BOARD OF STUDIES

NEW SOUTH WALES

2011

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–10
- All questions are of equal value

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Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use the Question 1 Writing Booklet.

(a) Evaluate $\sqrt[3]{\frac{651}{4\pi}}$ correct to four significant figures. **2**

(b) Simplify $\frac{n^2 - 25}{n - 5}$. **1**

(c) Solve $2^{2x+1} = 32$. **2**

(d) Differentiate $\ln(5x + 2)$ with respect to x. **2**

(e) Solve $2 - 3x \le 8$. **2**

(f) Rationalise the denominator of $\frac{4}{\sqrt{5} - \sqrt{3}}$. **2**

Give your answer in the simplest form.

(g) A batch of 800 items is examined. The probability that an item from this batch is defective is 0.02. **1**

How many items from this batch are defective?

Question 2 (12 marks) Use the Question 2 Writing Booklet.

(a) The quadratic equation $x^2 - 6x + 2 = 0$ has roots α and β.

(i) Find $\alpha + \beta$. **1**

(ii) Find $\alpha\beta$. **1**

(iii) Find $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$. **1**

(b) Find the exact values of x such that $2\sin x = -\sqrt{3}$, where $0 \le x \le 2\pi$. **2**

(c) Find the equation of the tangent to the curve $y = (2x+1)^4$ at the point where $x = -1$. **3**

(d) Find the derivative of $y = x^2 e^x$ with respect to x. **2**

(e) Find $\displaystyle\int \frac{1}{3x^2}\,dx$. **2**

Question 3 (12 marks) Use the Question 3 Writing Booklet.

(a) A skyscraper of 110 floors is to be built. The first floor to be built will cost \$3 million. The cost of building each subsequent floor will be \$0.5 million more than the floor immediately below.

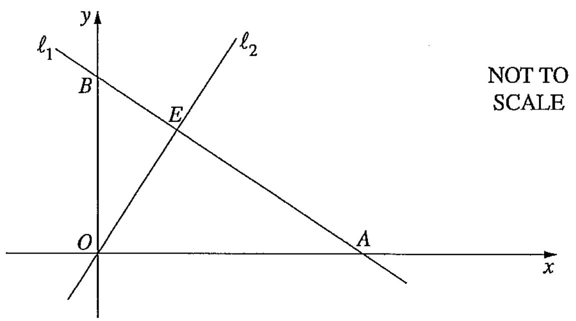
(i) What will be the cost of building the 25th floor? **2**

(ii) What will be the cost of building all 110 floors of the skyscraper? **2**

(b) A parabola has focus $(3, 2)$ and directrix $y = -4$. Find the coordinates of the vertex. **2**

(c) The diagram shows a line ℓ_1, with equation $3x + 4y - 12 = 0$, which intersects the y-axis at B.

A second line ℓ_2, with equation $4x - 3y = 0$, passes through the origin O and intersects ℓ_1 at E.



(i) Show that the coordinates of B are $(0, 3)$. **1**

(ii) Show that ℓ_1 is perpendicular to ℓ_2. **2**

(iii) Show that the perpendicular distance from O to ℓ_1 is $\dfrac{12}{5}$. **1**

(iv) Using Pythagoras' theorem, or otherwise, find the length of the interval BE. **1**

(v) Hence, or otherwise, find the area of $\triangle BOE$. **1**

Question 4 (12 marks) Use the Question 4 Writing Booklet.

(a) Differentiate $\dfrac{x}{\sin x}$ with respect to x. **2**

(b) Evaluate $\displaystyle\int_{e}^{e^3} \frac{5}{x}\,dx$. **2**

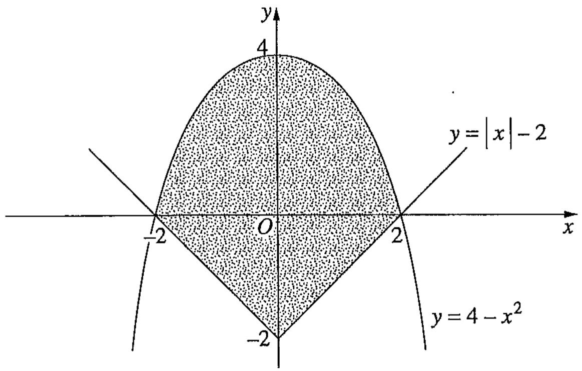
(c) The gradient of a curve is given by $\dfrac{dy}{dx} = 6x - 2$. The curve passes through the point $(-1, 4)$. **2**

What is the equation of the curve?

(d) (i) Differentiate $y = \sqrt{9 - x^2}$ with respect to x. **2**

(ii) Hence, or otherwise, find $\displaystyle\int \frac{6x}{\sqrt{9 - x^2}}\,dx$. **2**

(e) The diagram shows the graphs $y = |x| - 2$ and $y = 4 - x^2$. **2**



NOT TO SCALE

Write down inequalities that together describe the shaded region.

Question 5 (12 marks) Use the Question 5 Writing Booklet.

(a) The number of members of a new social networking site doubles every day. On Day 1 there were 27 members and on Day 2 there were 54 members.

(i) How many members were there on Day 12? **1**

(ii) On which day was the number of members first greater than 10 million? **2**

(iii) The site earns 0.5 cents per member per day. How much money did the site earn in the first 12 days? Give your answer to the nearest dollar. **2**

(b) Kim has three red shirts and two yellow shirts. On each of the three days, Monday, Tuesday and Wednesday, she selects one shirt at random to wear. Kim wears each shirt that she selects only once.

(i) What is the probability that Kim wears a red shirt on Monday? **1**

(ii) What is the probability that Kim wears a shirt of the same colour on all three days? **1**

(iii) What is the probability that Kim does not wear a shirt of the same colour on consecutive days? **2**

(c) The table gives the speed v of a jogger at time t in minutes over a 20-minute period. The speed v is measured in metres per minute, in intervals of 5 minutes. **3**

t	0	5	10	15	20
v	173	81	127	195	168

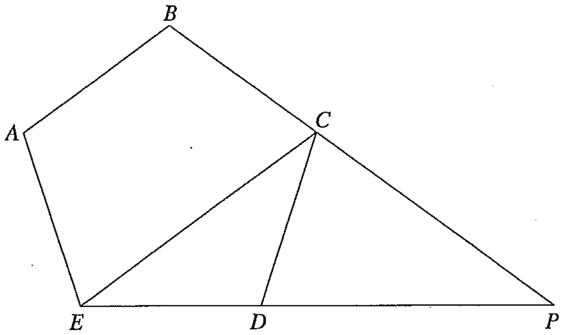
The distance covered by the jogger over the 20-minute period is given by

$$\int_0^{20} v\,dt.$$

Use Simpson's rule and the speed at each of the five time values to find the approximate distance the jogger covers in the 20-minute period.

Question 6 (12 marks) Use the Question 6 Writing Booklet.

(a) The diagram shows a regular pentagon $ABCDE$. Sides ED and BC are produced to meet at P.



Copy or trace the diagram into your writing booklet.

(i) Find the size of $\angle CDE$. **1**

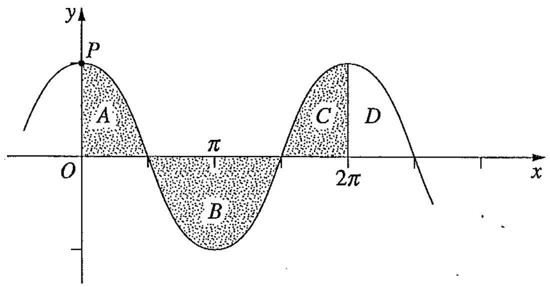
(ii) Hence, show that $\triangle EPC$ is isosceles. **2**

(b) A point $P(x, y)$ moves so that the sum of the squares of its distance from each of the points $A(-1, 0)$ and $B(3, 0)$ is equal to 40. **3**

Show that the locus of $P(x, y)$ is a circle, and state its radius and centre.

Question 6 continues on page 8

(c) The diagram shows the graph $y = 2\cos x$.



(i) State the coordinates of P. **1**

(ii) Evaluate the integral $\int_0^{\frac{\pi}{2}} 2\cos x \, dx$. **2**

(iii) Indicate which area in the diagram, A, B, C or D, is represented by the integral $\int_{\frac{3\pi}{2}}^{2\pi} 2\cos x \, dx$. **1**

(iv) Using parts (ii) and (iii), or otherwise, find the area of the region bounded by the curve $y = 2\cos x$ and the x-axis, between $x = 0$ and $x = 2\pi$. **1**

(v) Using the parts above, write down the value of $\int_{\frac{\pi}{2}}^{2\pi} 2\cos x \, dx$. **1**

End of Question 6

Question 7 (12 marks) Use the Question 7 Writing Booklet.

(a) Let $f(x) = x^3 - 3x + 2$.

(i) Find the coordinates of the stationary points of $y = f(x)$, and determine their nature. **3**

(ii) Hence, sketch the graph $y = f(x)$ showing all stationary points and the y-intercept. **2**

(b) The velocity of a particle moving along the x-axis is given by

$$\dot{x} = 8 - 8e^{-2t},$$

where t is the time in seconds and x is the displacement in metres.

(i) Show that the particle is initially at rest. **1**

(ii) Show that the acceleration of the particle is always positive. **1**

(iii) Explain why the particle is moving in the positive direction for all $t > 0$. **2**

(iv) As $t \to \infty$, the velocity of the particle approaches a constant. **1**

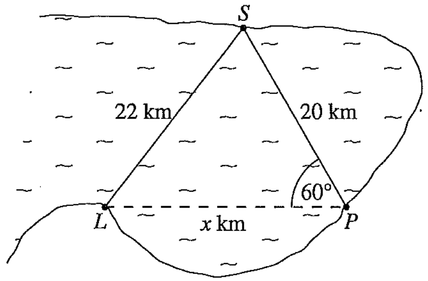
Find the value of this constant.

(v) Sketch the graph of the particle's velocity as a function of time. **2**

Question 8 (12 marks) Use the Question 8 Writing Booklet.

(a) In the diagram, the shop at S is 20 kilometres across the bay from the post office at P. The distance from the shop to the lighthouse at L is 22 kilometres and $\angle SPL$ is 60°.

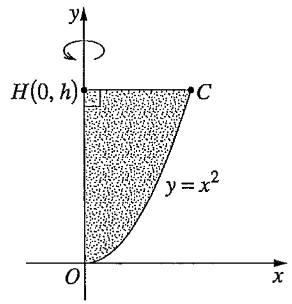
Let the distance PL be x kilometres.



(i) Use the cosine rule to show that $x^2 - 20x - 84 = 0$. **1**

(ii) Hence, find the distance from the post office to the lighthouse. Give your answer correct to the nearest kilometre. **2**

(b) The diagram shows the region enclosed by the parabola $y = x^2$, the y-axis and the line $y = h$, where $h > 0$. This region is rotated about the y-axis to form a solid called a paraboloid. The point C is the intersection of $y = x^2$ and $y = h$. The point H has coordinates $(0, h)$.



(i) Find the exact volume of the paraboloid in terms of h. **2**

(ii) A cylinder has radius HC and height h. **1**

What is the ratio of the volume of the paraboloid to the volume of the cylinder?

Question 8 continues on page 11

Question 8 (continued)

(c) When Jules started working she began paying \$100 at the beginning of each month into a superannuation fund.

The contributions are compounded monthly at an interest rate of 6% per annum.

She intends to retire after having worked for 35 years.

(i) Let \$$P$ be the final value of Jules's superannuation when she retires after 35 years (420 months). **2**

Show that $\$P = \$143\,183$ to the nearest dollar.

(ii) Fifteen years after she started working Jules read a magazine article about retirement, and realised that she would need \$800 000 in her fund when she retires. At the time of reading the magazine article she had \$29 227 in her fund. For the remaining 20 years she intends to work, she decides to pay a total of \$$M$ into her fund at the beginning of each month. The contributions continue to attract the same interest rate of 6% per annum, compounded monthly.

At the end of n months after starting the new contributions, the amount in the fund is \$$A_n$.

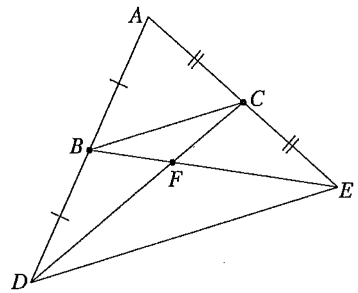
(1) Show that $A_2 = 29\,227 \times 1.005^2 + M(1.005 + 1.005^2)$. **1**

(2) Find the value of M so that Jules will have \$800 000 in her fund after the remaining 20 years (240 months). **3**

End of Question 8

Question 9 (12 marks) Use the Question 9 Writing Booklet.

(a) The diagram shows $\triangle ADE$, where B is the midpoint of AD and C is the midpoint of AE. The intervals BE and CD meet at F.



(i) Explain why $\triangle ABC$ is similar to $\triangle ADE$. **1**

(ii) Hence, or otherwise, prove that the ratio $BF : FE = 1:2$. **2**

(b) A tap releases liquid A into a tank at the rate of $\left(2 + \frac{t^2}{t+1}\right)$ litres per minute, where t is time in minutes. A second tap releases liquid B into the same tank at the rate of $\left(1 + \frac{1}{t+1}\right)$ litres per minute. The taps are opened at the same time and release the liquids into an empty tank.

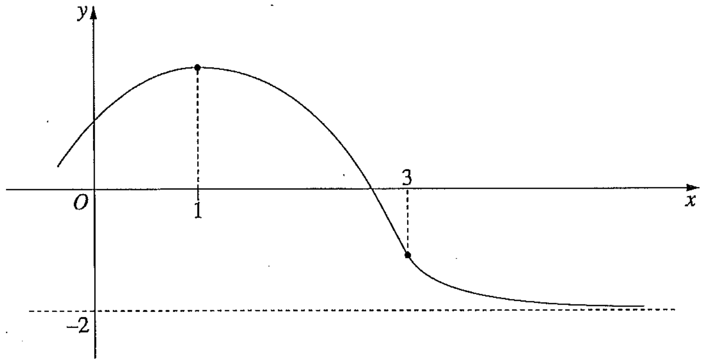
(i) Show that the rate of flow of liquid A is greater than the rate of flow of liquid B by t litres per minute. **1**

(ii) The taps are closed after 4 minutes. By how many litres is the volume of liquid A greater than the volume of liquid B in the tank when the taps are closed? **2**

Question 9 continues on page 13

Question 9 (continued)

(c) The graph $y = f(x)$ in the diagram has a stationary point when $x = 1$, a point of inflexion when $x = 3$, and a horizontal asymptote $y = -2$. **3**



Sketch the graph $y = f'(x)$, clearly indicating its features at $x = 1$ and at $x = 3$, and the shape of the graph as $x \to \infty$.

(d) (i) Rationalise the denominator in the expression **1**

$$\frac{1}{\sqrt{n}+\sqrt{n+1}},$$

where n is an integer and $n \geq 1$.

(ii) Using your result from part (i), or otherwise, find the value of the sum **2**

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{99}+\sqrt{100}}.$$

End of Question 9

Question 10 (12 marks) Use the Question 10 Writing Booklet.

(a) The intensity I, measured in watt/m^2, of a sound is given by

$$I = 10^{-12} \times e^{0.1L},$$

where L is the loudness of the sound in decibels.

(i) If the loudness of a sound at a concert is 110 decibels, find the intensity of the sound. Give your answer in scientific notation. **1**

(ii) Ear damage occurs if the intensity of a sound is greater than 8.1×10^{-9} watt/m^2. **2**

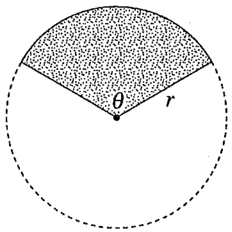
What is the maximum loudness of a sound so that no ear damage occurs?

(iii) By how much will the loudness of a sound have increased if its intensity has doubled? **2**

Question 10 continues on page 15

Question 10 (continued)

(b) A farmer is fencing a paddock using P metres of fencing. The paddock is to be in the shape of a sector of a circle with radius r and sector angle θ in radians, as shown in the diagram.



(i) Show that the length of fencing required to fence the perimeter of the paddock is $P = r(\theta + 2)$. **1**

(ii) Show that the area of the sector is $A = \frac{1}{2}Pr - r^2$. **1**

(iii) Find the radius of the sector, in terms of P, that will maximise the area of the paddock. **2**

(iv) Find the angle θ that gives the maximum area of the paddock. **1**

(v) Explain why it is only possible to construct a paddock in the shape of a sector if $\frac{P}{2(\pi + 1)} < r < \frac{P}{2}$. **2**

End of paper

2011 Higher School Certificate Solutions Mathematics

Question 1

(a) $\sqrt[3]{\dfrac{651}{4\pi}} = 3.72783809$

$= 3.728 \quad (4\text{ s.f.})$

(b) $\dfrac{n^2-25}{n-5} = \dfrac{(n-5)(n+5)}{(n-5)}$

$=(n+5).$

(c) $2^{2x+1} = 32$

$2^{2x+1} = 2^5$

$2x+1 = 5$ (comparing indices)

$2x = 4$

$x = 2.$

(d) $\dfrac{d}{dx}\ln(5x+2) = \dfrac{5}{5x+2}.$

(e) $2 - 3x \le 8$

$-3x \le 6$

$x \ge \dfrac{6}{-3}$

$x \ge -2.$

(f) $\dfrac{4}{\sqrt5-\sqrt3} \times \dfrac{\sqrt5+\sqrt3}{\sqrt5+\sqrt3} = \dfrac{4(\sqrt5+\sqrt3)}{5-3}$

$= \dfrac{4(\sqrt5+\sqrt3)}{2}.$

$= 2(\sqrt5+\sqrt3)$

(g) $800 \times 0.02 = 16$

There are 16 items that are defective.

Question 2

(a) (i) $\alpha+\beta = \dfrac{-(-6)}{1} = 6.$

(ii) $\alpha\beta = \dfrac{2}{1} = 2.$

(iii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta} = \dfrac{\alpha+\beta}{\alpha\beta}$

$= \dfrac{6}{2}$

$= 3.$

(b) $2\sin x = -\sqrt3$

$\sin x = -\dfrac{\sqrt3}{2}$

$\therefore$ 3rd and 4th quadrants

$x = \pi + \dfrac{\pi}{3},\ 2\pi - \dfrac{\pi}{3}$

$x = \dfrac{4\pi}{3},\ \dfrac{5\pi}{3}.$

(c) $y = (2x+1)^4$

$y' = 4(2x+1)^3 \times 2$

$= 8(2x+1)^3$

when $x = -1$:

$m = 8(2[-1]+1)^3 = 8(-1)^3 = -8$

$y = [2(-1)+1]^4 = [-1]^4 = 1$

Equation of the tangent:

$y-1 = -8(x-(-1))$

$y-1 = -8x-8$

$8x+y+7=0.$

(d) $y = x^2e^x$

$u = x^2 \qquad v = e^x$

$u' = 2x \qquad v' = e^x$

$\dfrac{dy}{dx} = 2xe^x + x^2e^x$

$= xe^x(x+2).$

(e) $\displaystyle\int \frac{1}{3x^2}\,dx = \frac{1}{3}\int x^{-2}\,dx$

$= \dfrac{1}{3} \times \dfrac{x^{-1}}{-1} + c$

$= -\dfrac{1}{3x} + c.$

Question 3

(a) (i) The total cost is $3 + 3.5 + 4 + \dots$ which is an arithmetic series with

$a=3,\quad d=0.5,\quad n=25$

$T_n = a+(n-1)d$

$T_{25} = 3+(25-1)\times0.5$

$=15$

$\therefore$ The cost of building the 25th floor is \$15 million.

(ii) $a=3,\quad d=0.5,\quad n=110$

$S_n = \dfrac{n}{2}[2a+(n-1)d]$

$S_{110} = \dfrac{110}{2}[2(3)+(110-1)\times0.5]$

$= 55[6+(109)\times0.5]$

$= 3327.5$

$\therefore$ It will cost \$3 327.5 million to build a 110 floor skyscraper.

(b) The vertex is the midpoint of $(3,2)$ and $(3,-4)$.

vertex $= \left(\dfrac{3+3}{2}, \dfrac{2+-4}{2}\right)$

$= (3,-1).$

(c) (i) At B, $x=0$:

$3(0)+4y-12=0$

$4y-12=0$

$4y=12$

$y=3$

$\therefore$ The coordinates of B are $(0,3)$.

(ii) For ℓ_1: $3x+4y-12=0$

$4y = -3x+12,$

$y = -\dfrac{3}{4}x+3,$

so $m_1 = -\dfrac{3}{4}.$

For ℓ_2: $4x-3y=0$

$-3y=-4x,$

$y = \dfrac{4}{3}x,$

so $m_2 = \dfrac{4}{3}.$

then $m_1 \times m_2 = -\dfrac{3}{4}\times\dfrac{4}{3}$

$=-1$

$\therefore \ell_1$ is perpendicular to ℓ_2.

(iii) $d = \dfrac{|3(0)+4(0)-12|}{\sqrt{3^2+4^2}}$

$= \dfrac{12}{5}$ units.

(iv) $OB=3,\quad OE=\dfrac{12}{5}$

$(BE)^2 = 3^2 - \left(\dfrac{12}{5}\right)^2$

$= 9 - \dfrac{144}{25}$

$= \dfrac{81}{25}$

$BE = \dfrac{9}{5}$ units.

(v) $\text{Area} = \frac{1}{2} \times \frac{12}{5} \times \frac{9}{5}$

$= \frac{54}{25}$

$= 2\frac{4}{25}$ units^2.

Question 4

(a) Let $u = x$, $v = \sin x$,

then $u' = 1$, $v' = \cos x$.

$$\frac{d}{dx}\frac{x}{\sin x} = \frac{1.\sin x - x\cos x}{\sin^2 x}$$

$$= \frac{\sin x - x\cos x}{\sin^2 x}.$$

(b)

$$\int_e^{e^3} \frac{5}{x}\,dx = 5\left[\ln x\right]_e^{e^3}$$

$= 5\left[\ln e^3 - \ln e\right]$

$= 5[3-1]$

$= 10.$

(c) Given $\frac{dy}{dx} = 6x - 2$,

$$y = \frac{6x^2}{2} - 2x + c.$$

When $x = -1$ and $y = 4$:

$$4 = \frac{6(-1)^2}{2} - 2(-1) + c$$

$4 = 3 + 2 + c$

$c = -1.$

The equation of the curve is

$y = 3x^2 - 2x - 1.$

(d) (i) $y = \sqrt{9 - x^2}$

$y = \left(9 - x^2\right)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}\left(9 - x^2\right)^{-\frac{1}{2}} \times -2x$$

$$= \frac{-x}{\sqrt{9 - x^2}}.$$

(ii) Using the result from part (i)

$$\int \frac{6x}{\sqrt{9-x^2}}\,dx = -6\int \frac{-x}{\sqrt{9-x^2}}\,dx$$

$$= -6\sqrt{9 - x^2} + c.$$

(e) $y = |x| - 2$ and $y = 4 - x^2$

Test at $(0,0)$

$0 > |0| - 2$ and $0 < 4 - 0^2$

True True

The inequalities that describe the shaded region are:

$y \geq |x| - 2$ and $y \leq 4 - x^2$.

Question 5

(a) (i) The number of members form a geometric sequence with

$a = 27$, $r = 2$, $n = 12$ (on day 12)

$T_n = ar^{n-1}$

$T_{12} = 27 \times 2^{11}$

$= 55\,296$ members.

(ii) $T_n > 10\,000\,000$

$27 \times 2^{n-1} > 10^7$

$$2^{n-1} > \frac{10^7}{27}$$

$$(n-1)\ln 2 > \ln\left(\frac{10^7}{27}\right)$$

$$(n-1) > \ln\left(\frac{10^7}{27}\right) \div \ln 2$$

$n - 1 > 18.49860916$

$n > 19.49860916$

so $n = 20$ as n is a whole number.

$\therefore$ It happens on the 20$^{\text{th}}$ day.

(iii) Geometric series with:

$a = 27$, $r = 2$, $n = 12$

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$S_{12} = \frac{27\left(2^{12} - 1\right)}{2 - 1}$$

$= 110\,565$

Site earns $= 0.5 \times 110\,565$

$= 55\,282.5$ cents

$= \$552.83$

$= \$553$ (nearest dollar).

(b) (i) $P(R) = \frac{3}{5}$.

(ii) Since there are only 3 reds:

$$P(RRR) = \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3}$$

$$= \frac{1}{10}.$$

(iii) $$P(RYR \text{ or } YRY) = \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} + \frac{2}{5} \times \frac{3}{4} \times \frac{1}{3}$$

$$= \frac{1}{5} + \frac{1}{10}$$

$$= \frac{3}{10}.$$

(c) $$D = \frac{5}{3}\left[173 + 168 + 4 \times (81 + 195) + 2 \times 127\right]$$

$= 2831.\dot{6}$

$= 2832$ m (nearest metre)

$\therefore$ the distance is 2832 m (to the nearest metre).

Question 6

(a) (i) Angle sum of a pentagon is 540°.

Since it is a regular pentagon:

$$\angle CDE = \frac{540}{5} = 108^\circ$$

(ii) $DE = DC$ (sides of regular pentagon)

$\therefore \Delta DEC$ is isosceles

$\angle DEC = \angle ECD$ (base $\angle$s isosceles Δ)

$$\angle CED = \frac{180 - 108}{2} = 36^\circ \text{ (}\angle\text{ sum of }\Delta\text{)}$$

In quadrilateral $ABPE$:

$\angle ABC = \angle EAB = \angle DEA = 108^0$

(vertices of a regular pentagon)

$\angle BPE = 360^\circ - 3 \times 108^\circ = 36^\circ$

(Angle sum of quadrilateral)

$\therefore$ ∠CED = ∠BPE = 36^0

$\therefore$ ΔEPC is isosceles (2 equal angles).

(b) $AP = \sqrt{(-1-x)^2 + (0-y)^2}$

$AP^2 = 1 + 2x + x^2 + y^2$

$BP = \sqrt{(3-x)^2 + (0-y)^2}$

$BP^2 = 9 - 6x + x^2 + y^2$

But $AP^2 + BP^2 = 40$

so $1 + 2x + x^2 + y^2 + 9 - 6x + x^2 + y^2 = 40$

$2x^2 - 4x + 2y^2 + 10 = 40$

$x^2 - 2x + y^2 + 5 = 20$

$x^2 - 2x + 1 + y^2 = 16$

$(x-1)^2 + (y-0)^2 = 4^2$

This represents a circle with radius 4 and centre $(1, 0)$.

(c) (i) When $x = 0$, $y = 2\cos(0) = 2$.

The coordinates of P are $(0,2)$.

(ii) $$\int_0^{\frac{\pi}{2}} 2\cos x\,dx = 2\left[\sin x\right]_0^{\frac{\pi}{2}}$$

$= 2(1 - 0)$

$= 2.$

(iii) From the limits of integration it must be C.

(iv) Each section has the same area as (ii).

Area $= 2 + 2 \times 2 + 2$

$= 8$ units^2.

(v) $$\int_{\frac{\pi}{2}}^{2\pi} 2\cos x\,dx = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\cos x\,dx + \int_{\frac{3\pi}{2}}^{2\pi} 2\cos x\,dx$$

$= -4 + 2$

$= -2.$

Question 7

(a) (i) $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$

$f''(x) = 6x$

For $f'(x)=0$:

$$3x^2-3x=0$$
$$3(x+1)(x-1)=0$$
$$x=-1,\quad 1$$

When $x=-1$:

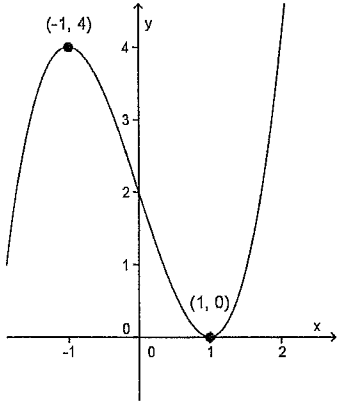
$$f(-1)=(-1)^3-3(-1)+2$$
$$=4$$
$$f''(-1)=6(-1)=-6$$
$$<0\quad\therefore\text{max.}$$

When $x=1$:

$$f(1)=(1)^3-3(1)+2$$
$$=0$$
$$f''(1)=6(1)=6$$
$$>0\quad\therefore\text{min.}$$

Thus $(-1,4)$ is a maximum turning point and $(1,0)$ is a minimum turning point.

(ii) At $x=0$, $f(0)=0-0+2=2$, so the y-intercept is 2.



(b) (i) When $t=0$:

$$\dot{x}=8-8e^0$$
$$=8-8$$
$$=0$$

$\therefore$ The particle is initially at rest.

(ii)
$$\ddot{x}=-8e^{-2t}\times-2$$
$$=16e^{-2t}$$

$e^{-2t}>0$ for all t

$\therefore 16e^{-2t}$ is always positive.

(iii) The particle is initially at rest [from (i)] and its acceleration is always positive [from (ii)] for all $t>0$, so it must always be moving in a positive direction.

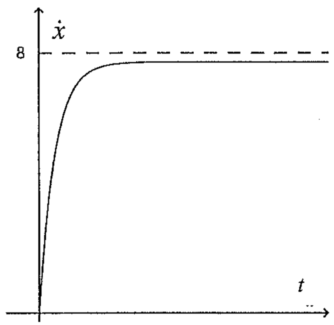
(iv)
$$\dot{x}=8-8e^{-2t}$$

$e^{-2t}\to 0$ as $t\to\infty$

$$\dot{x}\to 8-8(0)$$
$$\to 8$$

$\therefore$ The velocity approaches the constant value of 8 ms^{-1}.

(v)



Question 8

(a) (i) In ΔPLS,

$$p^2=x^2+l^2-2xl\cos\angle P$$
$$22^2=x^2+20^2-2\times x\times 20\cos 60°$$
$$484=x^2+400-40x\times\frac{1}{2}$$
$$84=x^2-20x$$
$$0=x^2-20x-84$$

$\therefore x^2-20x-84=0$ as required.

(a) (ii)
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{20\pm\sqrt{400-4\times1\times-84}}{2\times1}$$
$$=\frac{20\pm\sqrt{736}}{2}$$
$$=\frac{20\pm 27.1293...}{2}\quad\text{but } x>0$$
$$=\frac{20+27.1293...}{2}$$
$$=23.5646...$$

$\therefore$ The distance is 24 km.

(b) (i)
$$V=\int_0^h \pi x^2\,dy$$
$$=\pi\int_0^h y\,dy$$
$$=\pi\left[\frac{y^2}{2}\right]_0^h$$
$$=\frac{\pi h^2}{2}.$$

(ii) Let the cylinder have radius r.
Since (r, h) is on the parabola:

$$h=r^2$$
$$V_{\text{cylinder}}=\pi r^2 h$$
$$=\pi(h)h$$
$$=\pi h^2$$

$$\text{Ratio of volumes}=\frac{\pi h^2}{2}:\pi h^2$$
$$=\frac{1}{2}:1$$
$$=1:2.$$

(c) (i)
$$r=6\%\text{ p.a.}$$
$$=0.5\%\text{ per month}$$

Final value of each payment:

1^{st} payment $=100(1+0.005)^{420}$

2^{nd} payment $=100(1.005)^{419}$

3^{rd} payment $=100(1.005)^{418}$

$\vdots$

Last payment $=100(1.005)$

$$\text{Total}=100(1.005)+100(1.05)^2+...+100(1.005)^{420}$$
$$=100\left[1.005+1.005^2+...+1.005^{419}+1.005^{420}\right]$$

This is a geometric series with

$a=1.005,\ r=1.005$

$$\text{Total}=100\times\frac{1.005\left[1.005^{420}-1\right]}{(1.005-1)}$$
$$=\frac{100\times1.005}{0.005}\times\left[1.005^{420}-1\right]$$
$$=20100[7.12355...]$$
$$=143183.39$$

$\therefore P=\$143\ 183$ to the nearest dollar.

(ii) (1)
$$A_1=29227\times1.005+M\times1.005$$
$$A_2=A_1\times1.005+M\times1.005$$
$$A_2=\left[29227\times1.005+M\times1.005\right]\times1.005+M\times1.005$$
$$=29227\times1.005^2+M\times1.005^2+M\times1.005$$
$$=29227\times1.005^2+M(1.005+1.005^2)$$

(ii) (2)

Similarly

$$A_n=29227\times1.005^n+M(1.005+1.005^2+....+1.005^n)$$
$$A_n=29227\times1.005^n+M(1.005)\frac{\left[1.005^n-1\right]}{(1.005-1)}$$

when $n=420,\ A_{420}=80000$

$$80000=29227\times1.005^{240}+\frac{M(1.005)\left[1.005^{240}-1\right]}{0.005}$$
$$M\times1.005\times\frac{(1.005^{240}-1)}{0.005}=80000-29227\times1.005^{240}$$
$$=703252.6538$$
$$M=\frac{703252.65\times0.005}{1.005(1.005^{240}-1)}$$
$$=\$1514.48$$

$\therefore \$1514.48$ is the required value.

Question 9

(a) (i) In ΔABC and ΔADE

$\angle BAC$ is common

$$\frac{AB}{AD}=\frac{AC}{AE}=\frac{1}{2}$$

$AB=BD$ (given) and $AC = CE$ (given)

Two pairs of corresponding sides are proportional and the included angles are equal.

$\therefore$ ΔABC and ΔADE are similar.

(ii) $\dfrac{AB}{BD}=\dfrac{AC}{CE}$ (from (i))

$\therefore BC \parallel DE$ (transversals in the same ratio)

In ΔBFC and ΔDFE,

$\angle BFC=\angle DFE$ (vertically opposite)

$\angle FBC=\angle FED$ (alternate angles)

$\angle FCB=\angle FDE$ (alternate angles)

$\therefore \Delta BFC \,|||\, \Delta EFD$ (equiangular)

$\frac{AB}{BD} = \frac{AC}{CE} = \frac{1}{2}$ (from (i))

$\therefore \frac{BC}{DE} = \frac{1}{2}$

This applies now to ΔBFC and ΔDFE

$\therefore BF : FE = 1:2$

(corresponding sides in similar triangles).

(b) (i)

$$\frac{dV_A}{dt} = 2 + \frac{t^2}{t+1}$$

$$\frac{dV_B}{dt} = 1 + \frac{1}{t+1}$$

$$\frac{dV_A}{dt} - \frac{dV_B}{dt} = 2 + \frac{t^2}{t+1} - (1 + \frac{1}{t+1})$$

$$= 2 + \frac{t^2}{t+1} - 1 - \frac{1}{t+1}$$

$$= 1 + \frac{t^2}{t+1} - \frac{1}{t+1}$$

$$= \frac{t+1+t^2-1}{t+1}$$

$$= \frac{t+t^2}{t+1}$$

$$= \frac{t(t+1)}{t+1}$$

$$= t \text{ L/min.}$$

(ii) $\frac{d}{dt}(V_A - V_B) = t$

$$V_A - V_B = \int_0^4 t\,dt$$

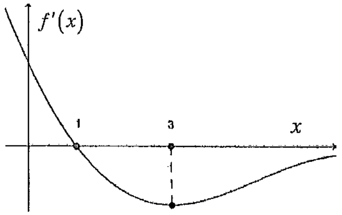
$$= \left[\frac{t^2}{2}\right]_0^4$$

$$= \frac{4^2}{2} - 0$$

$$= 8$$

The difference in volume is 8 L.

(c)



(d) (i)

$$\frac{1}{\sqrt{n}+\sqrt{n+1}} \times \frac{\sqrt{n}-\sqrt{n+1}}{\sqrt{n}-\sqrt{n+1}} = \frac{\sqrt{n}-\sqrt{n+1}}{n-(n+1)}$$

$$= \frac{\sqrt{n}-\sqrt{n+1}}{n-n-1}$$

$$= \frac{\sqrt{n}-\sqrt{n+1}}{-1}$$

$$= -\sqrt{n} + \sqrt{n+1}$$

$$= \sqrt{n+1} - \sqrt{n}$$

(ii) $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \ldots + \frac{1}{\sqrt{99}+\sqrt{100}}$

using part (i)

$$\sqrt{2} - \sqrt{1} + \sqrt{3} - \sqrt{2} + \ldots + \sqrt{100} - \sqrt{99} = \sqrt{100} - \sqrt{1}$$

$$= 10 - 1$$

$$= 9.$$

Question 10

(a) (i) $I = 10^{-12} \times e^{0.1L}$

When $L = 110$:

$$I = 10^{-12} \times e^{0.1 \times 110}$$

$$= 10^{-12} \times e^{11}$$

$$= 10^{-12} \times 59874.14172$$

$$= 5.9874 \times 10^{-8}$$

$$= 5.99 \times 10^{-8}.$$

(ii) Damage occurs when $I > 8.1 \times 10^{-9}$

$$10^{-12} \times e^{0.1L} > 8.1 \times 10^{-9}$$

$$e^{0.1L} > 8.1 \times 10^3$$

$$0.1 \times L > \ln 8100$$

$$L > \frac{\ln 8100}{0.1}$$

$$L > 89.99$$

$\therefore$ The maximum safe loudness is 89 decibels (to the nearest whole number) so that no ear damage occurs.

(iii) Assume the original intensity is I

The new intensity, I_1 where $I_1 = 2I$

$$10^{-12} \times e^{0.1L_1} = 2 \times 10^{-12} \times e^{0.1L}$$

$$e^{0.1L_1} = 2 \times e^{0.1L}$$

$$\frac{e^{0.1L_1}}{e^{0.1L}} = 2$$

$$e^{0.1(L_1 - L)} = 2$$

$$0.1(L_1 - L) = \ln 2$$

$$(L_1 - L) = \frac{\ln 2}{0.1}$$

$$L_1 - L = 6.931471806$$

When the intensity is doubled, loudness increased by 7 decibels.

(b) (i) Arc length $= r\theta$

Perimeter of the paddock:

$$P = r\theta + r + r$$

$$= r\theta + 2r$$

$$= r(\theta + 2).$$

(ii) Area of a sector $= \frac{1}{2}r^2\theta$

From part (i)

$$P = r\theta + 2r$$

$$r\theta = P - 2r$$

$$\theta = \frac{P-2r}{r}$$

$$A = \frac{1}{2} \times r^2 \times \theta$$

$$= \frac{1}{2} \times r^2 \times \frac{P-2r}{r}$$

$$= \frac{1}{2} \times r(P-2r)$$

$$A = \frac{1}{2}Pr - r^2.$$

(iii) $A = \frac{1}{2}Pr - r^2$

$$\frac{dA}{dr} = \frac{1}{2}P - 2r$$

For $\frac{dA}{dr} = 0$:

$$\frac{1}{2}P - 2r = 0$$

$$2r = \frac{1}{2}P$$

$$r = \frac{1}{4}P$$

and $\frac{d^2A}{dr^2} = -2 < 0$

$\therefore$ Maximum area when $r = \frac{1}{4}P$.

(iv) $\theta = \frac{P-2r}{r}$ (from part (ii))

For maximum area, $r = \frac{P}{4}$

$$\theta = \frac{P-2r}{r}$$

$$= \frac{P - 2\left(\frac{P}{4}\right)}{\left(\frac{P}{4}\right)}$$

$$= \frac{\frac{P}{2}}{\frac{P}{4}}$$

$$= \frac{P}{2} \times \frac{4}{P}$$

$$= 2 \text{ radians.}$$

(v) Given $P = r\theta + 2r$

To have a sector, $0 < \theta < 2\pi$

For $\theta > 0$:

$P > 2r$, then $\frac{P}{2} > r$

For $\theta < 2\pi$:

$$P < 2r + 2\pi r$$

$$P < 2r(1+\pi)$$

$$\frac{P}{2(1+\pi)} < r$$

Combining these results

$$\frac{P}{2(1+\pi)} < r < \frac{P}{2}.$$

End of Mathematics solutions