1997 HIGHER SCHOOL CERTIFICATE EXAMINATION PAPER

2/3 UNIT (COMMON) MATHEMATICS

QUESTION 1

Marks

(a) Find the value of **2**

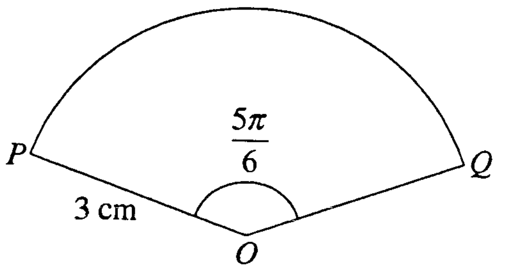
$$\frac{1}{7+5\times 3}$$

correct to three significant figures.

(b) Simplify $(2-3x)-(5-4x)$. **2**

(c) Write down the exact value of 135° in radians. **1**

(d) **1**



In the diagram, PQ is an arc of a circle with centre O. The radius $OP = 3$ cm and the angle POQ is $\frac{5\pi}{6}$ radians.

Find the length of the arc PQ.

(e) Using the table of standard integrals find $\int \sec 3x \tan 3x \, dx$. **1**

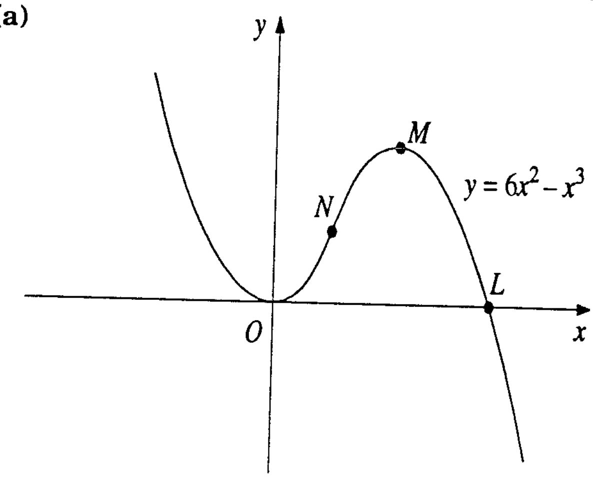
(f) Forty-five balls, numbered 1 to 45, are placed in a barrel, and one ball is drawn at random. What is the probability that the number on the ball drawn is even? **2**

(g) By rationalising the denominator, express $\frac{8}{3-\sqrt{5}}$ in the form $a+b\sqrt{5}$. **3**

QUESTION 2

Marks

(a) **6**



The diagram shows a sketch of the curve

$$y = 6x^2 - x^3.$$

The curve cuts the x axis at L, and has a local maximum at M and a point of inflection at N.

(i) Find the coordinates of L.

(ii) Find the coordinates of M.

(iii) Find the coordinates of N.

(b) The graph of $y=f(x)$ passes through the point (1, 4) and $f'(x) = 2x+7$. Find $f(x)$. **2**

(c) Find a primitive of $\frac{2x}{x^2+1}$. **1**

(d) Consider the parabola with equation **3**

$$x^2 = 4(y-1).$$

(i) Find the coordinates of the vertex of the parabola.

(ii) Find the coordinates of the focus of the parabola.

QUESTION 3

(a) Differentiate the following functions: **6**

(i) $\left(x^2+5\right)^3$ (ii) $\frac{\cos x}{x}$ (iii) $x^2 \ln x$.

(b) Let A and B be the points $(0, 1)$ and $(2, 3)$ respectively. **6**

(i) Find the coordinates of the midpoint of AB.

(ii) Find the slope of the line AB.

(iii) Find the equation of the perpendicular bisector of AB.

(iv) The point P lies on the line $y = 2x - 9$ and is equidistant from A and B. Find the coordinates of P.

QUESTION 4

(a) The table shows the values of a function $f(x)$ for five values of x. **3**

x	1	1·5	2	2·5	3
$f(x)$	5	1	−2	3	7

Use Simpson's rule with these five values to estimate $\int_1^3 f(x)\,dx$.

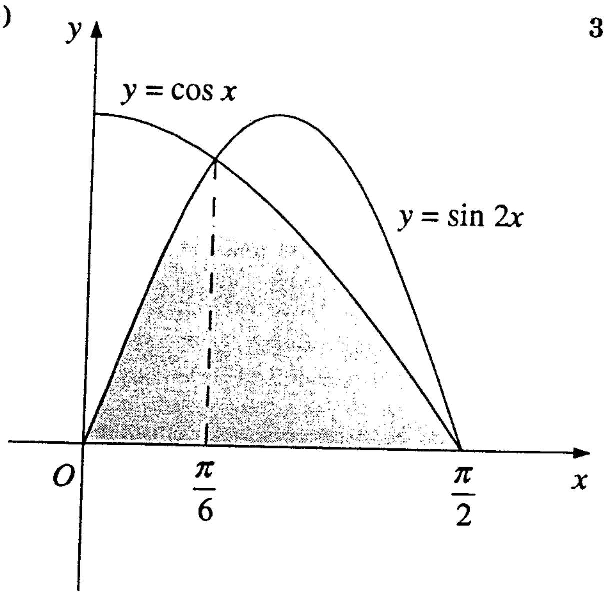
(b) (i) Sketch the graph of $y = x^2 - 6$, and label all intercepts with the axes. **6**

(ii) On the same set of axes, carefully sketch the graph of $y = |x|$.

(iii) Find the x coordinates of the two points where the graphs intersect.

(iv) Hence solve the inequality $x^2 - 6 \le |x|$.

(c) **3**



The diagram shows the graphs of the functions $y = \cos x$ and $y = \sin 2x$ between $x = 0$ and $x = \frac{\pi}{2}$.

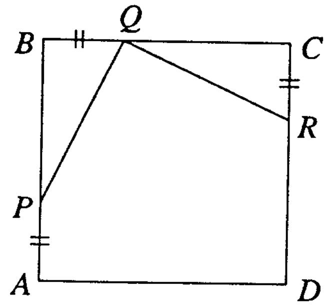
The two graphs intersect at $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$.

Calculate the area of the shaded region.

QUESTION 5

(a) Evaluate $\int_0^{\ln 7} e^{-x}\,dx$. **2**

(b) **4**



In the diagram, $ABCD$ is a square. The points P, Q, and R lie on AB, BC, and CD respectively, such that $AP = BQ = CR$.

(i) Prove that triangles PBQ and QCR are congruent.

(ii) Prove that $\angle PQR$ is a right angle.

(c) The rate of inflation measures the rate of change in prices. Between January 1996 and December 1996, prices were rising but the rate of inflation was falling. Draw a graph of prices as a function of time that fits this description. **2**

(d) A ball is dropped from a height of 2 metres onto a hard floor and bounces. After each bounce, the maximum height reached by the ball is 75% of the previous maximum height. Thus, after it first hits the floor, it reaches a height of 1·5 metres before falling again, and after the second bounce, it reaches a height of 1·125 metres before falling again. **4**

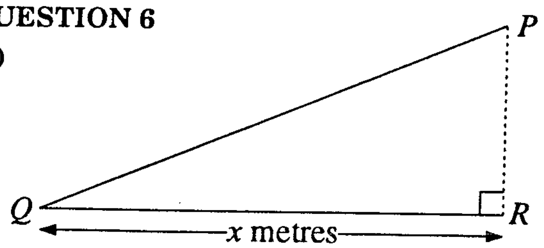
(i) What is the maximum height reached after the third bounce?

(ii) What kind of sequence is formed by the successive maximum heights?

(iii) What is the total distance travelled by the ball from the time it was first dropped until it eventually comes to rest on the floor?

QUESTION 6

(a) **6**



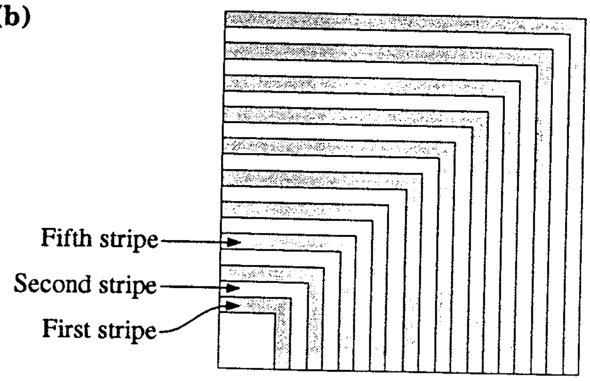
A wire of length 5 metres is to be bent to form the hypotenuse and base of a right-angled triangle PQR, as shown in the diagram. Let the length of the base QR be x metres.

(i) What is the length of the hypotenuse PQ in terms of x?

(ii) Show that the area of the triangle PQR is $\frac{1}{2}x\sqrt{25-10x}$ square metres.

(iii) What is the maximum possible area of the triangle?

(b) 6



A logo is made of 20 squares with a common corner, as shown in the diagram. The odd-numbered 'stripes' between successive squares are shaded in the diagram. The shaded stripes are painted in gold paint, which costs $9 per square centimetre.

The side length of the nth square is $(2n+4)$ cm. The nth stripe lies between the nth square and the $(n+1)$th square.

(i) Show that the area of the nth stripe is $(8n+20)\text{ cm}^2$.

(ii) Hence find the areas of the first and last stripes.

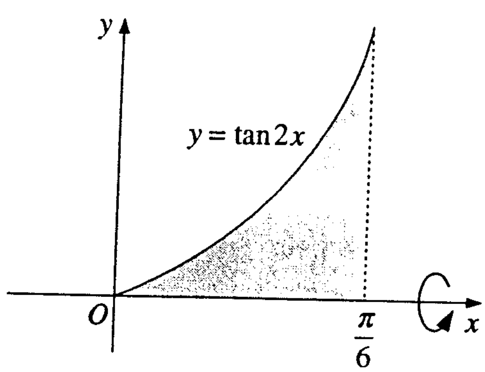
(iii) Hence find the total cost of the gold paint for the logo.

QUESTION 7

(a) By expressing $\sec\theta$ and $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$, show that 2

$$\sec^2\theta - \tan^2\theta = 1.$$

(b) 5



The diagram shows part of the graph of the function $y = \tan 2x$.

The shaded region is bounded by the curve, the x axis, and the line $x = \frac{\pi}{6}$. The region is rotated about the x axis to form a solid.

(i) Show that the volume of the solid is given by

$$V = \pi\int_0^{\frac{\pi}{6}}\left(\sec^2 2x - 1\right)\,dx.$$

You may use the result of part (a).

(ii) Find the exact volume of the solid.

(c) A ball is dropped into a long vertical tube filled with honey. The rate at which the ball decelerates is proportional to its velocity. 5

Thus $$\frac{dv}{dt} = -kv,$$

where v is the velocity in centimetres per second, t is the time in seconds, and k is a constant.

When the ball first enters the honey, at $t=0$, $v=100$. When $t=0{\cdot}25$, $v=85$.

(i) Show that $v = Ce^{-kt}$ satisfies the equation $\frac{dv}{dt} = -kv$.

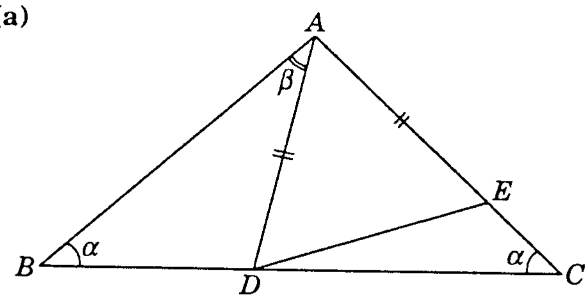
(ii) Find the value of the constant C.

(iii) Find the value of the constant k.

(iv) Find the velocity when $t=2$.

QUESTION 8

(a) 4



In the isosceles triangle ABC,

$$\angle ABC = \angle ACB = \alpha.$$

The points D and E lie on BC and AC, so that $AD = AE$, as shown in the diagram. Let $\angle BAD = \beta$.

(i) Explain why $\angle ADC = \alpha + \beta$.

(ii) Find $\angle DAC$ in terms of α and β.

(iii) Hence, or otherwise, find $\angle EDC$ in terms of β.

(b) A particle is moving along the x axis. Its position at time t is given by 8

$$x = t + \sin t.$$

(i) At what times during the period $0 < t < 3\pi$ is the particle stationary?

(ii) At what times during the period $0 < t < 3\pi$ is the acceleration equal to 0?

(iii) Carefully sketch the graph of $x = t + \sin t$ for $0 < t < 3\pi$.

Clearly label any stationary points and any points of inflection.

QUESTION 9

(a) A bag contains two red balls, one black ball, and one white ball. Andrew selects one ball from the bag and keeps it hidden. He then selects a second ball, also keeping it hidden. **5**

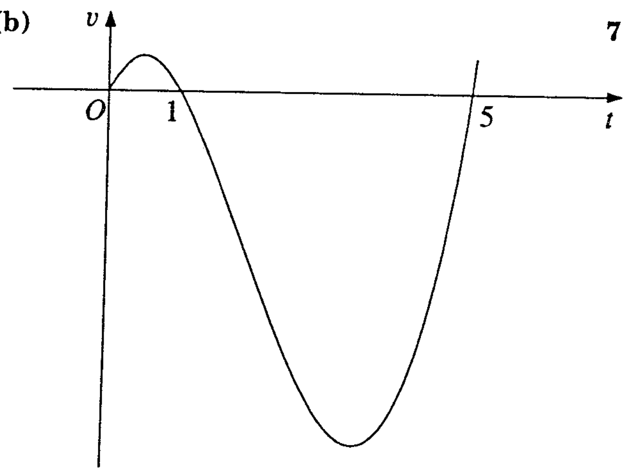
(i) Draw a tree diagram to show all the possible outcomes.

(ii) Find the probability that both the selected balls are red.

(iii) Find the probability that at least one of the selected balls is red.

(iv) Andrew drops one of the selected balls and we can see that it is red. What is the probability that the ball that is still hidden is also red?

(b) **7**



A pen moves along the x axis, ruling a line. The diagram shows the graph of the velocity of the tip of the pen as a function of time.

The velocity, in centimetres per second, is given by the equation

$$v = 4t^3 - 24t^2 + 20t,$$

where t is the time in seconds. When $t = 0$, the tip of the pen is at $x = 3$. That is, the tip is initially 3 centimetres to the right of the origin.

(i) Find an expression for x, the position of the tip of the pen, as a function of time.

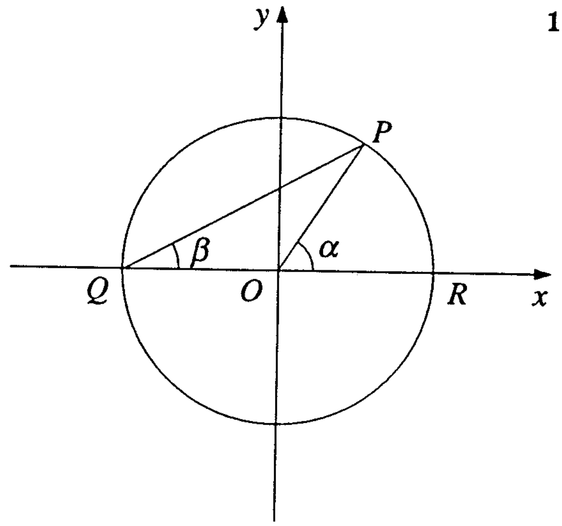
(ii) What feature will the graph of x as a function of t have at $t = 1$?

(iii) The pen uses 0·05 milligrams of ink per centimetre travelled. How much ink is used between $t = 0$ and $t = 2$?

QUESTION 10

(a) Graph the solution of $4x \le 15 \le -9x$ on a number line. **2**

(b) **10**



In the diagram, Q is the point $(-1, 0)$, R is the point $(1, 0)$, and P is another point on the circle with centre O and radius 1. Let $\angle POR = \alpha$ and $\angle PQR = \beta$, and let $\tan\beta = m$.

(i) Explain why ΔOPQ is isosceles, and hence deduce that $\alpha = 2\beta$.

(ii) Find the equation of the line PQ.

(iii) Show that the x coordinates of P and Q are solutions of the equation

$$\left(1 + m^2\right)x^2 + 2m^2x + m^2 - 1 = 0.$$

(iv) Using this equation, find the coordinates of P in terms of m.

(v) Hence deduce that

$$\tan 2\beta = \frac{2\tan\beta}{1 - \tan^2\beta}.$$

1997 HIGHER SCHOOL CERTIFICATE SOLUTIONS

2/3 UNIT (COMMON) MATHEMATICS

QUESTION 1

(a) $\frac{1}{7+5\times 3} = 0{\cdot}0\dot{4}\dot{5}$
$= 0{\cdot}0455$ (3 sig. figs)

(b) $(2-3x)-(5-4x) = 2-3x-5+4x$
$= x-3.$

(c) $180° = \pi^c$
$1° = \frac{\pi}{180}^c$
$\therefore\ 135° = 135\times\frac{\pi}{180}^c$
$= \frac{3\pi}{4}^c.$

(d) $PQ = r\theta$
$= 3\times\frac{5\pi}{6}$
$= \frac{5\pi}{2}$ cm.

(e) $\int \sec 3x \tan 3x\ dx = \frac{1}{3}\sec 3x + C.$

(f) There are 22 even number balls.
$\therefore\ P(\text{even}) = \frac{22}{45}$

(g) $\frac{8}{3-\sqrt{5}}\times\frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{24+8\sqrt{5}}{9-5}$
$= 6+2\sqrt{5}.$

QUESTION 2

(a) (i) At L, $y = 0$
$\therefore\ 6x^2 - x^3 = 0$
$x^2(6-x) = 0$
$\therefore\ x = 0$ or 6.
From the diagram the x coordinate at L is 6, $\therefore\ L = (6, 0)$.

(ii) At M, $\frac{dy}{dx} = 0$ as M is a stationary point.
$\frac{dy}{dx} = 12x - 3x^2$
$= 3x(4-x)$
$= 0$ when $x = 0$ or 4.
From the diagram the x coordinate at M is 4, $\therefore\ M = (4, 32)$.

(iii) N is a point of inflection. $\therefore\ \frac{d^2y}{dx^2} = 0$ and there is a change in concavity.
$\frac{d^2y}{dx^2} = 12 - 6x = 0$ when $x = 2$.

Testing:

x	$2-\varepsilon$	2	$2+\varepsilon$
y''	+	0	−

$\therefore\ N = (2, 16)$.

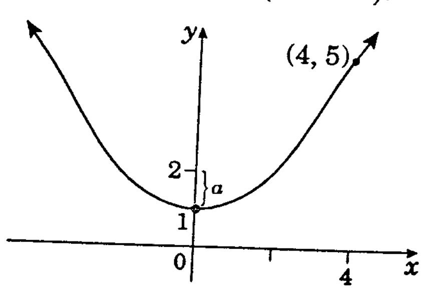
(b) $f'(x) = 2x+7$
$f(x) = x^2+7x+C$
$f(1) = 1+7+C = 4$
$C = -4$
$\therefore\ f(x) = x^2+7x-4.$

(c) $\int \frac{2x}{x^2+1}\,dx = \ln\left(x^2+1\right)+C.$

(d) We match $x^2 = 4(y-1)$ with the form
$(x-h)^2 = 4\times a\times(y-k)$,
giving $(x-0)^2 = 4\times 1\times(y-1)$.

(i) The vertex is at $(h, k) = (0, 1)$.

(ii) Focal length = 1 unit ($\because\ a = 1$).



From the diagram, the focus is at (0, 2).

QUESTION 3

(a) (i) $\frac{d}{dx}\left(x^2+5\right)^3 = 3\times\left(x^2+5\right)^2\times 2x$

$= 6x\left(x^2+5\right)^2.$

(ii) $\frac{d}{dx}\left(\frac{\cos x}{x}\right) = \frac{x\times(-\sin x)-\cos x\times 1}{x^2}$

$= \frac{-x\sin x-\cos x}{x^2}.$

(iii) $\frac{d}{dx}\left(x^2\ln x\right) = 2x\ln x + x^2\times\frac{1}{x}$

$= 2x\ln x + x$

$= x(2\ln x+1).$

(b) (i) Midpoint of $AB = \left(\frac{0+2}{2}, \frac{1+3}{2}\right) = (1,2).$

(ii) Slope of $AB = \frac{3-1}{2-0} = \frac{2}{2} = 1.$

(iii) Use $y-y_1 = m(x-x_1)$, with $(x_1, y_1) = (1,2)$, and $m=-1$.

$\therefore \quad y-2 = -1\times(x-1)$

$\therefore \quad y = -x+1+2$

i.e. $\quad y = -x+3.$

Alternative method:
Points on the perpendicular bisectors are equidistant from A and B. So if $P(x, y)$ is on the perpendicular bisector, then

$PA^2 = PB^2.$

i.e. $(x-0)^2+(y-1)^2 = (x-2)^2+(y-3)^2$

$\not{x^2}+\not{y^2}-2y+1$

$= \not{x^2}-4x+4+\not{y^2}-6y+9$

$4y = -4x+12$

$y = -x+3.$

(iv) Points equidistant from A and B lie on the perpendicular bisector, so the coordinates of P are the simultaneous solutions of

$y = -x+3 \quad$ —①

and $\quad y = 2x-9. \quad$ —②

Equating ① and ② to eliminate y:

$\therefore \quad 2x-9 = -x+3$

$\therefore \quad 3x = 12$

$\therefore \quad x = 4.$

In ①, $\quad y = -4+3$

$= -1$

$\therefore \quad P = (4,-1).$

QUESTION 4

(a)

x	1	1·5	2	2·5	3
$f(x)$	5	1	−2	3	7

$\int_1^3 f(x)\,dx \approx \frac{2-1}{6}[5+(-2)+4\times 1]$

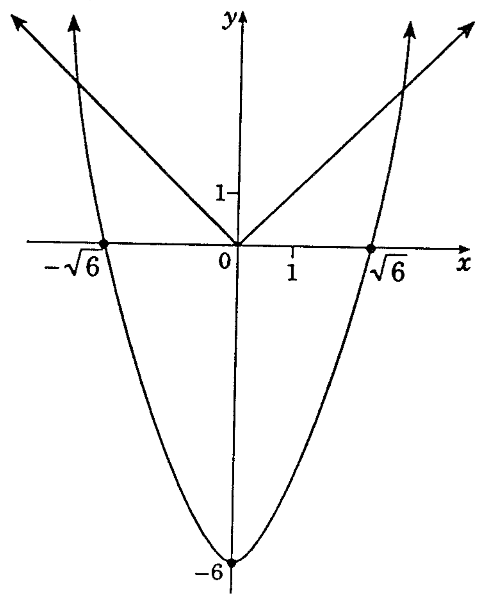
$+\frac{3-2}{6}(-2+7+4\times 3)$

$= \frac{1}{6}\times 7+\frac{1}{6}\times 17$

$= \frac{24}{6}$

$= 4.$

(b) (i) and (ii)



(iii) Solve $\quad x^2-6 = x \quad$ for $x>0$.

$x^2-x-6 = 0$

$(x-3)(x+2) = 0$

$\therefore \quad x = 3$ or -2.

Reject $x=-2$ (outside domain).

The x coordinates of the two points where the graphs intersect are 3 and −3.

(N.B. Both graphs are even functions.)

(iv) We want the x values for which the parabola is 'on' or below the straight lines, that is $-3\le x\le 3$.

(c) Area $= \int_0^{\frac{\pi}{6}} \sin 2x\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x\,dx$

$= \left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{6}} + \left[\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \frac{1}{2}\left(-\cos\frac{\pi}{3}+1\right)+\left(\sin\frac{\pi}{2}-\sin\frac{\pi}{6}\right)$

$= \frac{1}{2}\left(-\frac{1}{2}+1\right)+\left(1-\frac{1}{2}\right)$

$= \frac{1}{4}+\frac{1}{2}$

$= \frac{3}{4}.$

QUESTION 5

(a) $\int_0^{\ln 7} e^{-x}\,dx = \left[-e^{-x}\right]_0^{\ln 7}$

$= \left(-e^{-\ln 7} + e^0\right)$

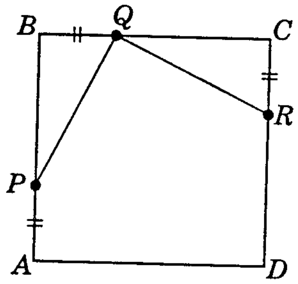
$= -e^{\ln 7^{-1}} + 1$

$= -7^{-1} + 1$

$= -\frac{1}{7} + 1$

$= \frac{6}{7}.$

(b)



(i)

$BA = BC$ ($ABCD$ is a square)

$PA = BQ$ (given)

$\therefore\ BA - PA = BC - BQ$

i.e. $BP = QC.$

So in triangles PBQ and QCR, we have

$BP = QC$ (from above)

$BQ = CR$ (given)

$\angle PBQ = \angle QCR$ (both 90°, since $ABCD$ is a square)

$\Delta PBQ \equiv \Delta QCR$ (SAS).

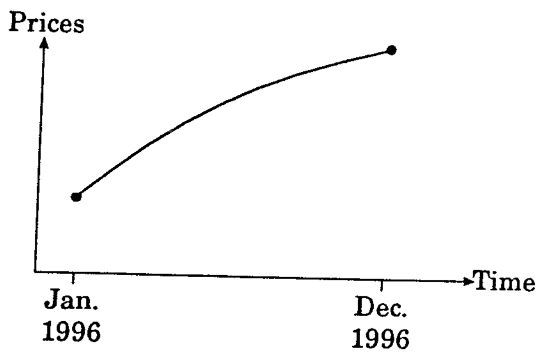
(ii) $\angle BQP + \angle BPQ = 90°$ ($\angle PBQ = 90°$, angle sum of Δ)

$\angle BPQ = \angle CQR$ (corresponding $\angle$s in congruent Δs)

$\therefore\ \angle BQP + \angle CQR = 90°$

$\therefore\ \angle PQR = 180° - 90° = 90°.$ $\left\{\angle CQR,\ \angle PQR,\ \angle BQP \text{ form a straight line.}\right\}$

(c)



N.B. The curve is increasing and concave down (gradient is decreasing).

(d) **(i)** Height $= 75\%$ of $1{\cdot}125$ metres

$= 0{\cdot}843\,75$ metres.

(ii) This is a geometric sequence, with common ratio $0{\cdot}75$.

(iii) The total distance travelled is 2 m for the first drop, plus twice the new height for every bounce (since the ball goes up and down), that is $2 + 2(1{\cdot}5 + 1{\cdot}125 + \cdots)$.

If we leave out the first drop, we can add up all the heights (limiting sum in brackets above) using

$$S_\infty = \frac{a}{1-r},$$

with $a = 1{\cdot}5$ and $r = 0{\cdot}75$.

$$S_\infty = \frac{1{\cdot}5}{1 - 0{\cdot}75}.$$

Total distance travelled $= 2 + 2 \times \left(\frac{1{\cdot}5}{1 - 0{\cdot}75}\right)$

$= 14$ m.

QUESTION 6

(a) **(i)** $PQ + x = 5$

$\therefore\ PQ = (5 - x)$ m.

(ii) $PR^2 = (5-x)^2 - x^2$

$= 25 - 10x + x^2 - x^2$

$= 25 - 10x.$

$\therefore\ PR = \sqrt{25 - 10x}$ m.

Area: $A = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times x \times \sqrt{25 - 10x}$

$= \frac{1}{2} x\sqrt{25 - 10x}$ m^2.

(iii) Differentiate to find maximum.

$A' = \frac{1}{2}\sqrt{25 - 10x} + \frac{1}{2}x.\frac{1}{2\sqrt{25-10x}}.(-10)$

$= \frac{1}{2}\sqrt{25-10x} - \frac{1}{2}.\frac{5x}{\sqrt{25-10x}}$

$= \frac{25 - 10x - 5x}{2\sqrt{25-10x}}$

$= \frac{25 - 15x}{2\sqrt{25-10x}}.$

$A' = 0$ when $25 - 15x = 0$, $\therefore\ x = \frac{5}{3}.$

If $x < \frac{5}{3}$, then $A' > 0$, and if $x > \frac{5}{3}$, then $A' < 0$, so this is a local maximum.

Since A is continuous on its domain $0 < x < 2{\cdot}5$, and since this is the only stationary point, this is also an absolute maximum.

$A_{\max} = \frac{1}{2} \times \frac{5}{3} \times \sqrt{25 - \frac{50}{3}}$

$= \frac{5}{6} \times \sqrt{\frac{25}{3}}$

$= \frac{25}{6\sqrt{3}} = \frac{25\sqrt{3}}{18}$ m^2.

(b) (i) Area of nth stripe

$= \text{area of } (n+1)^{\text{th}} \text{ square} - \text{area of } n^{\text{th}} \text{ square}$

$= [2(n+1)+4]^2 - (2n+4)^2$

$= (2n+4+2)^2 - (2n+4)^2$

$= (2n+4+2+2n+4)\times(2n+6-2n-4)$

$\left(\because\ a^2-b^2=(a+b)(a-b)\right)$

$= (4n+10)\times 2$

$= (8n+20)\text{ cm}^2.$

(ii) Area of 1st stripe $= (8\times 1+20)\text{ cm}^2$

$= 28\text{ cm}^2.$

Area of last stripe $(n=19) = 8\times 19+20$

$= 172\text{ cm}^2.$

(iii) There are 10 shaded stripes in arithmetic progression, so:

Area of shaded stripes

$= 1\text{st}+3\text{rd}+5\text{th}+\cdots+19\text{th}$

$= 28+44+60+\cdots+172$

$= \frac{n}{2}(a+\ell)$

$= \frac{10}{2}(28+172)$

$= 5\times 200$

$= 1000\text{ cm}^2.$

Cost of gold paint $= \$9\times 1000$

$= \$9000.$

QUESTION 7

(a) $\sec\theta = \dfrac{1}{\cos\theta}$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\therefore \quad \sec^2\theta - \tan^2\theta = \dfrac{1}{\cos^2\theta} - \dfrac{\sin^2\theta}{\cos^2\theta}$

$= \dfrac{1-\sin^2\theta}{\cos^2\theta}$

$= \dfrac{\cos^2\theta}{\cos^2\theta}$

$= 1.$

OR

$\cos^2\theta + \sin^2\theta = 1$

$\therefore \quad 1-\sin^2\theta = \cos^2\theta$

$\therefore \quad \dfrac{1-\sin^2\theta}{\cos^2\theta} = 1.$

$\therefore \quad \dfrac{1}{\cos^2\theta} - \dfrac{\sin^2\theta}{\cos^2\theta} = 1$

$\therefore \quad \sec^2\theta - \tan^2\theta = 1.$

(b) (i) $V = \int_0^{\frac{\pi}{6}} \pi y^2\, dx$

$= \pi\int_0^{\frac{\pi}{6}} \tan^2 2x\, dx$

$= \pi\int_0^{\frac{\pi}{6}} \left(\sec^2 2x - 1\right) dx$

$\left(\tan^2\theta = \sec^2\theta - 1 \text{ from (a)}\right).$

(ii) $V = \pi\left[\frac{1}{2}\tan 2x - x\right]_0^{\frac{\pi}{6}}$

$= \pi\left(\frac{1}{2}\tan\frac{\pi}{3} - \frac{\pi}{6}\right) - \pi(0-0)$

$= \pi\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)\text{ units}^3.$

(c) (i) $v = Ce^{-kt}$

$\dfrac{dv}{dt} = C\times(-k)e^{-kt}$

$= -k\times Ce^{-kt}$

$= -kv.$

(ii) Using $v=100$ when $t=0$,

$100 = Ce^0$

$\therefore \quad C = 100.$

(iii) $\therefore\ v = 100e^{-kt}$

Using $v=85$ when $t=0{\cdot}25$

$85 = 100e^{-0\cdot 25k}$

$0{\cdot}85 = e^{-0\cdot 25k}$

$\therefore \quad -0{\cdot}25k = \ln(0{\cdot}85)$

$\therefore \quad k = -4\ln(0{\cdot}85) \quad (\approx 0{\cdot}650).$

(iv) $v = 100e^{-0\cdot 65t}$

$= 100e^{-1\cdot 3}$ when $t=2$

$= 27{\cdot}25$ m/s.

OR

$v = 100e^{-(-4\ln 0\cdot 85)t}$

$= 100e^{8\ln 0\cdot 85}$ when $t=2$

$= 100e^{\ln 0\cdot 85^8}$

$= 100\times 0{\cdot}85^8$

$= 27{\cdot}25$ m/s.

QUESTION 8

(a) (i) $\angle ADC$ is the exterior angle of ΔBAD.

$\therefore\ \angle ADC = \angle DBA + \angle DAB$ (exterior $\angle$ equals sum of int. opp. $\angle$s.)

$= \alpha + \beta.$

OR

$\angle BDA = 180° - (\alpha+\beta)$ ($\angle$ sum of ΔBDA).

$\therefore\ \angle ADC = 180° - \angle BDA$ (BDC is a straight line.)

$= 180° - [180 - (\alpha+\beta)]$

$= \alpha+\beta.$

(ii) $\angle DAC = 180° - (\alpha+\beta) - \alpha$ ($\angle$ sum of ΔADC)

$= 180° - 2\alpha - \beta$

$= 180° - (2\alpha+\beta).$

(iii) $\angle ADE = \dfrac{180° - \angle DAE}{2}$ ($\angle$ sum of ΔADE, base $\angle$s of isosceles Δ are equal.)

$= \dfrac{2\alpha+\beta}{2}$

$= \alpha + \dfrac{\beta}{2}.$

$\angle EDC = \angle ADC - \angle ADE$

$= \alpha+\beta - \left(\alpha + \dfrac{\beta}{2}\right)$

$= \dfrac{\beta}{2}.$

(b) (i) $x = t + \sin t, \quad 0 < t < 3\pi$

$\dfrac{dx}{dt} = 1 + \cos t.$

The particle is stationary when $\dfrac{dx}{dt} = 0.$

$\therefore\quad \cos t = -1$

$\therefore\quad t = \pi.$

(N.B. Also at $t = 3\pi$, but this is not in the domain.)

(ii) $\dfrac{d^2x}{dt^2} = -\sin t.$

Acceleration zero means $\dfrac{d^2x}{dt^2} = 0.$

$\therefore\quad \sin t = 0$

$\therefore\quad t = \pi$ or $2\pi.$

(N.B. Also at 0 and 3π, but again not in the domain.)

(iii) At $t = \pi$ we have a stationary point and a point of inflection.

$x = \pi + \sin\pi$

$= \pi \qquad (\pi, \pi).$

At $t = 2\pi$ we have a point of inflection.

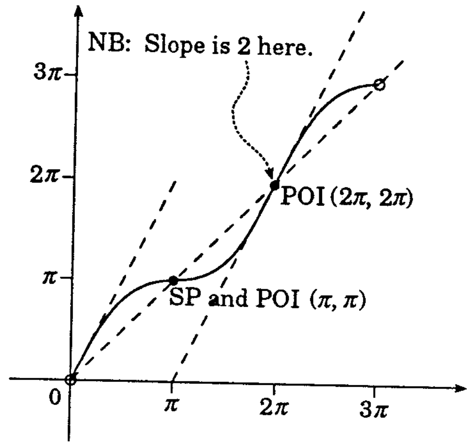
$x = 2\pi + \sin 2\pi$

$= 2\pi \qquad (2\pi, 2\pi).$

N.B. This is not a stationary point. It has a slope of $\dfrac{dx}{dt} = 1 + \cos 2\pi$

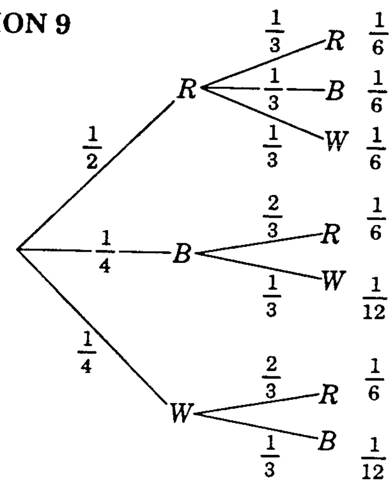
$= 1 + 1$

$= 2.$

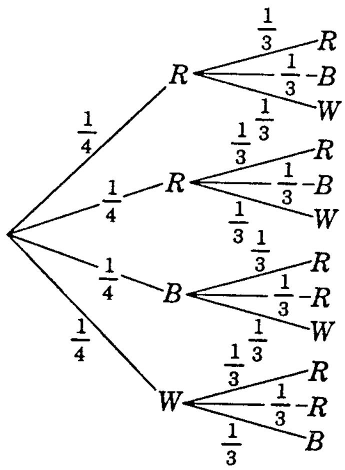


QUESTION 9

(a) (i)



OR



(ii) $P(\text{both red}) = \dfrac{1}{2} \times \dfrac{1}{3} = \dfrac{1}{6}.$

OR

$P(\text{both red}) = \dfrac{1}{12} + \dfrac{1}{12} = \dfrac{1}{6}.$

(iii) $P(\text{at least 1 red}) = \dfrac{10}{12} = \dfrac{5}{6}.$

OR

$P(\text{no red}) = P(BW) + P(WB)$

$= \dfrac{1}{4} \times \dfrac{1}{3} + \dfrac{1}{4} \times \dfrac{1}{3}$

$= \dfrac{1}{12} + \dfrac{1}{12} = \dfrac{1}{6}.$

$\therefore\ P(\text{at least 1 red}) = 1 - \dfrac{1}{6} = \dfrac{5}{6}.$

(iv) $P(2R \text{ given } 1R) = \frac{2}{10} = \frac{1}{5}$.

(b) (i) $\frac{dx}{dt} = 4t^3 - 24t^2 + 20t$.

$\therefore \quad x = t^4 - 8t^3 + 10t^2 + C$.

At $t = 0,\ x = 3, \quad \therefore\ C = 3$.

$x = t^4 - 8t^3 + 10t^2 + 3$.

(ii) At $t = 1,\ v = 0$ and changes sign from positive to negative:

/ — \

So at $t = 1$, the $x = f(t)$ graph has a maximum stationary point.

(iii) At $t = 1, \quad x = 1 - 8 + 10 + 3$
$= 6$.

At $t = 2, \quad x = 16 - 64 + 40 + 3$
$= -5$.

Distance travelled (from $t = 0$ to 1)
$= 6 - 3 = 3$.

Distance travelled (from $t = 1$ to 2)
$= |-5 - 6| = 11$.

$\therefore$ Total dist. travelled = 14 cm.
Ink used $= 0{\cdot}05 \times 14 = 0{\cdot}7$ milligrams.

OR

Distance travelled

$$= \int_0^1 v\, dt + \int_1^2 |v|\, dt$$

$$= \left[t^4 - 8t^3 + 10t^2\right]_0^1 + \left|\left[t^4 - 8t^3 + 10t^2\right]_1^2\right|$$

$$= (3 - 0) + |(-8 - 3)|$$

$$= 3 + 11$$

$$= 14 \text{ cm.}$$

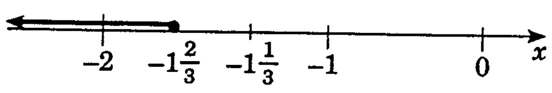
QUESTION 10

(a) $4x \le 15 \le -9x$.

i.e. $4x \le 15$ and $-9x \ge 15$.

$x \le \frac{15}{4} = 3\frac{3}{4}$ and $x \le -\frac{15}{9} = -1\frac{2}{3}$.

Since both conditions must be true, this simplifies to $x \le -1\frac{2}{3}$.



(b) (i) $OP = OQ$ (both radii equal to 1)

$\therefore$ OPQ is isosceles

$\therefore \angle OPQ = \beta$ (base $\angle$s of isosceles Δ are equal)

$\therefore \alpha = \angle OQP + \angle OPQ$ (exterior $\angle$ equals sum of interior opposite $\angle$s)
$= \beta + \beta$
$= 2\beta$.

(ii) PQ has slope $m = \tan\beta$ and goes through $Q(-1, 0)$.

$\therefore \quad y - 0 = m[x - (-1)]$
$y = m(x + 1)$
$y = mx + m$.

(iii) x coordinates of P and Q are the simultaneous solutions of

$$\left.\begin{aligned} y &= mx + m \\ x^2 + y^2 &= 1 \end{aligned}\right\}.$$

$\therefore \quad x^2 + [m(x+1)]^2 = 1$ (by substitution)

$\therefore \quad x^2 + m^2\left(x^2 + 2x + 1\right) = 1$

$\therefore \quad x^2 + m^2x^2 + 2m^2x + m^2 - 1 = 0$

$\therefore \quad \left(1 + m^2\right)x^2 + 2m^2x + m^2 - 1 = 0$.

(iv) Equation in (iii) is a quadratic equation. x coordinates of $P(x_P)$ and $Q(-1)$ must multiply to give $\frac{c}{a}$, where $c = m^2 - 1$ and $a = 1 + m^2$.

$\therefore \quad -1 \times x_P = \frac{m^2 - 1}{1 + m^2}$

$\therefore \quad x_P = \frac{1 - m^2}{1 + m^2}$

$$y_P = mx_P + m$$
$$= m\left(\frac{1 - m^2}{1 + m^2} + 1\right)$$
$$= m\left(\frac{1 - m^2 + 1 + m^2}{1 + m^2}\right)$$
$$= \frac{2m}{1 + m^2}.$$

$\therefore \quad P = \left(\frac{1 - m^2}{1 + m^2}, \frac{2m}{1 + m^2}\right)$.

OR

$$x = \frac{-2m^2 \pm \sqrt{4m^4 - 4(m^2 + 1)(m^2 - 1)}}{2(m^2 + 1)}$$

$$= \frac{-2m^2 \pm \sqrt{4m^4 - (4m^4 - 4)}}{2(m^2 + 1)}$$

$$= \frac{-2m^2 \pm 2}{2(m^2 + 1)}$$

$$= \frac{-m^2 \pm 1}{m^2 + 1}$$

$$= \frac{-(m^2 - 1)}{m^2 + 1} \text{ or } \frac{-(m^2 + 1)}{m^2 + 1}$$

$$= \frac{-(m^2 - 1)}{m^2 + 1} \text{ or } -1.$$

Reject -1 as it is the x coordinate of Q.

$$\therefore \quad P=\left[\frac{1-m^2}{1+m^2},\ m\left(\frac{1-m^2}{1+m^2}\right)+m\right]$$

$$=\left[\frac{1-m^2}{1+m^2},\ \frac{m-m^3+m+m^3}{1+m^2}\right]$$

$$=\left[\frac{1-m^2}{1+m^2},\ \frac{2m}{1+m^2}\right].$$

(v) $\tan 2\beta = \tan\alpha$ (from (i))

$$=\frac{y_P}{x_P}$$

$$=\frac{2m}{1+m^2}\div\frac{1-m^2}{1+m^2}$$

$$=\frac{2m}{1-m^2}$$

$$=\frac{2\tan\beta}{1-\tan^2\beta},\quad \text{since } m=\tan\beta.$$

END OF 2/3 UNIT (COMMON) MATHEMATICS SOLUTIONS