } t > 0.$$

$\therefore$ The acceleration is always negative.

(iv)



At $t = 0$, $\quad x = -\frac{3}{4}$.

As $t \to \infty$, $\frac{7}{t+4} \to 0$, so for $x = 1 - \frac{7}{t+4}$, $x \to 1$.

At $t = 3$, $\quad x = 0$.

(b) $A_n = 200\,000r^n - M(1 + r + r^2 + \cdots + r^{n-1})$

$r = 1{\cdot}006$

(i) If the loan is repaid in 300 instalments, $A_n = 0$, $n = 300$.

$$\therefore \quad 0 = 200\,000(1{\cdot}006)^{300} - M(1 + 1{\cdot}006 + 1{\cdot}006^2 + \cdots + 1{\cdot}006^{299}).$$

Now $1 + 1{\cdot}006 + 1{\cdot}006^2 + \cdots + 1{\cdot}006^{299}$ is a geometric series.

$$S_n = \frac{a(r^n - 1)}{r-1}, \quad a = 1,\ n = 300,\ r = 1{\cdot}006$$

$$= \frac{1(1{\cdot}006^{300} - 1)}{1{\cdot}006 - 1}$$
$$= 836{\cdot}199\,466\ldots$$
$$= 836{\cdot}1995 \text{ (to 4 d.p.)}.$$

$$\therefore \quad 0 = 2000 \times (1{\cdot}006)^{300} - 836{\cdot}1995M$$
$$836{\cdot}1995M = 200\,000 \times (1{\cdot}006)^{300}$$
$$836{\cdot}1995M = 1\,203\,439{\cdot}36\ldots$$
$$M = \frac{1\,203\,439{\cdot}36\ldots}{836{\cdot}1995}$$
$$= 1439{\cdot}1773\ldots$$

$\therefore$ The monthly repayment is \$1439·18.

(ii) $M = 2800$, $A_n = 0$. Find n.

$$0 = 200\,000(1{\cdot}006)^n - 2800(1 + 1{\cdot}006 + 1{\cdot}006^2 + \cdots + 1{\cdot}006^{n-1})$$
$$0 = 200\,000(1{\cdot}006)^n - 2800\left[\frac{1(1{\cdot}006^n - 1)}{1{\cdot}006 - 1}\right]$$
$$2800\left[\frac{1{\cdot}006^n - 1}{0{\cdot}006}\right] = 200\,000(1{\cdot}006)^n$$
$$1{\cdot}006^n - 1 = \frac{0{\cdot}006 \times 200\,000(1{\cdot}006)^n}{2800}$$
$$1{\cdot}006^n - 1 = \frac{3}{7}(1{\cdot}006)^n$$
$$7 \times 1{\cdot}006^n - 7 = 3 \times 1{\cdot}006^n$$
$$4 \times 1{\cdot}006^n = 7$$
$$1{\cdot}006^n = \frac{7}{4}$$
$$1{\cdot}006^n = 1{\cdot}75.$$

METHOD 1

$$\ln(1{\cdot}006)^n = \ln 1{\cdot}75$$
$$n\ln(1{\cdot}006) = \ln 1{\cdot}75$$
$$n = \frac{\ln 1{\cdot}75}{\ln 1{\cdot}006}$$
$$n = 93{\cdot}5488\ldots$$

$\therefore$ The loan will be repaid after 94 months, but the 94th payment will not be a full payment.

METHOD 2

Guess and check.

$1{\cdot}006^n = 1{\cdot}75$

n	$1{\cdot}006^n$	
50	1·348 …	n must be greater
100	1·818 …	n must be smaller
95	1·765 …	n must be smaller
94	1·754 …	n must be smaller
93	1·744 …	n must be greater

$\therefore$ The loan will be paid off after 94 monthly instalments, but the 94th payment will not be a full payment.

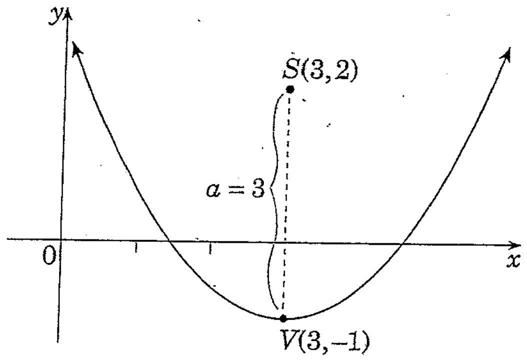
QUESTION 9

(a)

$$12y = x^2 - 6x - 3$$
$$x^2 - 6x = 12y + 3$$
$$x^2 - 6x + (-3)^2 = 12y + 3 + (-3)^2$$
$$(x-3)^2 = 12y + 12$$
$$(x-3)^2 = 4(3)(y+1), \text{ which is in the}$$

form $(x-h)^2 = 4a(y-k)$, where $4a = 12$
$a = 3.$

Vertex is $(3, -1)$ and focal length is 3.



$\therefore$ Focus $(3, 2)$.

(b) (i) $\frac{dV}{dt} = 120 + 26t - t^2$

Initial rate when $t = 0$,

$\therefore \frac{dV}{dt} = 120$ L/min.

$\therefore$ Twice the initial rate $= 240$ L/min.

Finding t when $\frac{dV}{dt} = 240$:

$$240 = 120 + 26t - t^2$$
$$t^2 - 26t + 120 = 0$$
$$(t-6)(t-20) = 0.$$

$\therefore t = 6, 20$

$\therefore$ The tank fills at twice the initial rate after 6 minutes and after 20 minutes.

(ii) $V = \int \frac{dV}{dt}\, dt$

$\therefore V = \int (120 + 26t - t^2)\, dt$

$= 120t + 13t^2 - \frac{1}{3}t^3 + c$, where c is a constant.

(iii) When $t = 0$, $V = 1500$

$\therefore 1500 = 120(0) + 13(0)^2 - \frac{1}{3}(0)^3 + c$

ie. $c = 1500.$

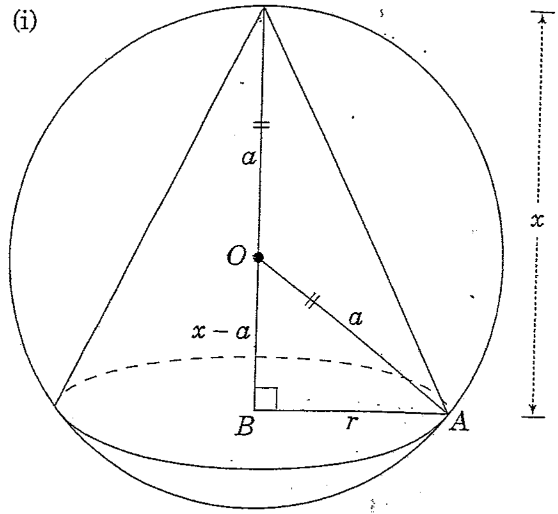
When $t = 30$,

$V = 120(30) + 13(30)^2 - \frac{1}{3}(30)^3 + 1500$
$= 7800.$

But the tank has only 7000 L capacity.

$\therefore$ Water lost $= 7800 - 7000$ L
$= 800$ L.

(c) (i)



Volume of cone $= \frac{1}{3}\pi r^2 h$, where $h = x$

$\therefore \quad V = \frac{1}{3}\pi r^2 x \quad *$

Using Pythagoras' theorem in ΔOBA:

$$a^2 = r^2 + (x-a)^2$$
$$a^2 = r^2 + x^2 - 2ax + a^2$$
$$0 = r^2 + x^2 - 2ax$$

$\therefore \quad r^2 = 2ax - x^2.$

Substituting into $*$:

$V = \frac{1}{3}\pi(2ax - x^2)x$

$\therefore V = \frac{1}{3}\pi(2ax^2 - x^3)$, as required.

(ii) For maximum volume,

$\frac{dV}{dx} = 0$ and $\frac{d^2V}{dx^2} < 0.$

$\frac{dV}{dx} = \frac{1}{3}\pi(4ax - 3x^2)$

and $\frac{d^2V}{dx^2} = \frac{1}{3}\pi(4a - 6x).$

Now if $\frac{dV}{dx} = 0$: $\frac{1}{3}\pi(4ax - 3x^2) = 0$

$$4ax - 3x^2 = 0$$
$$x(4a - 3x) = 0$$

$\therefore \quad x = 0 \quad$ or $\quad 4a - 3x = 0$

$$4a = 3x$$
$$x = \frac{4a}{3}.$$

But if $x = 0$, $V = 0$ and there is no cone.

If $x = \frac{4a}{3}$: $\frac{d^2V}{dx^2} = \frac{1}{3}\pi\left[4a - 6\left(\frac{4a}{3}\right)\right]$

$= \frac{1}{3}\pi(4a - 8a)$

$= \frac{1}{3}\pi(-4a)$

$= \frac{-4}{3}\pi a$, but $a > 0$,

$$\therefore \quad \frac{d^2V}{dx^2} < 0.$$

$\therefore$ The maximum volume occurs when $x = \dfrac{4a}{3}$.

QUESTION 10

(a) *METHOD* 1

Using $\displaystyle\int_a^b f(x)\ dx$

$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

x	0·5	1	1·5
$f(x)$	$\log_e 0{\cdot}5$	$\log_e 1$	$\log_e 1{\cdot}5$

$$\int_{0\cdot5}^{1\cdot5} \left(\log_e x\right)^3\ dx$$

$$= \frac{1{\cdot}5 - 0{\cdot}5}{6}\left[\left(\log_e 0{\cdot}5\right)^3 + 4\left(\log_e 1\right)^3 + \left(\log_e 1{\cdot}5\right)^3\right]$$

$$= \frac{1}{6}\left[-0{\cdot}333\,02 + 0 + 0{\cdot}066\,659\right]$$

$= -0{\cdot}0439$

$= -0{\cdot}044$ to 3 decimal places.

METHOD 2

Using $\displaystyle\int_a^b f(x)\ dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$, $h = \dfrac{b-a}{2}$.

$$\int_{0\cdot5}^{1\cdot5} \left(\log_e x\right)^3\ dx = \frac{\frac{1}{2}}{3}\left[\left(\log_e 0{\cdot}5\right)^3 + 4\left(\log_e 1\right)^3 + \left(\log_e 1{\cdot}5\right)^3\right]$$

$$= \frac{1}{6}\left(-0{\cdot}266\ldots\right)$$

$= -0{\cdot}044$ to 3 decimal places.

METHOD 3 Using the weights method:

x	0·5	1	1·5	Σ
y	$(\log_e 0{\cdot}5)^3$	$(\log_e 1)^3$	$(\log_e 1{\cdot}5)^3$	
weights:	1	4	1	6
$y \times$ wts	$(\log_e 0{\cdot}5)^3$	4×0	$(\log_e 1{\cdot}5)^3$	$(\log_e 0{\cdot}5)^3 + (\log_e 1{\cdot}5)^3$

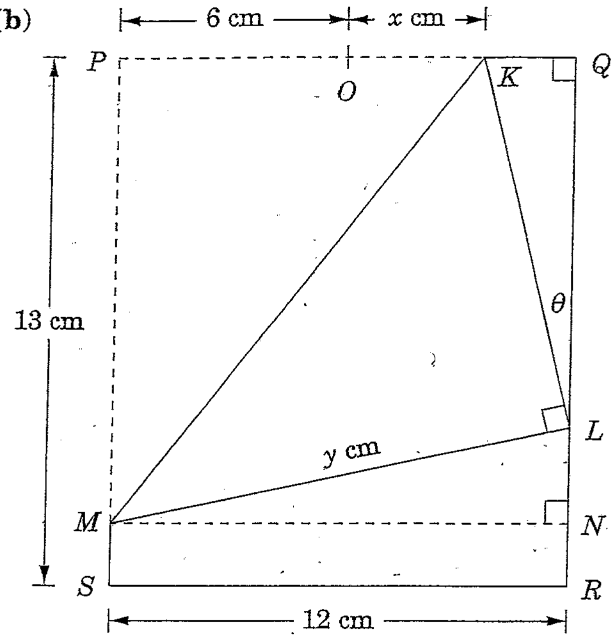
$$\int_a^b f(x)\ dx \doteqdot \frac{\text{total width}}{\sum \text{wts}} \times \sum\left(y \times \text{wts}\right)$$

$$\doteqdot \frac{1{\cdot}5 - 0{\cdot}5}{6}\left[\left(\log_e 0{\cdot}5\right)^3 + \left(\log_e 1{\cdot}5\right)^3\right]$$

$= -0{\cdot}044\,39$

$= -0{\cdot}044$ to 3 decimal places.

(b)



(i) If P folds to L, then $PK = KL$

$\therefore \quad KL = 6 + x$.

Also, $\quad x + KQ = 6$

$\therefore \quad KQ = 6 - x$.

Using Pythagoras' theorem in ΔKQL:

$$KL^2 = KQ^2 + QL^2$$
$$(6+x)^2 = (6-x)^2 + QL^2$$
$$36 + 12x + x^2 = 36 - 12x + x^2 + QL^2$$
$$24x = QL^2$$
$$\therefore \quad QL^2 = 24x.$$

(ii) In ΔQKL and ΔNLM,

$\angle KQL = \angle LNM = 90°$ (given).

Let $\angle KLQ = \theta$,

$\angle MLN = 180° - 90° - \theta$ (straight $\angle$)

$\therefore \quad \angle MLN = 90° - \theta$

$\angle LKQ = 90° - \theta$ ($\angle$ sum $\Delta KQL = 180°$)

$\therefore \quad \angle MLN = \angle LKQ$

$\therefore \quad \Delta QKL \,|||\, \Delta NLM$ (a pair of corresponding angles equal)

$\therefore \quad \dfrac{ML}{KL} = \dfrac{MN}{QL}$ (corresponding sides of similar Δs in same ratio)

$$\frac{y}{6+x} = \frac{12}{\sqrt{24x}}$$

$$y = \frac{12(6+x)}{\sqrt{24x}}$$

$$= \frac{12(6+x)}{2\sqrt{6} \times \sqrt{x}}$$

$$= \frac{6(6+x)}{\sqrt{6} \times \sqrt{x}}$$

$$\therefore \quad y = \frac{\sqrt{6}\,(6+x)}{\sqrt{x}}.$$

(iii) Area $\Delta KLM = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times KL \times ML$$

$$= \frac{1}{2} \times (6+x) \times \frac{\sqrt{6}(6+x)}{\sqrt{x}}.$$

$$\therefore \quad \text{Area} = \frac{\sqrt{6}(6+x)^2}{2\sqrt{x}}.$$

(iv) $12 \le y \le 13$.

Possible values of x: $y = \frac{\sqrt{6}(6+x)}{\sqrt{x}}$

For $y = 12$, $12 = \frac{\sqrt{6}(6+x)}{\sqrt{x}}$

$$12\sqrt{x} = \sqrt{6}(6+x)$$
$$144x = 6(6+x)^2$$
$$144x = 6(36+12x+x^2)$$
$$24x = 36+12x+x^2$$
$$0 = 36-12x+x^2$$
$$0 = (6-x)^2$$
$$\therefore \quad x = 6.$$

For $y = 13$, $13 = \frac{\sqrt{6}(6+x)}{\sqrt{x}}$

$$13\sqrt{x} = \sqrt{6}(6+x)$$
$$169x = 6(6+x)^2$$
$$169x = 6(36+12x+x^2)$$
$$169x = 216+72x+6x^2$$
$$0 = 216-97x+6x^2$$
$$\therefore \quad 0 = 6x^2-97x+216.$$

$$\Delta = (-97)^2 - 4\times 6 \times 216$$
$$= \sout{4425}.\ 4225$$

$$\therefore \quad x = \frac{97 \pm \sqrt{\sout{4425}\ 4225}}{12}, \text{ quadratic formula}$$

$$x = \frac{97 \pm 65}{12}$$

$$\therefore \quad x = 13\frac{1}{2} \text{ or } 2\frac{2}{3}.$$

But $x \ne 13\frac{1}{2}$ as $0 \le x \le 6$,

$\therefore$ for $12 \le y \le 13$, $2\frac{2}{3} \le x \le 6$.

(v) Minimum value of area ΔKLM:

$$A = \frac{\sqrt{6}(6+x)^2}{2\sqrt{x}}$$

$$A = \frac{\sqrt{6}}{2}(6+x)^2 \times x^{-\frac{1}{2}}$$

$$\frac{dA}{dx} = \frac{\sqrt{6}}{2}\left[2(6+x)\times 1 \times x^{-\frac{1}{2}} + \left(-\frac{1}{2}x^{-\frac{3}{2}}\right)(6+x)^2\right]$$

$$\frac{dA}{dx} = \frac{\sqrt{6}}{2}\left[\frac{2(6+x)}{\sqrt{x}} - \frac{(6+x)^2}{2x\sqrt{x}}\right]$$

$$= \frac{\sqrt{6}}{2}(6+x)\left[\frac{2}{\sqrt{x}} - \frac{(6+x)}{2x\sqrt{x}}\right]$$

for stationary point $\frac{dA}{dx} = 0$.

$$\therefore \quad \frac{\sqrt{6}}{2}(6+x)\left[\frac{2}{\sqrt{x}} - \frac{(6+x)}{2x\sqrt{x}}\right] = 0$$

$6 + x = 0$ or $\frac{2}{\sqrt{x}} - \frac{(6+x)}{2x\sqrt{x}} = 0$

$x = -6$: (no solution)

$$2 - \frac{(6+x)}{2x} = 0$$
$$4x - 6 - x = 0$$
$$3x - 6 = 0$$
$$3x = 6$$
$$x = 2.$$

This does not give a valid solution since $2\frac{2}{3} \le x \le 6$.

Therefore the minimum area must lie at one of the endpoints,

ie. $x = 2\frac{2}{3}$ or $x = 6$.

When $x = 6$: $A = \frac{\sqrt{6}(6+6)^2}{2\sqrt{6}}$

$$A = \frac{144}{2}$$
$$A = 72.$$

When $x = 2\frac{2}{3}$: $A = \dfrac{\sqrt{6}\left(6+2\frac{2}{3}\right)^2}{2\sqrt{\frac{8}{3}}}$

$$= \frac{\sqrt{6}\left(6+2\frac{2}{3}\right)^2}{2\sqrt{8}} \times \sqrt{3}$$

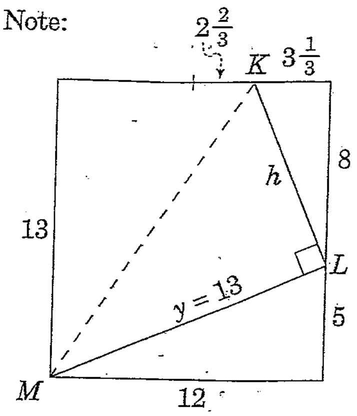
$$= \frac{3\sqrt{2}\left(6+2\frac{2}{3}\right)^2}{4\sqrt{2}}$$

$$= \frac{3}{4}\left(6+2\frac{2}{3}\right)^2$$

$$= 56\frac{1}{3}.$$

$\therefore$ The minimum area is $56\frac{1}{3}$ cm^2.

Note:



ΔKLM will have minimum area when M lies on S, ie. when $y = 13$.

When $y = 13$, $h^2 = \left(3\frac{1}{3}\right)^2 + 8^2$

$= 75\frac{1}{9}$

$\therefore$ $h = 8\frac{2}{3}$.

$\therefore$ Area $= \frac{1}{2} \times 13 \times 8\frac{2}{3}$

$= 56\frac{1}{3}$.

$\therefore$ The minimum area is $56\frac{1}{3}$ cm^2.

END OF MATHEMATICS SOLUTIONS