BOARD OF STUDIES
NEW SOUTH WALES

2004

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–10
- All questions are of equal value

Total marks – 120
Attempt Questions 1–10
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) The radius of Mars is approximately 3 397 000 m. Write this number in scientific notation, correct to two significant figures. **2**

(b) Differentiate $x^4 + 5x^{-1}$ with respect to x. **2**

(c) Solve $\frac{x-5}{3} - \frac{x+1}{4} = 5$. **2**

(d) Find integers a and b such that $(3-\sqrt{2})^2 = a - b\sqrt{2}$. **2**

(e) A packet contains 12 red, 8 green, 7 yellow and 3 black jellybeans. One jellybean is selected from the packet at random. **2**

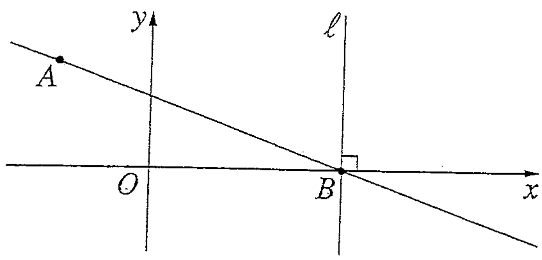
What is the probability that the selected jellybean is red or yellow?

(f) Find the values of x for which $|x+1| \leq 5$. **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the points $A(-1, 3)$ and $B(2, 0)$.
The line ℓ is drawn perpendicular to the x-axis through the point B.



NOT TO SCALE

(i) Calculate the length of the interval AB. **1**

(ii) Find the gradient of the line AB. **1**

(iii) What is the size of the acute angle between the line AB and the line ℓ? **1**

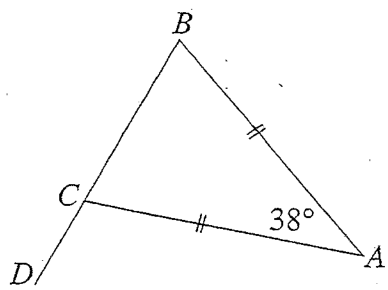
(iv) Show that the equation of the line AB is $x + y - 2 = 0$. **1**

(v) Copy the diagram into your writing booklet and shade the region defined by $x + y - 2 \leq 0$. **1**

(vi) Write down the equation of the line ℓ. **1**

(vii) The point C is on the line ℓ such that AC is perpendicular to AB. Find the coordinates of C. **2**

(b) **2**



NOT TO SCALE

In the diagram, ABC is an isosceles triangle with $AB = AC$ and $\angle BAC = 38°$.
The line BC is produced to D.

Copy or trace the diagram into your writing booklet.

Find the size of $\angle ACD$. Give reasons for your answer.

(c) For what values of k does $x^2 - kx + 4 = 0$ have no real roots? **2**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Differentiate with respect to x:

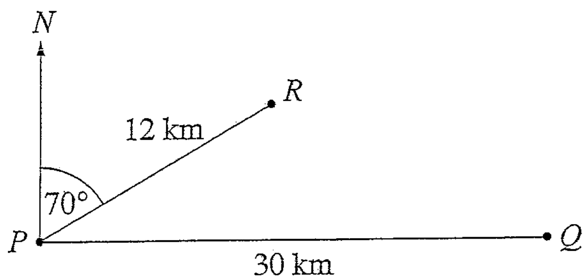
(i) $x^2 \log_e x$ — **2**

(ii) $(1+\sin x)^5$. — **2**

(b) (i) Evaluate $\int_1^2 e^{3x}\,dx$. — **2**

(ii) Find $\int \frac{x}{x^2-3}\,dx$. — **2**

(c)



NOT TO SCALE

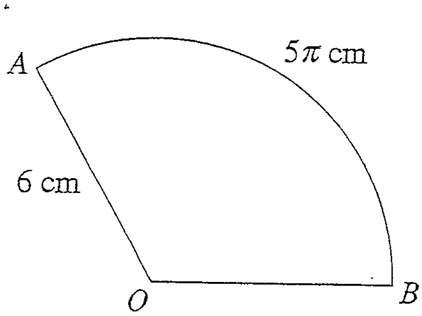
The diagram shows a point P which is 30 km due west of the point Q.
The point R is 12 km from P and has a bearing from P of 070°.

(i) Find the distance of R from Q. — **2**

(ii) Find the bearing of R from Q. — **2**

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) 2



AOB is a sector of a circle, centre O and radius 6 cm.
The length of the arc AB is 5π cm.

Calculate the exact area of the sector AOB.

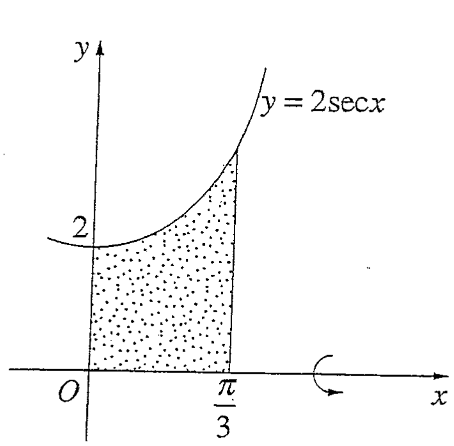
(b) Consider the function $f(x) = x^3 - 3x^2$.

(i) Find the coordinates of the stationary points of the curve $y = f(x)$ and determine their nature. 3

(ii) Sketch the curve showing where it meets the axes. 2

(iii) Find the values of x for which the curve $y = f(x)$ is concave up. 2

(c) 3



In the diagram, the shaded region is bounded by the curve $y = 2\sec x$, the coordinate axes and the line $x = \frac{\pi}{3}$. The shaded region is rotated about the x-axis.

Calculate the exact volume of the solid of revolution formed.

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Clare is learning to drive. Her first lesson is 30 minutes long. Her second lesson is 35 minutes long. Each subsequent lesson is 5 minutes longer than the lesson before.

(i) How long will Clare's twenty-first lesson be? **1**

(ii) How many hours of lessons will Clare have completed after her twenty-first lesson? **2**

(iii) During which lesson will Clare have completed a total of 50 hours of driving lessons? **2**

(b) A particle moves along a straight line so that its displacement, x metres, from a fixed point O is given by $x = 1 + 3\cos 2t$, where t is measured in seconds.

(i) What is the initial displacement of the particle? **1**

(ii) Sketch the graph of x as a function of t for $0 \le t \le \pi$. **2**

(iii) Hence, or otherwise, find when AND where the particle first comes to rest after $t = 0$. **2**

(iv) Find a time when the particle reaches its maximum speed. What is this speed? **2**

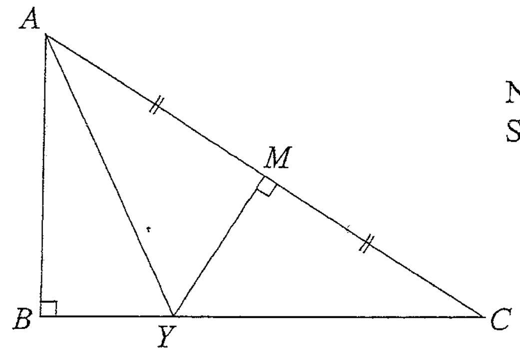
Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Solve the following equation for x: **2**

$$e^{2x} + 3e^{x} - 10 = 0.$$

(b)



The diagram shows a right-angled triangle ABC with $\angle ABC = 90°$. The point M is the midpoint of AC, and Y is the point where the perpendicular to AC at M meets BC.

(i) Show that $\Delta AYM \equiv \Delta CYM$. **2**

(ii) Suppose that it is also given that AY bisects $\angle BAC$. Find the size of $\angle YCM$ and hence find the exact ratio $MY : AC$. **3**

(c) In a game, a turn involves rolling two dice, each with faces marked 0, 1, 2, 3, 4 and 5. The score for each turn is calculated by multiplying the two numbers uppermost on the dice.

(i) What is the probability of scoring zero on the first turn? **2**

(ii) What is the probability of scoring 16 or more on the first turn? **1**

(iii) What is the probability that the sum of the scores in the first two turns is less than 45? **2**

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\sum_{n=2}^{4} n^2$. **1**

(b) At the beginning of 1991 Australia's population was 17 million. At the beginning of 2004 the population was 20 million.

Assume that the population P is increasing exponentially and satisfies an equation of the form $P = Ae^{kt}$, where A and k are constants, and t is measured in years from the beginning of 1991.

(i) Show that $P = Ae^{kt}$ satisfies $\frac{dP}{dt} = kP$. **1**

(ii) What is the value of A? **1**

(iii) Find the value of k. **2**

(iv) Predict the year during which Australia's population will reach 30 million. **2**

(c) Betty decides to set up a trust fund for her grandson, Luis. She invests \$80 at the beginning of each month. The money is invested at 6% per annum, compounded monthly.

The trust fund matures at the end of the month of her final investment, 25 years after her first investment. This means that Betty makes 300 monthly investments.

(i) After 25 years, what will be the value of the first \$80 invested? **2**

(ii) By writing a geometric series for the value of all Betty's investments, calculate the final value of Luis' trust fund. **3**

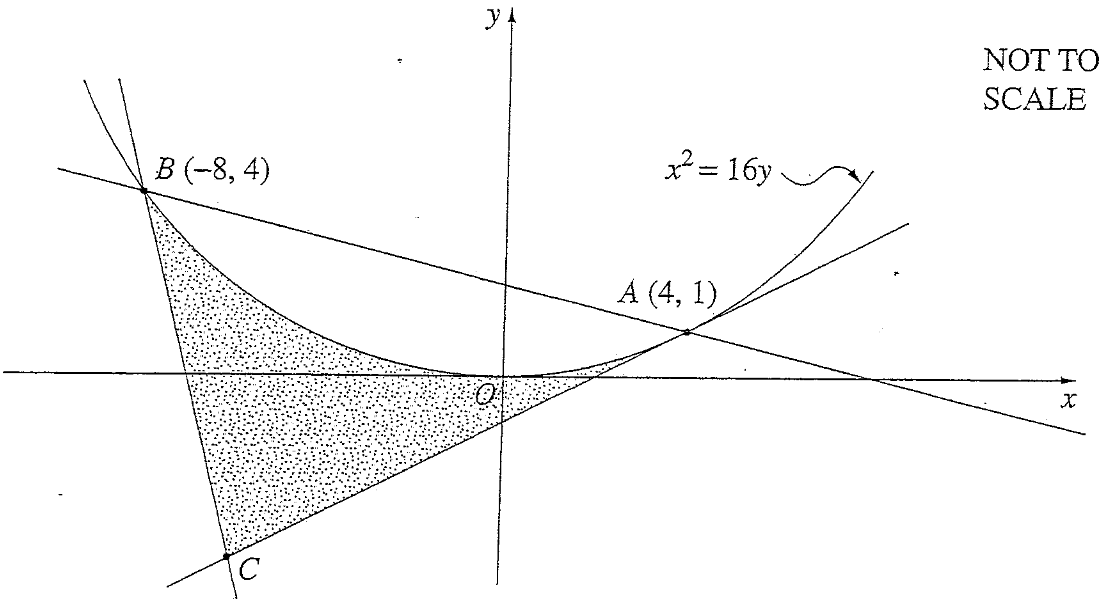
Marks

Question 8 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\cos\theta\tan\theta = \sin\theta$. **1**

(ii) Hence solve $8\sin\theta\cos\theta\tan\theta = \operatorname{cosec}\theta$ for $0 \le \theta \le 2\pi$. **2**

(b) The diagram shows the graph of the parabola $x^2 = 16y$. The points $A\,(4, 1)$ and $B\,(-8, 4)$ are on the parabola, and C is the point where the tangent to the parabola at A intersects the directrix.



(i) Write down the equation of the directrix of the parabola $x^2 = 16y$. **1**

(ii) Find the equation of the tangent to the parabola at the point A. **2**

(iii) Show that C is the point $(-6, -4)$. **1**

(iv) Given that the equation of the line AB is $y = 2 - \frac{x}{4}$, find the area bounded by the line AB and the parabola. **2**

(v) Hence, or otherwise, find the shaded area bounded by the parabola, the tangent at A and the line BC. **3**

Marks

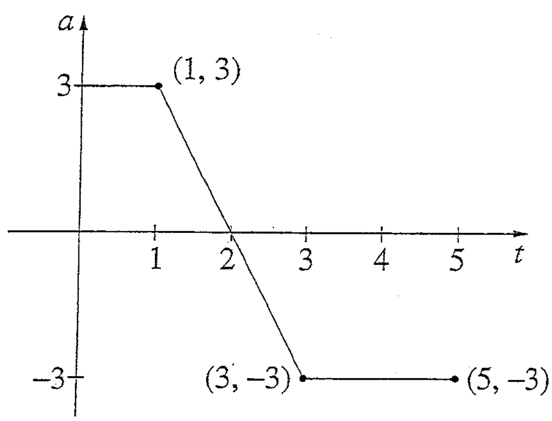
Question 9 (12 marks) Use a SEPARATE writing booklet.

(a) Consider the geometric series $1-\tan^2\theta+\tan^4\theta-\ldots$

(i) When the limiting sum exists, find its value in simplest form. **2**

(ii) For what values of θ in the interval $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ does the limiting sum of the series exist? **2**

(b) A particle moves along the x-axis. Initially it is at rest at the origin. The graph shows the acceleration, a, of the particle as a function of time t for $0\le t\le 5$.



(i) Write down the time at which the velocity of the particle is a maximum. **1**

(ii) At what time during the interval $0\le t\le 5$ is the particle furthest from the origin? Give brief reasons for your answer. **2**

(c) Consider the function $f(x)=\dfrac{\log_e x}{x}$, for $x>0$.

(i) Show that the graph of $y=f(x)$ has a stationary point at $x=e$. **2**

(ii) By considering the gradient on either side of $x=e$, or otherwise, show that the stationary point is a maximum. **1**

(iii) Use the fact that the maximum value of $f(x)$ occurs at $x=e$ to deduce that $e^x\ge x^e$ for all $x>0$. **2**

Marks

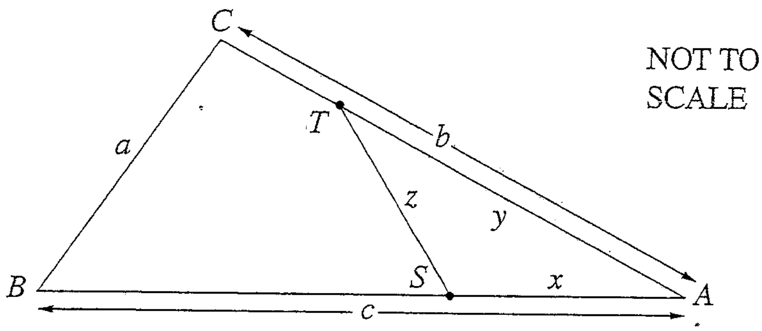
Question 10 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Use Simpson's rule with 3 function values to find an approximation to the area under the curve $y=\frac{1}{x}$ between $x=a$ and $x=3a$, where a is positive. **2**

(ii) Using the result in part (i), show that **1**

$$\ln 3 \doteqdot \frac{10}{9}.$$

(b)



The diagram shows a triangular piece of land ABC with dimensions $AB=c$ metres, $AC=b$ metres and $BC=a$ metres, where $a \le b \le c$.

The owner of the land wants to build a straight fence to divide the land into two pieces of equal area. Let S and T be points on AB and AC respectively so that ST divides the land into two pieces of equal area.

Let $AS=x$ metres, $AT=y$ metres and $ST=z$ metres.

(i) Show that $xy=\frac{1}{2}bc$. **1**

(ii) Use the cosine rule in triangle AST to show that **2**

$$z^2 = x^2 + \frac{b^2c^2}{4x^2} - bc\cos A.$$

(iii) Show that the value of z^2 in the equation in part (ii) is a minimum when $x=\sqrt{\frac{bc}{2}}$. **4**

(iv) Show that the minimum length of the fence is $\sqrt{\frac{(P-2b)(P-2c)}{2}}$ metres, where $P=a+b+c$. **2**

(You may assume that the value of x given in part (iii) is feasible.)

End of paper

2004 Higher School Certificate Solutions

MATHEMATICS

QUESTION 1

(a) $3\,397\,000 = 3{\cdot}397 \times 10^6$
$= 3{\cdot}4 \times 10^6$ (2 significant figures).

(b) $\frac{d}{dx}\left(x^4 + 5x^{-1}\right) = 4x^3 - 5x^{-2}.$

(c)
$$\begin{aligned} \frac{x-5}{3} - \frac{x+1}{4} &= 5. \\ 12 \times \frac{(x-5)}{3} - 12 \times \frac{(x+1)}{4} &= 12 \times 5 \\ 4(x-5) - 3(x+1) &= 60 \\ 4x - 20 - 3x - 3 &= 60 \\ x - 23 &= 60 \\ x &= 83. \end{aligned}$$

(d)
$$\begin{aligned} \left(3 - \sqrt{2}\right)^2 &= 9 - 2(3)\sqrt{2} + \left(\sqrt{2}\right)^2 \\ &= 9 - 6\sqrt{2} + 2 \\ &= 11 - 6\sqrt{2}. \end{aligned}$$

Need to find values a, b such that
$\left(3 - \sqrt{2}\right)^2 = a - b\sqrt{2}.$
$\therefore\ a = 11,\ b = 6.$

(e)
$$\begin{aligned} \text{P(red or yellow)} &= \text{P(red)} + \text{P(yellow)} \\ &= \frac{12}{30} + \frac{7}{30} \\ &= \frac{19}{30}. \end{aligned}$$

(f) $|x+1| \le 5.$

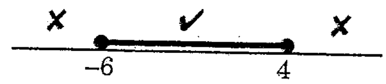
METHOD 1 $\quad x+1 \le 5$ or $-(x+1) \le 5$
$\quad x \le 4 \qquad x+1 \ge -5$
$\qquad x \ge -6.$

$\therefore -6 \le x \le 4.$

METHOD 2
$|x+1| \le 5. \qquad -5 \le x+1 \le 5$
$\therefore \qquad -6 \le x \le 4.$

METHOD 3
Let $x+1 = 5$ or $x+1 = -5$
$x = 4 \qquad x = -6.$

Test:



$\therefore\ -6 \le x \le 4.$

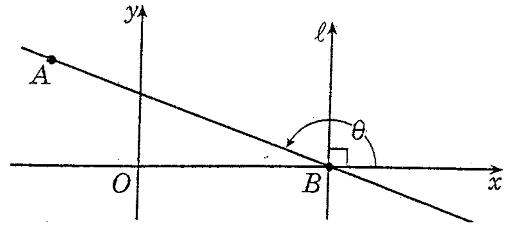
QUESTION 2

(a) (i)
$$\begin{aligned} AB &= \sqrt{(2 - -1)^2 + (0-3)^2} \\ &= \sqrt{3^2 + (-3)^2} \\ &= \sqrt{9+9} \\ &= \sqrt{18} \\ &= 3\sqrt{2} \text{ units.} \end{aligned}$$

(ii) *METHOD* 1 $\quad$ Gradient $AB = \frac{3-0}{-1-2} = \frac{3}{-3} = -1.$

METHOD 2 $\quad$ Gradient $AB = \frac{\text{rise}}{\text{run}} = \frac{-3}{3} = -1.$

(iii)

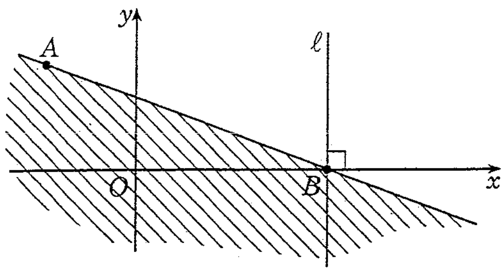


Let θ be the angle between the line AB and the positive direction of the x-axis.
$m = \tan\theta$
but gradient $AB = -1$
so $\tan\theta = -1$
$\theta = 135°.$

$\therefore$ The acute angle between ℓ and line AB is $135° - 90° = 45°.$

(iv) Equation of line AB: $\quad y - y_1 = m(x - x_1).$
Given $m_{AB} = -1,\ B(2, 0)$
$y - 0 = -1(x-2)$
$y = -x + 2.$
$\therefore\ x + y - 2 = 0.$

(v)

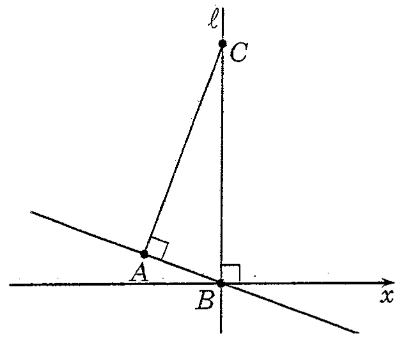


Test (0, 0): LHS $= x + y - 2$
$= 0 + 0 - 2$
$= -2.$

ie. (0, 0) satisfies $x + y - 2 \leq 0$.
$\therefore$ (0, 0) lies in the region $x + y - 2 \leq 0$.

(vi) ℓ passes through $B(2, 0)$.
$\therefore$ Equation of line ℓ is $x = 2$.

(vii)



METHOD 1
Equation of line AC: $y - y_1 = m(x - x_1)$.
Now $AC \perp AB$.
So if $m_{AB} = -1$, then $m_{AC} = 1$.
Line passes through $A(-1, 3)$.

$$y - 3 = 1(x + 1)$$
$$\therefore \quad y = x + 4.$$

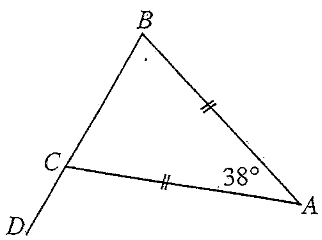
At C, $x = 2$: $y = 2 + 4$
$\therefore$ $y = 6$.
$\therefore$ Coordinates are $C(2, 6)$.

METHOD 2
Since C lies on the line $x = 2$, let C have coordinates $(2, y)$. Now $AC \perp AB$.
So if $m_{AB} = -1$, then $m_{AC} = 1$.

$$m_{AC} = \frac{y-3}{2--1}$$
$$1 = \frac{y-3}{3}$$
$$3 = y - 3$$
$$\therefore \quad y = 6.$$

$\therefore$ Coordinates are $C(2, 6)$.

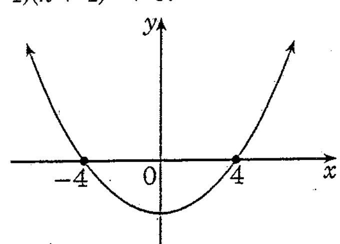
(b)



METHOD 1
$\angle ABC + \angle ACB + 38° = 180°$ ($\angle$ sum of ΔABC)
$\angle ABC + \angle ACB = 142°$.
But ΔABC is isosceles.
$\therefore$ $\angle ABC = \angle ACB$ ($\angle$s opposite equal sides are equal)
ie. $2 \times \angle ACB = 142°$
$\angle ACB = 71°$
$\therefore$ $\angle ACD = 180° - 71°$ (straight angle)
$= 109°$.

METHOD 2
Let $\angle ABC = x°$.
Since ΔABC is isosceles,
$\angle ACB = x°$ ($\angle$s opposite equal sides are equal)
In ΔABC,
$x + x + 38 = 180$ ($\angle$ sum of ΔABC)
$2x + 38 = 180$
$2x = 142$
$x = 71$
ie. $\angle ACB = 71°$.
$\therefore$ $\angle ACD = 180° - 71°$ (straight $\angle$)
$= 109°$.

(c) $x^2 - kx + 4 = 0$.
$\Delta = b^2 - 4ac$
$= (-k)^2 - 4 \times 1 \times 4$
$= k^2 - 16$.
For no real roots $\Delta < 0$.
$\therefore$ $k^2 - 16 < 0$
$(k-4)(k+4) < 0$.



$\therefore$ $-4 < k < 4$.

QUESTION 3

(a) (i) $\frac{d}{dx}\left(x^2 \log_e x\right) = x^2 \times \frac{1}{x} + \log_e x \times 2x$
$= x + 2x \log_e x$.

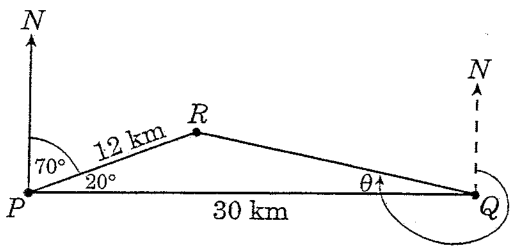
(ii) $\frac{d}{dx}\left(1 + \sin x\right)^5 = 5\left(1 + \sin x\right)^4 \times \left(\cos x\right)$
$= 5 \cos x \left(1 + \sin x\right)^4$.

(b) (i) $\int_1^2 e^{3x}\, dx = \left[\frac{1}{3}e^{3x}\right]_1^2$
$= \frac{1}{3}e^6 - \frac{1}{3}e^3$
$= \frac{1}{3}\left(e^6 - e^3\right)$.

(ii) $\displaystyle\int \frac{x}{x^2-3}\,dx = \frac{1}{2}\int \frac{2x}{x^2-3}\,dx$

$\displaystyle= \frac{1}{2}\log_e\left(x^2-3\right)+C.$

(c)



(i) $\angle RPQ = 90° - 70°$
$= 20°.$
Using the cosine rule,
$a^2 = b^2 + c^2 - 2bc\cos A$
$RQ^2 = 12^2 + 30^2 - 2\times 12\times 30\times\cos 20°$
$= 367{\cdot}421\,313\ldots$
$\therefore\ RQ = 19{\cdot}168\,237\,09\ldots$
$= 19{\cdot}2$ km (correct to 1 decimal place).

(ii) Let $\angle RPQ = \theta$.
Using the sine rule,
$\displaystyle\frac{\sin P}{p} = \frac{\sin Q}{q}$
$\displaystyle\frac{\sin\theta}{12} = \frac{\sin 20°}{RQ}$
$\displaystyle\sin\theta = \frac{12\times\sin 20°}{19{\cdot}168\,237\,09\ldots}$
$= 0{\cdot}214\,116\,806\ldots$
$\theta = 12°\,21'\,49{\cdot}38''$
$\theta = 12°\,22'$ (nearest minute).

$\therefore$ Bearing of R from Q
$= 270° + 12°\,22'$
$= 282°\,22'$
$= 282°$ (nearest degree).

QUESTION 4

(a) *METHOD* 1

$\ell = r\theta$ (θ in radians)
$5\pi = 6\theta$
$\displaystyle\theta = \frac{5\pi}{6}.$
$\displaystyle A = \frac{1}{2}r^2\theta$ (θ in radians)
$\displaystyle= \frac{1}{2}\times 6^2\times\frac{5\pi}{6}$
$\displaystyle= \frac{1}{2}\times 36\times\frac{5\pi}{6}$
$= 15\pi.$
$\therefore$ Area is 15π cm^2.

METHOD 2

Length of arc $\displaystyle= \frac{\theta}{360}\times 2\pi r$ (θ in degrees)

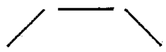
$\displaystyle 5\pi = \frac{\theta}{360}\times 2\pi\times 6$
$\displaystyle 5\pi = \frac{\theta\pi}{30}$
$\therefore\ \theta = 150°.$
$\displaystyle A = \frac{\theta}{360}\times\pi r^2$ (θ in degrees)
$\displaystyle A = \frac{150}{360}\times\pi\times 6^2$
$= 15\pi.$
$\therefore$ Area is 15π cm^2.

(b) (i) $f(x) = x^3 - 3x^2$
$f'(x) = 3x^2 - 6x.$
Stationary points when $f'(x) = 0.$
$3x^2 - 6x = 0$
$3x(x-2) = 0$
$\therefore\ x = 0$ or $x = 2.$
At $x = 0$, $y = 0^3 - 3(0)^2$
$= 0.$
At $x = 2$, $y = 2^3 - 3(2)^2$
$= -4.$
$\therefore$ Stationary points are $(0, 0)$ and $(2, -4)$.

METHOD 1

Using first derivative, test $x = 0$:

x	-1	0	1
$f'(x)$	9	0	-3



$\therefore$ Maximum turning point at $(0, 0)$.

Test $x = 2$:

x	1	2	3
$f'(x)$	-3	0	9

$\therefore$ Minimum turning point at $(2, -4)$.

METHOD 2

Using second derivative, $f''(x) = 6x - 6.$

Test $x = 0$: $f''(0) = 6(0) - 6$
$= -6.$

ie. $f''(0) < 0 \Rightarrow$ concave down 

$\therefore$ Maximum turning point at $(0, 0)$.

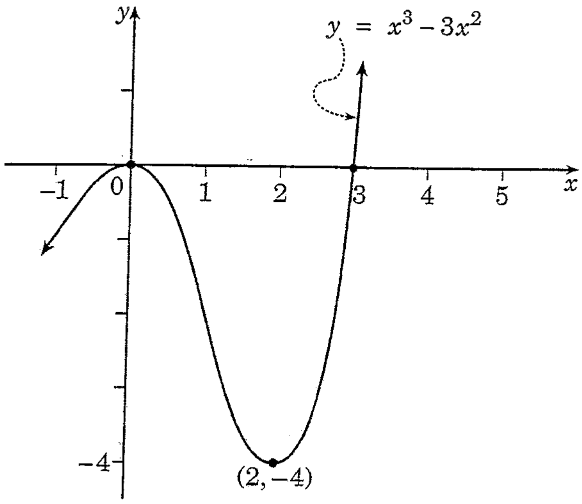
Test $x = 2$: $f''(2) = 6(2) - 6$
$= 6.$

ie. $f''(2) > 0 \Rightarrow$ concave up 

$\therefore$ Minimum turning point at $(2, -4)$.

(ii) Cuts x-axis when $y = 0$.

$$x^3 - 3x^2 = 0$$
$$x^2(x-3) = 0$$
$$\therefore\ x = 0 \text{ or } x = 3.$$



(iii) $f''(x) = 6x - 6$.

Curve is concave up when $f''(x) > 0$.

$$6x - 6 > 0$$
$$6x > 6$$
$$\therefore\ x > 1.$$

(c) Volume $= \pi\int_a^b y^2\,dx$

$$= \pi\int_0^{\frac{\pi}{3}} (2\sec x)^2\,dx$$
$$= \pi\int_0^{\frac{\pi}{3}} 4\sec^2 x\,dx$$
$$= 4\pi\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$$
$$= 4\pi\Big[\tan x\Big]_0^{\frac{\pi}{3}}$$
$$= 4\pi\left(\tan\frac{\pi}{3} - \tan 0\right)$$
$$= 4\pi\left(\sqrt{3} - 0\right)$$
$$= 4\pi\sqrt{3} \text{ units}^3.$$

QUESTION 5

(a) Lesson 1 = 30 minutes
Lesson 2 = 35 minutes
Lesson 3 = 40 minutes
$\therefore$ 30, 35, 40, 45, ...
This is an arithmetic sequence with $a = 30$, $d = 5$.

(i)
$$T_n = a + (n-1)d$$
$$\therefore\ T_{21} = 30 + (21-1)\times 5$$
$$= 30 + 20\times 5$$
$$= 130.$$

$\therefore$ The 21st lesson will be 130 minutes long (ie. 2 h 10 min).

(ii) *METHOD* 1

$$S_n = \frac{n}{2}(a + \ell)$$

$a = 30$, $\ell = 130$, $n = 21$.

$$\therefore\ S_{21} = \frac{21}{2}(30 + 130)$$
$$= 1680 \text{ min}$$
$$= 28 \text{ hours.}$$

$\therefore$ After 21 lessons, Claire will have completed 28 hours.

METHOD 2

$$S_n = \frac{n}{2}[2a + (n-1)d].$$

$a = 30$, $d = 5$, $n = 21$.

$$\therefore\ S_{21} = \frac{21}{2}[2\times 30 + (21-1)\times 5]$$
$$= \frac{21}{2}(60 + 20\times 5)$$
$$= 1680 \text{ min}$$
$$= 28 \text{ hours.}$$

$\therefore$ After 21 lessons, Claire will have completed 28 hours.

(iii) 50 hours $= 50\times 60$ minutes $= 3000$ minutes.

$a = 30$, $d = 5$, $S_n = 3000$, $n = ?$

Using $S_n = \frac{n}{2}[2a + (n-1)d]$

$$3000 = \frac{n}{2}[60 + (n-1)\times 5]$$
$$6000 = n(60 + 5n - 5)$$
$$6000 = n(55 + 5n)$$
$$6000 = 55n + 5n^2.$$
$$\therefore\ 5n^2 + 55n - 6000 = 0$$
$$n^2 + 11n - 1200 = 0.$$

Using the quadratic formula,

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-11 \pm \sqrt{11^2 - 4\times 1\times(-1200)}}{2\times 1}$$
$$= \frac{-11 \pm \sqrt{4921}}{2}$$

$n = 29{\cdot}574\,919\,81\ldots$ or
$n = -40{\cdot}574\,919\,81\ldots$

(Ignore the negative answer as $n > 0$.)

$\therefore$ More than 29 lessons needed as $n > 29$.

ie. Clare will have completed 50 hours of lessons during her 30th lesson.

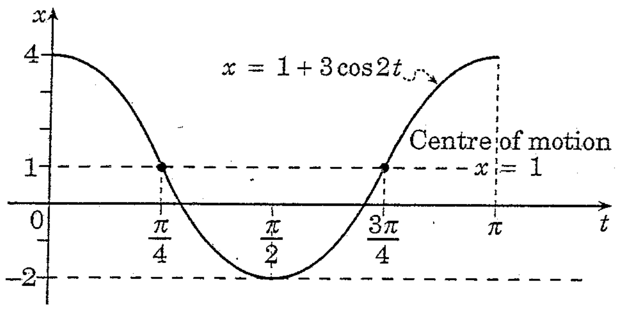
(b) $x = 1 + 3\cos 2t$

(i) Initial displacement when $t = 0$.

$$\begin{aligned} x &= 1 + 3\cos 2(0) \\ &= 1 + 3\cos 0 \\ &= 1 + 3 \times 1 \\ &= 4. \end{aligned}$$

$\therefore$ Initial displacement is 4 metres to the right of the point 0.

(ii)



(iii) *METHOD* 1

From the graph, the particle comes to rest when $v = 0$, ie. at stationary points.

$\therefore$ The next stationary point after $t = 0$ is when $t = \frac{\pi}{2}$ seconds, which occurs when $x = -2$.

$\therefore$ The particle first comes to rest after $t = 0$ when $t = \frac{\pi}{2}$ and $x = -2$, (ie. when the particle is 2 metres to the left of point 0).

METHOD 2

The particle comes to rest when $v = 0$.

$$\begin{aligned} v &= \frac{dx}{dt} \\ &= -6\sin 2t \\ 0 &= -6\sin 2t \\ 0 &= \sin 2t \\ 2t &= 0,\ \pi,\ 2\pi,\ \ldots \\ t &= 0,\ \frac{\pi}{2},\ \pi,\ \ldots \end{aligned}$$

The next time $v = 0$ after $t = 0$ will be when $t = \frac{\pi}{2}$ seconds.

$$\begin{aligned} \therefore \quad x &= 1 + 3\cos\left(2 \times \frac{\pi}{2}\right) \\ &= 1 + 3\cos\pi \\ &= -2. \end{aligned}$$

$\therefore$ The particle first comes to rest after $t = 0$ when $t = \frac{\pi}{2}$ and $x = -2$, (ie. when the particle is 2 metres to the left of point 0).

(iv) *METHOD* 1

From the graph, the particle reaches maximum speed at points of inflexion.

A point of inflexion occurs when $t = \frac{\pi}{4}$.

$$\begin{aligned} v &= -6\sin 2t \\ &= -6\sin\left(2 \times \frac{\pi}{4}\right) \\ &= -6. \end{aligned}$$

$$\begin{aligned} \therefore \text{ Maximum speed} &= |\text{velocity}| \\ &= |-6| \\ &= 6 \text{ m/s}. \end{aligned}$$

$\therefore$ When $t = \frac{\pi}{4}$, the particle reaches its maximum speed of 6 m/s.

METHOD 2

The particle reaches maximum speed when $a = 0$, ie. maximum speed is reached when $\frac{dv}{dt} = 0$.

$$\begin{aligned} v &= -6\sin 2t \\ \frac{dv}{dt} &= -12\cos 2t. \end{aligned}$$

When $\frac{dv}{dt} = 0$, $\quad 0 = -12\cos 2t$

$$\begin{aligned} \cos 2t &= 0 \\ 2t &= \frac{\pi}{2},\ \frac{3\pi}{2},\ \ldots \\ t &= \frac{\pi}{4},\ \frac{3\pi}{4},\ \ldots \end{aligned}$$

Taking $t = \frac{\pi}{4}$, $\quad v = -6\sin 2\left(\frac{\pi}{4}\right) = -6$ m/s.

$$\begin{aligned} \therefore \text{ Maximum speed} &= |-6 \text{ m/s}| \\ &= 6 \text{ m/s}. \end{aligned}$$

$\therefore$ When $t = \frac{\pi}{4}$, the particle reaches its maximum speed of 6 m/s.

QUESTION 6

(a)

$$\begin{aligned} e^{2x} + 3e^x - 10 &= 0 \\ \left(e^x\right)^2 + 3e^x - 10 &= 0 \end{aligned}$$

Let $m = e^x$: $\quad m^2 + 3m - 10 = 0$

$(m+5)(m-2) = 0$

$\therefore\ m = -5 \quad$ or $\quad m = 2$.

$\therefore\ e^x = -5 \quad$ or $\quad e^x = 2$.

No solution, since $e^x > 0$ for all x. $\quad | \quad x = \log_e 2$

$\therefore$ The only solution is $x = \log_e 2$.

(b) (i) In ΔAYM and ΔCYM,

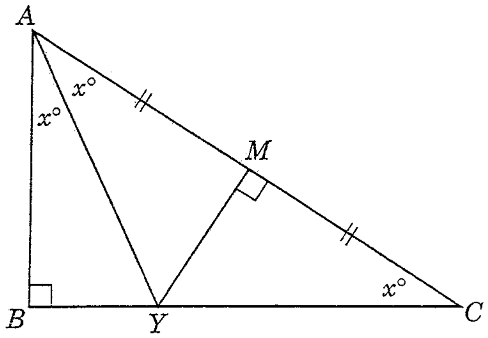
YM is common

$\angle YMA = \angle YMC$ $(YM \perp AC)$

$AM = CM$ $(M$ is midpoint of $AC)$

$\therefore \Delta AYM \equiv \Delta CYM$ **(SAS)**.

(ii)



Let $\angle YAM = x°$.

$\angle YAM = \angle YCM$ (corresponding $\angle$s of congruent Δs AYM and CYM)

$= x°$

$\angle BAY = \angle YAM$ $(AY$ bisects $\angle BAM)$

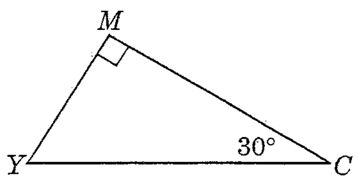
$= x°$.

In ΔBAC, $3x + 90 = 180$ $(\angle$ sum $\Delta BAC)$

$3x = 90$

$x = 30$.

$\therefore \angle YCM = 30°$.



In ΔCYM:

$$\tan 30° = \frac{MY}{MC}$$

$$\frac{1}{\sqrt{3}} = \frac{MY}{MC}$$

$$MC = \sqrt{3}MY$$

$\therefore$ $2MC = 2\sqrt{3}MY$.

$AC = 2\sqrt{3}MY$ $(M$ is midpoint of AC, so $2MC = AC)$

$$\frac{AC}{AC} = \frac{2\sqrt{3}MY}{AC}$$

$$1 = \frac{2\sqrt{3}MY}{AC}$$

$$\frac{1}{2\sqrt{3}} = \frac{MY}{AC}$$

$$\therefore \frac{MY}{AC} = \frac{1}{2\sqrt{3}}$$

$\therefore$ $MY : AC = 1 : 2\sqrt{3}$.

(c) Sample space is shown in the table.

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	6	8	10
3	0	3	6	9	12	15
4	0	4	8	12	16	20
5	0	5	10	15	20	25

$\therefore$ 36 possible outcomes.

(i) $P(0 \text{ on first turn}) = \frac{11}{36}$.

(ii) $P(16 \text{ or more on first turn}) = \frac{4}{36} = \frac{1}{9}$.

(iii) There are only 3 ways to get a sum of 45 or more on 2 throws:

$20 + 25$ or $25 + 20$ or $25 + 25$.

$$P(\text{sum} \geq 45)$$

$$= P(20+25) + P(25+20) + P(25+25)$$

$$= \left(\frac{2}{36} \times \frac{1}{36}\right) + \left(\frac{1}{36} \times \frac{2}{36}\right) + \left(\frac{1}{36} \times \frac{1}{36}\right)$$

$$= \frac{5}{1296}.$$

$$\therefore P(\text{sum} < 45) = 1 - P(\text{sum} \geq 45)$$

$$= 1 - \frac{5}{1296}$$

$$= \frac{1291}{1296}.$$

QUESTION 7

(a) $$\sum_{n=2}^{4} n^2 = 2^2 + 3^2 + 4^2$$

$$= 4 + 9 + 16$$

$$= 29.$$

(b) Let $t = 0$ for 1991 when $P = 17$ million

$t = 13$ for 2004 when $P = 20$ million.

(i) $$P = Ae^{kt}$$

$$\frac{dP}{dt} = k \times Ae^{kt}$$

$$= k \times P.$$

$\therefore$ $P = Ae^{kt}$ satisfies $\frac{dP}{dt} = kP$.

(ii) When $t = 0$, $P = 17$ million.

$17 \text{ million} = Ae^{k(0)}$

$A = 17$ million

$= 17\ 000\ 000$.

(iii) When $t = 13$, $P = 20$ million,

$20\ 000\ 000 = 17\ 000\ 000\, e^{13(k)}$

$20 = 17e^{13k}$

$$\frac{20}{17} = e^{13k}$$
$$\log_e\left(\frac{20}{17}\right) = \log_e e^{13k}$$
$$\log_e\left(\frac{20}{17}\right) = 13k\log_e e$$
$$13k = \log_e\left(\frac{20}{17}\right)$$
$$k = \frac{1}{13}\log_e\left(\frac{20}{17}\right)$$
$$\therefore \quad k = 0{\cdot}012\ 501\ 456\ \ldots$$
($k \doteqdot 0{\cdot}0125$ to 4 dec. places).

(iv) Require the value of t when $P = 30$ million.
$$P = 17\ 000\ 000\, e^{kt}$$
$$30\ 000\ 000 = 17\ 000\ 000\, e^{kt}$$
$$\frac{30}{17} = e^{kt}$$
$$\log_e\left(\frac{30}{17}\right) = \log_e e^{kt}$$
$$\log_e\left(\frac{30}{17}\right) = kt\log_e e$$
$$kt = \log_e\left(\frac{30}{17}\right)$$
$$t = \log_e\left(\frac{30}{17}\right) \div k$$
$$= \log_e\left(\frac{30}{17}\right) \div \frac{1}{13}\log_e\left(\frac{20}{17}\right)$$
$$= 45{\cdot}433\ 430\ 5\ \ldots$$
$\therefore$ Required year is $45{\cdot}43\ \ldots$ years after 1991.
$$1991 + 45{\cdot}43\ \ldots = 2036{\cdot}43\ \ldots$$
$\therefore$ During 2036 the population will reach 30 million.

(c) (i) $A = P(1+r)^n$.

$P = 80$, $\quad r = 6\%$ per annum $= \frac{6\%}{12}$ per month $= 0{\cdot}5\% = 0{\cdot}005$,

$n = 25$ years $= 300$ months.

$$A = 80(1+0{\cdot}005)^{300}$$
$$= 80(1{\cdot}005)^{300}$$
$$= 357{\cdot}197\ 585\ \ldots$$
$= \$357{\cdot}20$ (to the nearest cent).

$\therefore$ After 25 years, the value of the first \$80 invested will be \$357·20.

(ii) Let A_n be the value of the nth instalment at the end of the term.

1st \$80 invested: $\quad A_1 = 80(1{\cdot}005)^{300}$

2nd \$80 invested: $\quad A_2 = 80(1{\cdot}005)^{299}$

3rd \$80 invested: $\quad A_3 = 80(1{\cdot}005)^{298}$

$\vdots$

last \$80 invested: $\quad A_{300} = 80(1{\cdot}005)^{1}$.

$$\text{Total} = A_{300} + A_{299} + A_{298} + \cdots + A_1$$
$$= 80(1{\cdot}005) + 80(1{\cdot}005)^2 + 80(1{\cdot}005)^3 + \cdots + 80(1{\cdot}005)^{300}$$
$$= 80\left(1{\cdot}005 + 1{\cdot}005^2 + 1{\cdot}005^3 + \cdots + 1{\cdot}005^{300}\right)$$
$= 80 \times S_n$, $\quad S_n$ is a geometric series, with $a = 1{\cdot}005$, $r = 1{\cdot}005$, $n = 300$.
$$= 80 \times \frac{a(r^n - 1)}{r-1}$$
$$= 80 \times \frac{1{\cdot}005\left(1{\cdot}005^{300} - 1\right)}{1{\cdot}005 - 1}$$
$$= 55\ 716{\cdot}714\ 58\ \ldots$$
$= \$55\ 716{\cdot}71$ (to the nearest cent).

$\therefore$ Final value of the trust fund will be \$55 716·71.

QUESTION 8

(a) (i) $\cos\theta\tan\theta = \sin\theta$.
$$\text{LHS} = \cos\theta\tan\theta$$
$$= \cos\theta \cdot \frac{\sin\theta}{\cos\theta}$$
$$= \sin\theta$$
$$= \text{RHS}.$$

(ii) $8\sin\theta\,\underbrace{\cos\theta\tan\theta} = \operatorname{cosec}\theta$
$$8\sin\theta \times \sin\theta = \operatorname{cosec}\theta$$
$$8\sin^2\theta = \operatorname{cosec}\theta$$
$$8\sin^2\theta = \frac{1}{\sin\theta}$$
$$8\sin^3\theta = 1$$
$$\sin^3\theta = \frac{1}{8}$$
$$\sin\theta = \sqrt[3]{\frac{1}{8}}$$
$$\sin\theta = \frac{1}{2}.$$

[The related angle is $\frac{\pi}{6}$.]

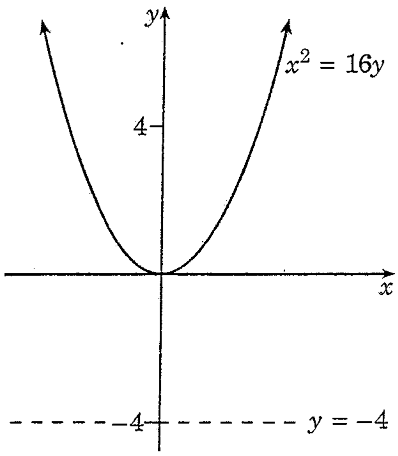
In the 1st quadrant, $\theta = \dfrac{\pi}{6}$.

In the 2nd quadrant, $\theta = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$.

$\therefore \quad \theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$.

(b) (i) $x^2 = 16y$ is of the form $x^2 = 4ay$.

$$4a = 16$$
$$a = 4.$$



$\therefore$ Directrix is $y = -4$.

(ii)
$$x^2 = 16y$$
$$y = \frac{1}{16}x^2$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{8}x.$$

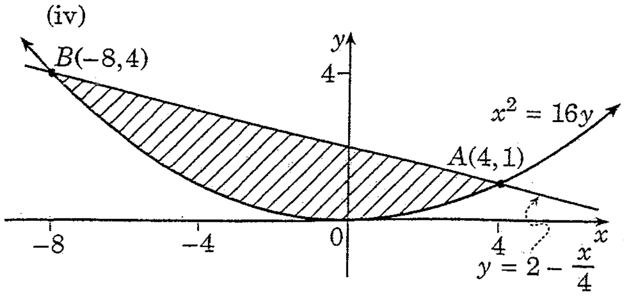
At A, $x = 4$: $m_{\text{tangent}} = \dfrac{1}{8} \times 4 = \dfrac{1}{2}$.

Equation of tangent at $A(4, 1)$:
$$y - y_1 = m(x - x_1)$$
$$y - 1 = \frac{1}{2}(x - 4)$$
$$2y - 2 = x - 4$$
$$x - 2y - 2 = 0.$$

(iii) C lies on the directrix. So at the point C, $y = -4$. But C also lies on the tangent
$$x - 2y - 2 = 0.$$
$$\therefore \quad x - 2(-4) - 2 = 0$$
$$x = -6.$$

$\therefore$ C is the point $(-6, -4)$.

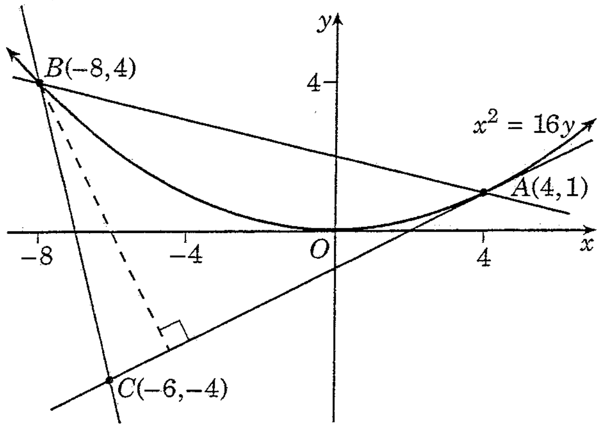
(iv)



$$\text{Area} = \int_{-8}^{4} \left[\left(2 - \frac{x}{4}\right) - \frac{x^2}{16} \right] dx$$
$$= \left[2x - \frac{x^2}{8} - \frac{x^3}{48} \right]_{-8}^{4}$$
$$= \left[2(4) - \frac{4^2}{8} - \frac{4^3}{48} \right] - \left[2(-8) - \frac{(-8)^2}{8} - \frac{(-8)^3}{48} \right]$$
$$= 4\frac{2}{3} - \left(-13\frac{1}{3}\right).$$

$\therefore$ Area $= 18$ units^2.

(v) *METHOD* 1



Length of AC:
$$AC = \sqrt{(4 - -6)^2 + (1 - -4)^2}$$
$$= \sqrt{10^2 + 5^2}$$
$$= \sqrt{125}.$$

Perpendicular distance from B to AC:
$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

The equation of tangent AC is $x - 2y - 2 = 0$.

Use $B(-8, 4)$ as (x_1, y_1):
$$d = \frac{|1 \times (-8) - 2 \times 4 - 2|}{\sqrt{1^2 + (-2)^2}}$$
$$= \frac{|-18|}{\sqrt{5}}$$
$$= \frac{18}{\sqrt{5}}.$$

Area of ΔABC: $\quad A = \dfrac{1}{2}bh$

where $b = \sqrt{125}$, $h = \dfrac{18}{\sqrt{5}}$.
$$A = \frac{1}{2} \times \sqrt{125} \times \frac{18}{\sqrt{5}}$$
$$= 45.$$

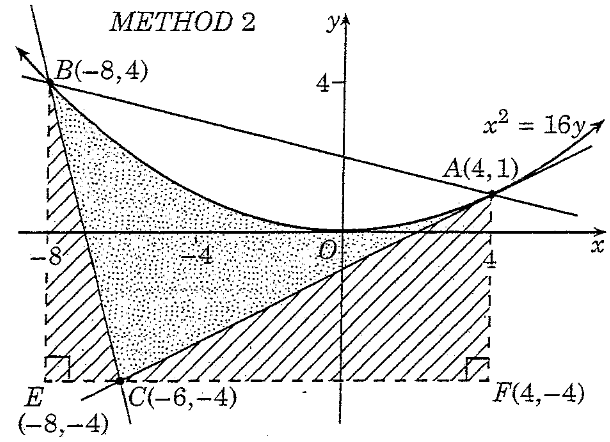
$\therefore$ Shaded area

$= \text{area } \Delta ABC - \text{area from part (iv)}$

$= 45 - 18$

$= 27 \text{ units}^2.$

METHOD 2



E and F are on the directrix $\left(\text{ie. } E \text{ and } F \text{ lie on } y = -4\right)$.

E is $(-8, -4)$ and F is $(4, -4)$.

For a trapezium, $A = \dfrac{1}{2}h(a+b)$.

$\therefore$ Area of trapezium $EBAF$

$$= \frac{1}{2} \times EF \times \left(EB + AF\right)$$

$$= \frac{1}{2} \times 12 \times (8+5)$$

$$= 78 \text{ units}^2.$$

Area ΔABC

$= \text{area trapezium } EBAF - \text{area } \Delta BEC - \text{area } \Delta ACF$

$$= 78 - \left(\frac{1}{2} \times 2 \times 8\right) - \left(\frac{1}{2} \times 10 \times 5\right)$$

$$= 78 - 8 - 25$$

$$= 45 \text{ units}^2.$$

$\therefore$ Shaded area

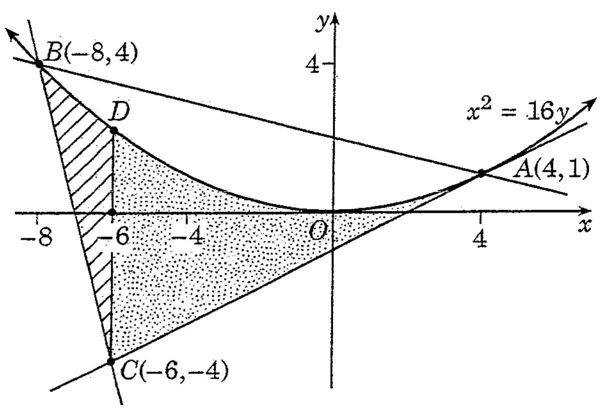
$= \text{area } \Delta ABC - \text{area from part (iv)}$

$= 45 - 18$

$= 27 \text{ units}^2.$

METHOD 3

Consider the following, where a vertical line through C meets the parabola at D.



Gradient of line BC: $m_{BC} = \dfrac{4 - -4}{-8 - -6} = \dfrac{8}{-2} = -4.$

Equation of line BC:

$$\begin{aligned} y - 4 &= -4(x - -8) \\ y - 4 &= -4x - 32 \\ y &= -4x - 28. \end{aligned}$$

Area of region BDC

$$= \int_{-8}^{-6} \left[\frac{1}{16}x^2 - \left(-4x - 28\right)\right] dx$$

$$= \int_{-8}^{-6} \left(\frac{1}{16}x^2 + 4x + 28\right) dx$$

$$= \left[\frac{1}{48}x^3 + 2x^2 + 28x\right]_{-8}^{-6}$$

$$= \left[\frac{1}{48}(-6)^3 + 2(-6)^2 + 28(-6)\right] - \left[\frac{1}{48}(-8)^3 + 2(-8)^2 + 28(-8)\right]$$

$$= -100\frac{1}{2} + 106\frac{2}{3}$$

$$= 6\frac{1}{6} \text{ units}^2.$$

Equation of tangent AC is

$$\begin{aligned} x - 2y - 2 &= 0 \\ 2y &= x - 2 \\ y &= \frac{1}{2}x - 1. \end{aligned}$$

Area of region ADC

$$= \int_{-6}^{4} \left[\frac{1}{16}x^2 - \left(\frac{1}{2}x - 1\right)\right] dx$$

$$= \int_{-6}^{4} \left(\frac{1}{16}x^2 - \frac{1}{2}x + 1\right) dx$$

$$= \left[\frac{1}{48}x^3 - \frac{1}{4}x^2 + x\right]_{-6}^{4}$$

$$= \left[\frac{1}{48}(4)^3 - \frac{1}{4}(4)^2 + 4\right] - \left[\frac{1}{48}(-6)^3 - \frac{1}{4}(-6)^2 + (-6)\right]$$

$$= 1\frac{1}{3} - -19\frac{1}{2}$$

$$= 20\frac{5}{6}.$$

Required shaded area

$= \text{area of region } BDC + \text{area of region } ADC$

$$= 6\frac{1}{6} + 20\frac{5}{6}$$

$$= 27 \text{ units}^2.$$

QUESTION 9

(a) (i) $1 - \tan^2\theta + \tan^4\theta - \ldots$

This is a geometric series with $a = 1$, $r = -\tan^2\theta$.

When the limiting sum exists,

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\left(-\tan^2\theta\right)} = \frac{1}{1+\tan^2\theta} = \frac{1}{\sec^2\theta} = \cos^2\theta.$$

(ii) For the limiting sum to exist, $|r| < 1$.

ie. $-1 < r < 1$

$-1 < -\tan^2\theta < 1$ $\left(\text{since } r = -\tan^2\theta\right)$

$1 > \tan^2\theta > -1$

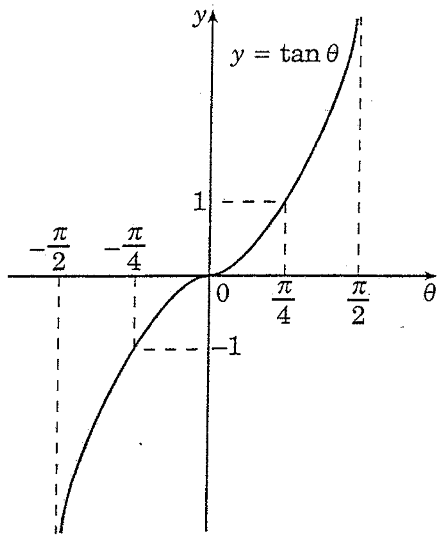
ie. $-1 < \tan^2\theta < 1$.

Since $\tan^2\theta > 0$, only need to solve $\tan^2\theta < 1$

$\therefore\ -1 < \tan\theta < 1.$

When $\tan\theta = \pm 1$, $\theta = \pm\frac{\pi}{4}$.

Since $y = \tan\theta$ is an increasing function, this means the solution is $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.



(b) (i) Velocity is maximum when $a = 0$.

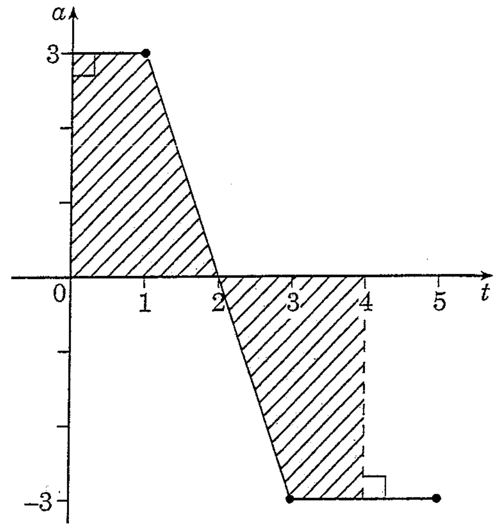
From the graph, this occurs when $t = 2$.

(ii) *METHOD* 1

During $0 \le t \le 5$, a is initially positive, then decreases and becomes negative.

At $t = 0$, the particle is at rest and $a = 3$, so the velocity is positive.

$\therefore$ The particle starts moving to the right with increasing velocity.



After a time it slows down and eventually stops, then begins moving to the left (ie. velocity is negative).

Now $v = \int a\ dt$.

By observing the graph, the following can be noted: $v > 0$ for $0 \le t < 4$

$\left(\int a\ dt \text{ is positive since the area above the } x\text{-axis is greater than the area below the } x\text{-axis.}\right)$

$v = 0$ for $t = 4$

$\left(\int a\ dt = 0 \text{ since the area above the } x\text{-axis is equal to the the area below the } x\text{-axis.}\right)$

$v < 0$ for $4 < t \le 5$

$\left(\int a\ dt < 0 \text{ since the area is below the } x\text{-axis.}\right)$

Hence, from $t = 0$ until $t = 4$, the particle has positive velocity and is travelling away from the origin. At $t = 4$, $v = 0$ and the particle is at rest. After $t = 4$, the velocity is negative and the particle is returning to the origin.

$\therefore$ The particle is furthest from the origin at $t = 4$.

METHOD 2

Sketching the velocity curve:

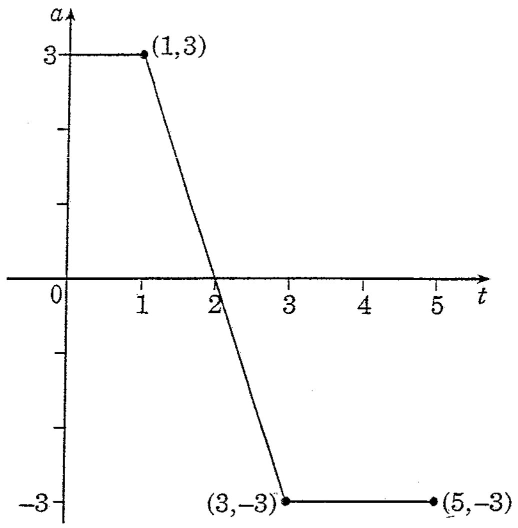
$0 \le t \le 1$, $a = 3$ and is constant.

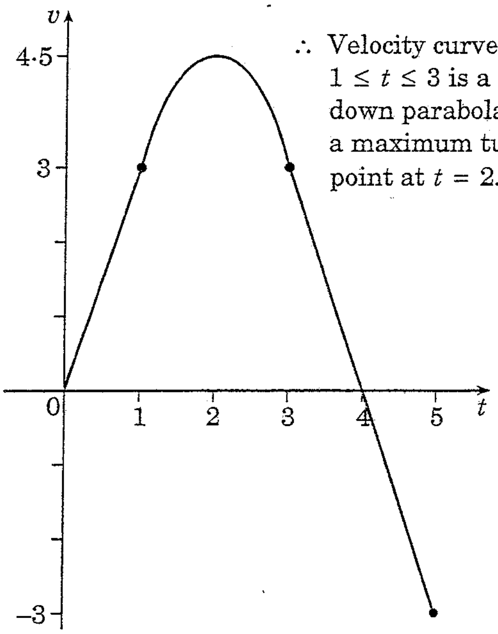
$\therefore$ The graph of velocity is a straight line starting at $v = 0$ (particle initially at rest), with gradient 3.

$1 \le t \le 3$, acceleration is linear, with negative gradient.

Note that at $t = 2$, $a = \frac{dv}{dt} = 0$.

ie. velocity has a stationary point at $t = 2$.





$\therefore$ Velocity curve for $1 \le t \le 3$ is a concave down parabola with a maximum turning point at $t = 2$.

$3 \le t \le 5$, $a = -3$ and is constant

$\therefore$ The graph of velocity is a straight line, with gradient -3.

Note from the velocity graph that for $t < 4$, $v > 0$ (ie. the particle is travelling away from the origin), while for $t > 4$, $v < 0$ (ie. the particle is returning to the origin). At $t = 4$, $v = 0$ and the particle is at rest.

$\therefore$ The particle is furthest from the origin at $t = 4$.

METHOD 3

Alternatively, the velocity graph for $0 \le t \le 5$ could be determined using calculus.

When $0 \le t \le 1$, $a = 3$.

$$v = \int 3\ dt$$

$$v = 3t + C$$

$$t = 0,\ v = 0$$

$$\therefore \quad v = 3t. \qquad \ldots ①$$

When $1 \le t \le 3$, the graph of a is a straight line with $m = -3$, $b = 6$, ie. the equation is $a = -3t + 6$.

$$v = \int -3t + 6\ dt$$

$$v = -\frac{3}{2}t^2 + 6t + C.$$

From ①, when $t = 1$, $v = 3$.

$$3 = -\frac{3}{2} + 6 + C$$

$$C = -\frac{3}{2}.$$

$$\therefore \quad v = -\frac{3}{2}t^2 + 6t - \frac{3}{2}. \qquad \ldots ②$$

When $3 \le t \le 5$, $a = -3$.

$$v = \int -3\ dt$$

$$v = -3t + C.$$

From ②, $v = 3$ when $t = 3$.

$$3 = -9 + C$$

$$12 = C.$$

$$\therefore \quad v = -3t + 12. \qquad \ldots ③$$

The velocity graph for $0 \le t \le 5$ could then be drawn and conclusions reached. For graph and conclusions, refer to *METHOD* 2.

(c) (i)

$$f(x) = \frac{\log_e x}{x}, \quad x > 0$$

$$f'(x) = \frac{x \times \frac{1}{x} - \log_e x \times 1}{x^2}$$

$$= \frac{1 - \log_e x}{x^2}.$$

When $x = e$, $\quad f'(e) = \dfrac{1 - \log_e e}{e^2} = \dfrac{1 - 1}{e^2} = 0.$

When $x = e$, $\quad f(e) = \dfrac{\log_e e}{e} = \dfrac{1}{e}.$

$\therefore$ There is a stationary point at $\left(e, \dfrac{1}{e}\right)$.

(ii) Using the first derivative test:

x	$x < e$ say $x = 2$	e	$x > e$ say $x = 3$
$f'(x)$	$\dfrac{1 - \log_e 2}{4}$ $\doteqdot 0{\cdot}077$	0	$\dfrac{1 - \log_e 3}{9}$ $\doteqdot -0{\cdot}01$

／ — ＼

$\therefore$ The stationary point $\left(e, \dfrac{1}{e}\right)$ is a maximum.

(iii) Since maximum occurs at $\left(e, \frac{1}{e}\right)$,

$$f(x) \le \frac{1}{e} \quad \text{for all } x > 0$$

$$\text{ie.} \quad \frac{\log_e x}{x} \le \frac{1}{e} \quad \text{for all } x > 0$$

$$\log_e x \le \frac{x}{e}$$

$$e^{\log_e x} \le e^{\frac{x}{e}}$$

$$x \le e^{\frac{x}{e}}$$

$$x^e \le \left(e^{\frac{x}{e}}\right)^e$$

$$x^e \le e^x$$

$$\therefore \quad e^x \ge x^e \quad \text{for all } x > 0.$$

QUESTION 10

(a) (i)

x	a	$2a$	$3a$
y	$\frac{1}{a}$	$\frac{1}{2a}$	$\frac{1}{3a}$

$$A \doteqdot \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\doteqdot \frac{3a-a}{6}\left(\frac{1}{a} + 4 \times \frac{1}{2a} + \frac{1}{3a}\right)$$

$$\doteqdot \frac{a}{3}\left(\frac{3+6+1}{3a}\right)$$

$$\doteqdot \frac{a}{3} \times \frac{10}{3a}$$

$$\therefore \text{ Area} \doteqdot \frac{10}{9} \text{ units}^2.$$

(ii) $$\int_a^{3a} \frac{1}{x}\,dx = \Big[\ln x\Big]_a^{3a}$$

$$= \ln 3a - \ln a$$

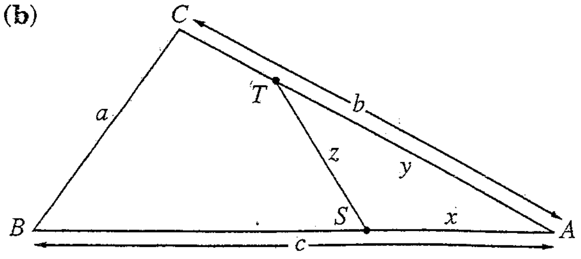
$$= \ln \frac{3a}{a}$$

$$= \ln 3.$$

Using the result of part (i),

$\therefore \quad \ln 3 \doteqdot \frac{10}{9}$ (since Simpson's rule gives an approximation for the area under the curve).

(b)



(i) Area of $\Delta ABC = \frac{1}{2} bc \sin A.$

Area of $\Delta AST = \frac{1}{2} xy \sin A.$

But given area of $\Delta AST = \frac{1}{2}$ area of ΔABC

$$\text{ie.} \quad \frac{1}{2} xy \sin A = \frac{1}{2} \times \frac{1}{2} bc \sin A$$

$$\therefore \quad xy = \frac{1}{2} bc.$$

(ii) Using the cosine rule in ΔAST,

$$z^2 = x^2 + y^2 - 2xy \cos A$$

$$= x^2 + \left(\frac{bc}{2x}\right)^2 - 2\left(\frac{1}{2}bc\right)\cos A,$$

using $xy = \frac{1}{2}bc,\ y = \frac{bc}{2x}.$

$$\therefore \quad z^2 = x^2 + \frac{b^2c^2}{4x^2} - bc\cos A.$$

(iii) $$z^2 = x^2 + \left(\frac{b^2c^2}{4}\right)x^{-2} - bc\cos A$$

$$\frac{d}{dx}\left(z^2\right) = 2x + \left(\frac{b^2c^2}{4}\right) \times \left(-2x^{-3}\right)$$

$$\therefore \quad \frac{d}{dx}\left(z^2\right) = 2x - \frac{b^2c^2}{2x^3}$$

$$\frac{d^2}{dx^2}\left(z^2\right) = 2 - \left(\frac{b^2c^2}{2}\right) \times \left(-3x^{-4}\right)$$

$$\therefore \quad \frac{d^2}{dx^2}\left(z^2\right) = 2 + \frac{3b^2c^2}{2x^4}.$$

But $2 + \frac{3b^2c^2}{2x^4} > 0$ for all x,

ie. $\frac{d^2}{dx^2}\left(z^2\right) > 0 \Rightarrow$ concave up.

So any stationary point must be a minimum.

$\therefore$ Minimum value when

$$\frac{d}{dx}\left(z^2\right) = 0$$

$$2x - \frac{b^2c^2}{2x^3} = 0$$

$$2x = \frac{b^2c^2}{2x^3}$$

$$x^4 = \frac{b^2c^2}{4}$$

$$x^2 = \frac{bc}{2}.$$

$\therefore \quad x = \sqrt{\frac{bc}{2}}$ (as x is the length of a side, $x > 0$).

(iv) Minimum z^2 (and hence minimum length of fence, z) is when $x = \sqrt{\frac{bc}{2}}$.

$$z^2 = x^2 + \frac{b^2c^2}{4x^2} - bc\cos A.$$

When $x = \sqrt{\dfrac{bc}{2}}$,

$$z^2 = \frac{bc}{2} + \frac{b^2c^2}{4} \times \frac{2}{bc} - bc\cos A$$
$$= \frac{bc}{2} + \frac{bc}{2} - bc\cos A$$
$$= bc - bc\cos A$$
$$= bc(1 - \cos A)$$
$$= bc\left[1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)\right],$$

since $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$$= bc\left(\frac{2bc - b^2 - c^2 + a^2}{2bc}\right)$$
$$= \frac{1}{2}\left(2bc - b^2 - c^2 + a^2\right)$$
$$= \frac{1}{2}\left[a^2 - \left(b^2 - 2bc + c^2\right)\right]$$
$$= \frac{1}{2}\left[a^2 - (b - c)^2\right]$$
$$= \frac{1}{2}(a - b + c)(a + b - c)$$
$$= \frac{1}{2}(a + b + c - 2b)(a + b + c - 2c)$$

ie. $z^2 = \dfrac{1}{2}(P - 2b)(P - 2c)$,

where $P = a + b + c$.

$\therefore \quad z = \sqrt{\dfrac{(P - 2b)(P - 2c)}{2}}$.

$\therefore$ The minimum length of the fence is $\sqrt{\dfrac{(P - 2b)(P - 2c)}{2}}$ metres.

END OF MATHEMATICS SOLUTIONS