BOARD OF STUDIES
NEW SOUTH WALES

2008

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Total marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) The polynomial x^3 is divided by $x+3$. Calculate the remainder. **2**

(b) Differentiate $\cos^{-1}(3x)$ with respect to x. **2**

(c) Evaluate $\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\,dx$. **2**

(d) Find an expression for the coefficient of x^8y^4 in the expansion of $(2x+3y)^{12}$. **2**

(e) Evaluate $\int_0^{\frac{\pi}{4}} \cos\theta \sin^2\theta \, d\theta$. **2**

(f) Let $f(x) = \log_e[(x-3)(5-x)]$. **2**

What is the domain of $f(x)$?

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $u = \log_e x$ to evaluate $\int_e^{e^2} \frac{1}{x(\log_e x)^2}\,dx$. **3**

(b) A particle moves on the x-axis with velocity v. The particle is initially at rest at $x = 1$. Its acceleration is given by $\ddot{x} = x + 4$. **3**

Using the fact that $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, find the speed of the particle at $x = 2$.

(c) The polynomial $p(x)$ is given by $p(x) = ax^3 + 16x^2 + cx - 120$, where a and c are constants. **3**

The three zeros of $p(x)$ are –2, 3 and α.

Find the value of α.

(d) The function $f(x) = \tan x - \log_e x$ has a zero near $x = 4$. **3**

Use one application of Newton's method to obtain another approximation to this zero. Give your answer correct to two decimal places.

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

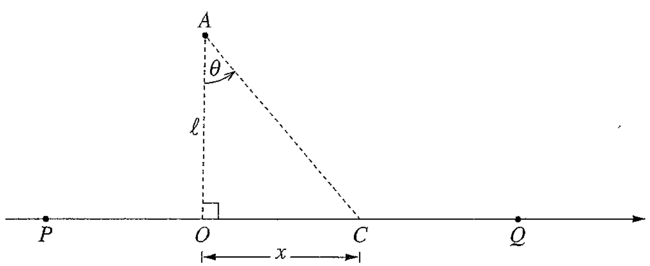
(a) (i) Sketch the graph of $y = |2x - 1|$. **1**

(ii) Hence, or otherwise, solve $|2x - 1| \le |x - 3|$. **3**

(b) Use mathematical induction to prove that, for integers $n \ge 1$, **3**

$$1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n+2) = \frac{n}{6}(n+1)(2n+7).$$

(c)



A race car is travelling on the x-axis from P to Q at a constant velocity, v. A spectator is at A which is directly opposite O, and $OA = \ell$ metres. When the car is at C, its displacement from O is x metres and $\angle OAC = \theta$, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(i) Show that $\dfrac{d\theta}{dt} = \dfrac{v\ell}{\ell^2 + x^2}$. **2**

(ii) Let m be the maximum value of $\dfrac{d\theta}{dt}$. **1**

Find the value of m in terms of v and ℓ.

(iii) There are two values of θ for which $\dfrac{d\theta}{dt} = \dfrac{m}{4}$. **2**

Find these two values of θ.

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) A turkey is taken from the refrigerator. Its temperature is 5°C when it is placed in an oven preheated to 190°C.

Its temperature, T°C, after t hours in the oven satisfies the equation

$$\frac{dT}{dt} = -k(T - 190).$$

(i) Show that $T = 190 - 185e^{-kt}$ satisfies both this equation and the initial condition. **2**

(ii) The turkey is placed into the oven at 9 am. At 10 am the turkey reaches a temperature of 29°C. The turkey will be cooked when it reaches a temperature of 80°C. **3**

At what time (to the nearest minute) will it be cooked?

(b) Barbara and John and six other people go through a doorway one at a time.

(i) In how many ways can the eight people go through the doorway if John goes through the doorway after Barbara with no-one in between? **1**

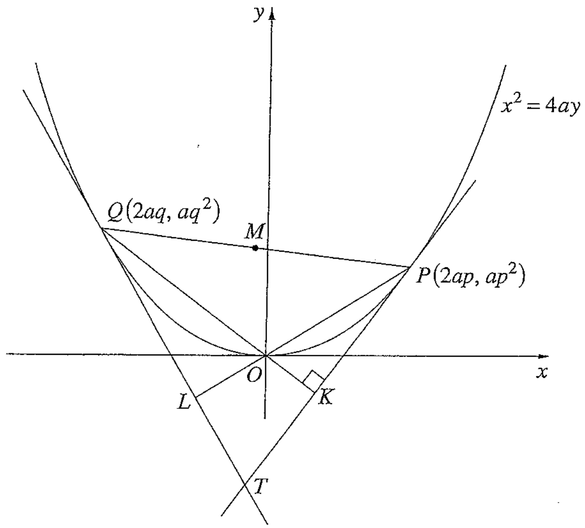
(ii) Find the number of ways in which the eight people can go through the doorway if John goes through the doorway after Barbara. **1**

Question 4 continues on page 7

Question 4 (continued)

Marks

(c)



The points $P(2ap, ap^2)$, $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The tangents to the parabola at P and Q intersect at T. The chord QO produced meets PT at K, and $\angle PKQ$ is a right angle.

(i) Find the gradient of QO, and hence show that $pq = -2$. **2**

(ii) The chord PO produced meets QT at L. Show that $\angle PLQ$ is a right angle. **1**

(iii) Let M be the midpoint of the chord PQ. By considering the quadrilateral $PQLK$, or otherwise, show that $MK = ML$. **2**

End of Question 4

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Let $f(x) = x - \frac{1}{2}x^2$ for $x \le 1$. This function has an inverse, $f^{-1}(x)$.

(i) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. (Use the same scale on both axes.) **2**

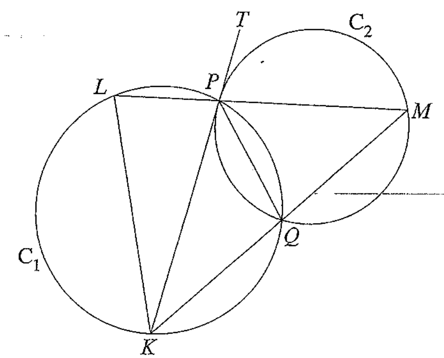
(ii) Find an expression for $f^{-1}(x)$. **3**

(iii) Evaluate $f^{-1}\left(\frac{3}{8}\right)$. **1**

(b) A particle is moving in simple harmonic motion in a straight line. Its maximum speed is 2 m s^{-1} and its maximum acceleration is 6 m s^{-2}. **3**

Find the amplitude and the period of the motion.

(c) **3**



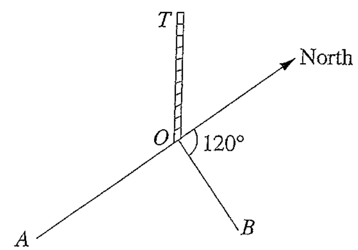
Two circles C_1 and C_2 intersect at P and Q as shown in the diagram. The tangent TP to C_2 at P meets C_1 at K. The line KQ meets C_2 at M. The line MP meets C_1 at L.

Copy or trace the diagram into your writing booklet.

Prove that ΔPKL is isosceles.

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) From a point A due south of a tower, the angle of elevation of the top of the tower T, is 23°. From another point B, on a bearing of 120° from the tower, the angle of elevation of T is 32°. The distance AB is 200 metres.



(i) Copy or trace the diagram into your writing booklet, adding the given information to your diagram. **1**

(ii) Hence find the height of the tower. **3**

(b) It can be shown that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ for all values of θ. (Do NOT prove this.) **3**

Use this result to solve $\sin 3\theta + \sin 2\theta = \sin\theta$ for $0 \le \theta \le 2\pi$.

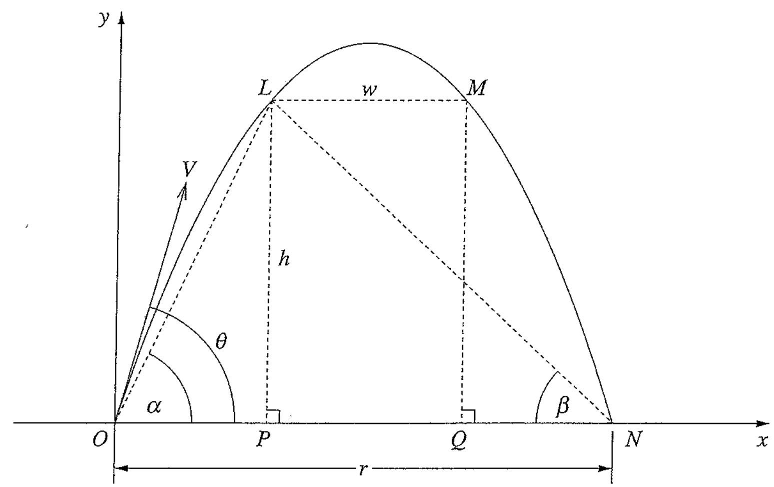
(c) Let p and q be positive integers with $p \le q$.

(i) Use the binomial theorem to expand $(1+x)^{p+q}$, and hence write down the term of $\dfrac{(1+x)^{p+q}}{x^q}$ which is independent of x. **2**

(ii) Given that $\dfrac{(1+x)^{p+q}}{x^q} = (1+x)^p\left(1+\dfrac{1}{x}\right)^q$, apply the binomial theorem and the result of part (i) to find a simpler expression for **3**

$$1+\binom{p}{1}\binom{q}{1}+\binom{p}{2}\binom{q}{2}+\cdots+\binom{p}{p}\binom{q}{p}.$$

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**



A projectile is fired from O with velocity V at an angle of inclination θ across level ground. The projectile passes through the points L and M, which are both h metres above the ground, at times t_1 and t_2 respectively. The projectile returns to the ground at N.

The equations of motion of the projectile are

$$x = Vt\cos\theta \text{ and } y = Vt\sin\theta - \frac{1}{2}gt^2. \text{ (Do NOT prove this.)}$$

(a) Show that $t_1 + t_2 = \dfrac{2V}{g}\sin\theta$ AND $t_1t_2 = \dfrac{2h}{g}$. **2**

Question 7 continues on page 11

Marks

Question 7 (continued)

Let $\angle LON = \alpha$ and $\angle LNO = \beta$. It can be shown that

$$\tan\alpha = \frac{h}{Vt_1\cos\theta} \text{ and } \tan\beta = \frac{h}{Vt_2\cos\theta}.$$ (Do NOT prove this.)

(b) Show that $\tan\alpha + \tan\beta = \tan\theta$. **2**

(c) Show that $\tan\alpha\tan\beta = \dfrac{gh}{2V^2\cos^2\theta}$. **1**

Let $ON = r$ and $LM = w$.

(d) Show that $r = h(\cot\alpha + \cot\beta)$ and $w = h(\cot\beta - \cot\alpha)$. **2**

Let the gradient of the parabola at L be $\tan\phi$.

(e) Show that $\tan\phi = \tan\alpha - \tan\beta$. **3**

(f) Show that $\dfrac{w}{\tan\phi} = \dfrac{r}{\tan\theta}$. **2**

End of paper

2008 Higher School Certificate Solutions
Mathematics Extension 1

Question 1

(a) Let $P(x)=x^3$

By the remainder theorem,

remainder $=P(-3)$

$=(-3)^3$

$=-27$.

(b) $\frac{d}{dx}\left[\cos^{-1}(3x)\right]$

$=\frac{-1}{\sqrt{1-(3x)^2}}\times 3$

$=\frac{3}{\sqrt{1-9x^2}}$.

(c) *METHOD 1*

$\int_{-1}^{1}\frac{1}{\sqrt{4-x^2}}\,dx$

$=\left[\sin^{-1}\frac{x}{2}\right]_{-1}^{1}$

$=\sin^{-1}\frac{1}{2}-\sin^{-1}\left(-\frac{1}{2}\right)$

$=\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)$

$=\frac{\pi}{3}$.

METHOD 2

$\frac{1}{\sqrt{4-x^2}}$ is an even function

$\therefore \int_{-1}^{1}\frac{1}{\sqrt{4-x^2}}\,dx$

$=2\int_0^1\frac{1}{\sqrt{4-x^2}}.dx$

$=2\left[\sin^{-1}\frac{x}{2}\right]_0^1$

$=2\left(\sin^{-1}\frac{1}{2}-\sin^{-1}0\right)$

$=2\left(\frac{\pi}{6}\right)$

$=\frac{\pi}{3}$.

(d) $(2x+3y)^{12}=\sum_{k=0}^{12}{}^{12}C_k(2x)^{12-k}(3y)^k$

Coefficient of x^8y^4 is when $k=4$.

$\therefore$ Coefficient $={}^{12}C_4(2)^8(3)^4$.

(e) Let $u=\sin\theta$

$\frac{du}{d\theta}=\cos\theta$

When $\theta=0,\ u=0$

When $\theta=\frac{\pi}{4},\ u=\frac{1}{\sqrt{2}}$

$\therefore \int_0^{\frac{\pi}{4}}\cos\theta\sin^2\theta=\int_0^{\frac{1}{\sqrt{2}}}u^2du$

$=\left[\frac{u^3}{3}\right]_0^{\frac{1}{\sqrt{2}}}$

$=\frac{1}{3}.\left(\frac{1}{\sqrt{2}}\right)^3-0$

$=\frac{1}{6\sqrt{2}}$.

(f) $f(x)=\log_e[(x-3)(5-x)]$

Domain: $(x-3)(5-x)>0$

$\therefore x-3>0$ and $5-x>0$

$\therefore 3<x<5$.

Question 2

(a) Let $u=\log_e x$

$\frac{du}{dx}=\frac{1}{x}$

when $x=e,u=1$

when $x=e^2,\ u=2$

$\therefore \int_e^{e^2}\frac{1}{x(\log_e x)^2}dx=\int_1^2\frac{du}{u^2}$

$=\int_1^2 u^{-2}du$

$=\left[\frac{u^{-1}}{-1}\right]_1^2$

$=-\left[\frac{1}{u}\right]_1^2$

$=-\left(\frac{1}{2}-1\right)=\frac{1}{2}$.

(b) $\ddot{x}=x+4$

and $\ddot{x}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right)=x+4$

$\frac{1}{2}v^2=\int x+4\,dx$

$=\frac{x^2}{2}+4x+c$

When $x=1,v=0$

By substitution,

$0=\frac{1^2}{2}+4(1)+c$

$\therefore c=-\frac{9}{2}$

$\therefore \frac{1}{2}v^2=\frac{x^2}{2}+4x-\frac{9}{2}$

$v^2=x^2+8x-9$

When $x=2$,

$v^2=2^2+8(2)-9$

$=11$

$v=\pm\sqrt{11}$

Since speed $=|v|$, the speed of the particle when $x=2$ is $\sqrt{11}$.

(c) *METHOD 1*

$p(x)=ax^3+16x^2+cx-120$

Zeros of the polynomial are -2, 3 and α.

$\therefore \alpha+(-2)+3=\frac{-b}{a}$

$\alpha+1=\frac{-16}{a}$ —— ①

and $\alpha(-2)(3)=\frac{c}{a}$

$-6\alpha=\frac{120}{a}$ —— ②

Rearranging ②, $a=\frac{-20}{\alpha}$ —— ③

Substituting ③ into ①,

$$\alpha+1=\frac{-16}{\left(\frac{-20}{\alpha}\right)}$$

$$\alpha+1=-16\times\left(\frac{\alpha}{-20}\right)$$

$$\alpha+1=\frac{4\alpha}{5}$$

$$5\alpha+5=4\alpha$$

$$\therefore \alpha=-5.$$

METHOD 2

$p(x)=ax^3+16x^2+cx-120$

Zeros of the polynomial are $-2, 3$ and α.

Using the factor theorem,

$$p(-2)=a(-2)^3+16(-2)^2+c(-2)-120$$
$$=0$$

$\therefore 4a+c=-28$ ——①

$$p(3)=a(3)^3+16(3)^2+c(3)-120=0$$

$\therefore 9a+c=-8$ ——②

Solving simultaneously ②−①,

$$5a=20$$
$$\therefore a=4$$

Substituting $a=4$ into ①,

$$4(4)+c=-28$$
$$\therefore c=-44$$
$$\therefore p(x)=4x^3+16x^2-44x-120$$

Using the product of roots,

$$\alpha(-2)(3)=\frac{-e}{a}$$

$$-6\alpha=-\frac{(-120)}{4}$$

$$-6\alpha=30$$

$$\therefore \alpha=-5.$$

(d) $f(x)=\tan x-\log_e x$

$$f'(x)=\sec^2 x-\frac{1}{x}$$

Using Newton's method,

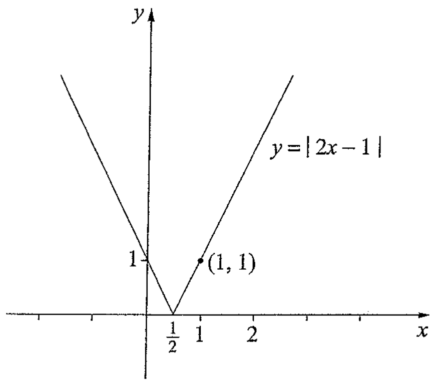
$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

Let $x_1=4$

$$x_2=4-\frac{\tan 4-\log_e 4}{\sec^2 4-\frac{1}{4}}$$
$$=4.1092\ldots$$
$$=4.11 \text{ (to 2 decimal places).}$$

Question 3

(a) (i)

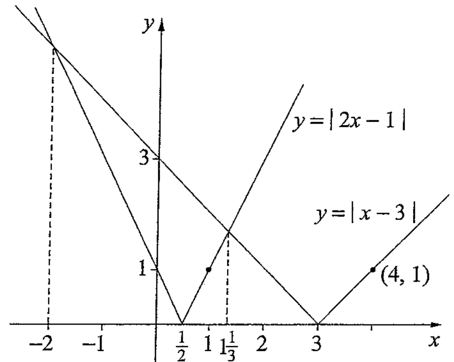


(ii) Solve $|2x-1|=|x-3|$

$$2x-1=x-3 \quad\text{or}\quad 2x-1=-(x-3)$$

$$x=-2 \qquad\qquad x=\frac{4}{3}$$

Sketch $y=|x-3|$ on the same plane.



$\therefore$ From the graph,

$$|2x-1|\le|x-3| \text{ for } -2\le x\le 1\tfrac{1}{3}.$$

(b) Let $T_n=n(n+2)$

and $S_n=\frac{n}{6}(n+1)(2n+7)$

When $n=1$

LHS $=T_1$
$=1\times 3$
$=3$

RHS $=S_1$
$=\frac{1}{6}(1+1)(2\times 1+7)$
$=3$
$=$ LHS

$\therefore$ The statement is true for $n=1$.
Let k be a value for which the statement is true,

i.e. $S_k=\frac{k}{6}(k+1)(2k+7)$

We need to prove that the statement is then true for $n=k+1$,

i.e.

$$S_{k+1}=\frac{k+1}{6}(k+1+1)(2(k+1)+7)$$
$$=\frac{k+1}{6}(k+2)(2k+9)$$

Now,

$$S_{k+1}=S_k+T_{k+1}$$
$$=\frac{k}{6}(k+1)(2k+7)+(k+1)(k+3)$$
$$=\frac{k+1}{6}\left[k(2k+7)+6(k+3)\right]$$
$$=\frac{k+1}{6}\left[2k^2+13k+18\right]$$
$$=\frac{k+1}{6}(k+2)(2k+9)$$

$\therefore$ The statement is true for $n=k+1$ if it is true for $n=k$.

Hence by the principle of mathematical induction, the statement is true for all integers $n\ge 1$.

(c) (i) $\tan\theta=\frac{x}{\ell}$

$$\therefore \theta=\tan^{-1}\left(\frac{x}{\ell}\right)$$

$$\frac{d\theta}{dt}=\frac{d\theta}{dx}\times\frac{dx}{dt}$$

$$\frac{d\theta}{dx}=\frac{1}{1+\left(\frac{x}{\ell}\right)^2}\times\frac{1}{\ell}$$
$$=\frac{1}{\frac{\ell^2+x^2}{\ell^2}}\times\frac{1}{\ell}$$
$$=\frac{\ell}{\ell^2+x^2}$$

$$\frac{dx}{dt}=\text{velocity } v$$

By substitution,

$$\frac{d\theta}{dt}=\frac{\ell}{\ell^2+x^2}\times v$$
$$=\frac{v\ell}{\ell^2+x^2}.$$

(ii) Since v and ℓ are constants,

$\frac{d\theta}{dt}$ is a maximum (m) when

$\ell^2 + x^2$ is a minimum,

i.e. when $x = 0$.

$$\therefore\ m = \frac{v\ell}{\ell^2}$$
$$= \frac{v}{\ell}.$$

(iii)
$$\frac{d\theta}{dt} = \frac{m}{4}$$
$$\therefore \frac{v\ell}{\ell^2 + x^2} = \frac{v}{4\ell} \quad \text{from (i) and (ii)}$$
$$4v\ell^2 = v\ell^2 + vx^2$$
$$3v\ell^2 = vx^2$$

Since $v \neq 0$,
$$x^2 = 3\ell^2$$

Now $\quad x = \ell \tan\theta \quad$ from (i)
$$\therefore \ell^2 \tan^2\theta = 3\ell^2$$
$$\tan^2\theta = 3 \quad (\ell \neq 0)$$
$$\tan\theta = \pm\sqrt{3}$$
$$\therefore \theta = \pm\frac{\pi}{3}.$$

Question 4

(a) (i)
$$T = 190 - 185e^{-kt}$$
$$\therefore -185e^{-kt} = T - 190 \quad \text{——①}$$

and
$$\frac{dT}{dt} = -k\left(-185e^{-kt}\right)$$
$$\therefore \frac{dT}{dt} = -k(T - 190)$$
using ①, as required

Also, when $t = 0$
$$T = 190 - 185e^0$$
$$= 5, \text{ as required}$$

$\therefore T = 190 - 185e^{-kt}$ satisfies both the equation $\frac{dT}{dt} = -k(T - 190)$ and the initial condition.

(ii) At 9 am, let $t = 0$

$\therefore$ At 10 am, $t = 1$ and $T = 29$.

Substituting.
$$29 = 190 - 185e^{-k(1)}$$
$$185e^{-k} = 190 - 29$$
$$e^{-k} = \frac{161}{185}$$

Taking $\log_e$ of both sides,
$$-k = \log_e\left(\frac{161}{185}\right)$$
$$\therefore k = -\log_e\left(\frac{161}{185}\right)$$
$$= 0.1389\ldots$$

Substituting $T = 80$ to find the time taken,
$$80 = 190 - 185e^{-kt}$$
$$185e^{-kt} = 190 - 80$$
$$= 110$$
$$e^{-kt} = \frac{110}{185}$$

Taking $\log_e$ of both sides,
$$-kt = \log_e\left(\frac{110}{185}\right)$$
$$\therefore t = \frac{\log_e\left(\frac{110}{185}\right)}{-k}$$
$$= 3.7414\ldots \text{ hours}$$
$$= 3 \text{ h } 44 \text{ minutes}$$
(to nearest minute)

$\therefore$ The turkey will be cooked at 12:44 pm.

(b) (i) Regarding John and Barbara as one unit,
Number of ways = 7!
= 5040.

(ii) *METHOD 1*

Without restrictions, 8 people can go through the doorway in 8! ways i.e. 40 320 ways.

Now the number of ways John can go through after Barbara = the number of ways Barbara can go through after John.

$\therefore$ Number of ways John can go through the doorway after Barbara
$$= \frac{1}{2} \times 8!$$
$$= \frac{1}{2} \times 40\,320$$
$$= 20\,160.$$

METHOD 2

If Barbara goes through the door first, John can follow her in 7 other positions

i.e. B _ _ _ _ _ _ _

If Barbara goes through the door in second position, John can follow her in 6 other positions,

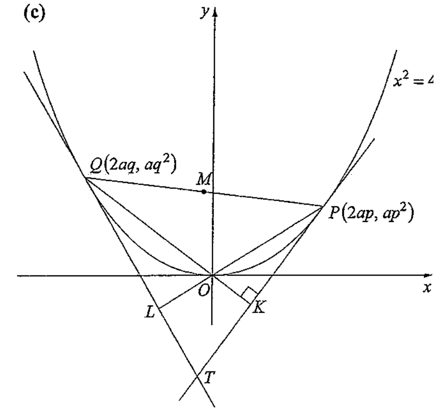
i.e. _ B _ _ _ _ _ _ etc.

$\therefore$ Number of ways John can follow Barbara
$= 7 + 6 + 5 + 4 + 3 + 2 + 1$
$= 28$

For each way John can follow Barbara, the number of ways the other people can go through the door = 6!

$\therefore$ Number of ways in total
$= 28 \times 6!$
$= 20\,160.$

(c)



(i) Gradient of QO
$$= m_{QO} = \frac{aq^2 - 0}{2aq - 0}$$
$$= \frac{q}{2}$$

To show $Pq = -2$, we find m_{PT}, the gradient of the tangent at P.
$$x^2 = 4ay$$
$$y = \frac{x^2}{4a}$$
$$\therefore \frac{dy}{dx} = \frac{2x}{4a}$$
$$= \frac{x}{2a}$$

At P, $\quad x = 2ap$
$$\therefore \frac{dy}{dx} = \frac{2ap}{2a}$$
$$= p$$
$$\therefore m_{PT} = p$$

Since $QO \perp PT$,
$$m_{QO} \times m_{PT} = -1$$
$$\therefore \frac{q}{2} \times p = -1$$
$$pq = -2 \text{ as required.}$$

(ii) To show $\angle PLQ = 90°$
we must show $QL \perp PL$

m_{QL} = gradient of tangent at Q

$= \dfrac{x}{2a}$ from (i)

$= \dfrac{2aq}{2a}$

$= q$

m_{PL} = gradient of $PL = m_{PO}$

$= \dfrac{ap^2 - 0}{2ap - 0}$

$= \dfrac{p}{2}$

$\therefore m_{QL} \times m_{PL} = q \times \dfrac{p}{2}$

$= \dfrac{pq}{2}$

$= \dfrac{-2}{2}$ since $pq = -2$ from (i)

$= -1$

$\therefore QL \perp PL$

$\therefore \angle PLQ = 90°.$

(iii) Since $\angle PLQ = \angle PKQ = 90°$,
$PQLK$ is a cyclic quadrilateral
($\angle$s at the circumference standing on chord PQ are equal).

PQ is a diameter
($\angle$s in a semicircle are 90°).

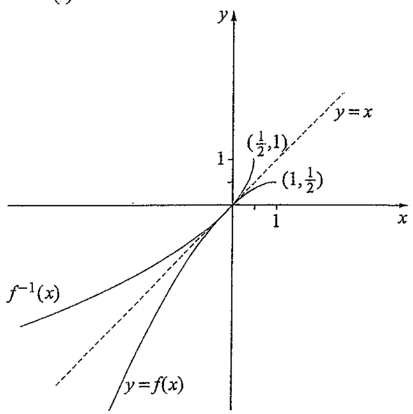
Since M is the midpoint of PQ,
M is the centre of the circle.

$\therefore MK = ML$ (radii of a circle).

Question 5

(a) $f(x) = x^2 - \frac{1}{2}x^2$ for $x \le 1$.

(i)



(ii) Interchanging x and y gives

$x = y - \frac{1}{2}y^2$

$\therefore y^2 - 2y = -2x$

$y^2 - 2y + 1 = 1 - 2x$

$(y-1)^2 = 1 - 2x$

$y = 1 \pm \sqrt{1-2x}$

Since $y \le 1$,

$y = 1 - \sqrt{1-2x}$

i.e. $f^{-1}(x) = 1 - \sqrt{1-2x}$.

(iii) $f^{-1}\left(\frac{3}{8}\right) = 1 - \sqrt{1 - 2\left(\frac{3}{8}\right)}$

$= 1 - \sqrt{\frac{1}{4}}$

$= \frac{1}{2}$.

(b) *METHOD 1*

Using $v^2 = n^2(A^2 - x^2)$,

when $x = 0$,

$v = 2$

$\therefore n^2A^2 = 4$ ——①

Using $\ddot{x} = -n^2x$,

when $x = A$,

$|\ddot{x}| = n^2A$

$\therefore n^2A = 6$ ——②

Solving ① and ②,

$4 = 6A$

$A = \frac{2}{3}$

$n = 3$

$\therefore$ Amplitude $= \frac{2}{3}$ m

Period $= \dfrac{2\pi}{n}$

$= \dfrac{2\pi}{3}$ seconds.

METHOD 2

$x = A\sin nt$

$v = \dot{x}$

$= nA\cos nt$

acc. $= \ddot{x}$

$= -n^2A\sin nt$

Maximum $v = 2$ $\quad\therefore nA = 2$ ——①

Maximum acc. $= 6$ $\quad\therefore n^2A = 6$ ——②

Solving ① and ②,

$2n = 6$

$n = 3$

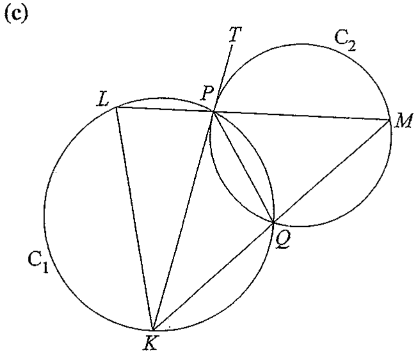
$A = \frac{2}{3}$

$\therefore$ Period $T = \dfrac{2\pi}{n}$

$= \dfrac{2\pi}{3}$ seconds

Amplitude $= \frac{2}{3}$ m.

(c)



$\angle TPM = \angle LPK$ (vertically opposite $\angle$s)

$\angle TPM = \angle PQM$ ($\angle$ in the alternative segment in circle C_2 with TP tangent)

$\therefore \angle LPK = \angle PQM$

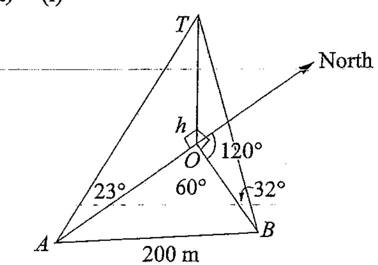
$\angle PQM = \angle PLK$ (exterior $\angle$ to a cyclic quadrilateral in circle C_1 is equal to the opposite interior $\angle$)

$\therefore \angle LPK = \angle PLK$

$\therefore \Delta PKL$ is isosceles.

Question 6

(a) (i)



(ii) Let $TO = h$ metres.

In ΔAOT, $\tan 23° = \dfrac{h}{AO}$

$\therefore AO = \dfrac{h}{\tan 23°}$

In ΔBOT, $\tan 32° = \dfrac{h}{BO}$

$\therefore BO = \dfrac{h}{\tan 32°}$

From the diagram,

$\angle AOB = 180° - 120°$

$= 60°$

Using the cosine rule in ΔAOB,

AB^2

$= AO^2 + BO^2 - 2(AO)(BO)\cos 60°$

By substitution,

$$200^2 = \left(\frac{h}{\tan 23°}\right)^2 + \left(\frac{h}{\tan 32°}\right)^2 - 2\times\left(\frac{h}{\tan 23°}\right)\times\left(\frac{h}{\tan 32°}\right) \times \cos 60°$$

$$= \frac{h^2}{\tan^2 23°} + \frac{h^2}{\tan^2 32°} - \frac{2h^2}{\tan 23° \tan 32°}\times\frac{1}{2}$$

Rearranging,

$$h^2\left(\frac{1}{\tan^2 23°} + \frac{1}{\tan^2 32°} - \frac{1}{\tan^2 23° \tan^2 32°}\right) = 40\,000$$

$$\therefore h^2 = \frac{40\,000}{\left(\frac{1}{\tan^2 23°} + \frac{1}{\tan^2 32°} - \frac{1}{\tan 23° \tan 32°}\right)}$$

$$\therefore h = \sqrt{\frac{40\,000}{4.3409\ldots}}$$

$= 95.9924\ldots \text{m}$

$= 96\,\text{m}$ (to nearest m)

$\therefore$ Height of the tower is 96 m.

(b) $\sin 3\theta + \sin 2\theta = \sin\theta$ —① for $0 \le \theta \le 2\pi$

Substituting $\sin 3\theta = \sin 3\theta - 4\sin^3\theta$ into ①,

$3\sin\theta - 4\sin^3\theta + \sin 2\theta = \sin\theta$

Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$3\sin\theta - 4\sin^3\theta + 2\sin\theta\cos\theta = \sin\theta$

$3\sin\theta - 4\sin^3\theta + 2\sin\theta\cos\theta - \sin\theta = 0$

$2\sin\theta - 4\sin^3\theta + 2\sin\theta\cos\theta = 0$

$2\sin\theta(1 - 2\sin^2\theta + \cos\theta) = 0$

Since $\sin^2\theta = 1 - \cos^2\theta$,

$2\sin\theta(1 - 2(1-\cos^2\theta) + \cos\theta) = 0$

$2\sin\theta(1 - 2 + 2\cos^2\theta + \cos\theta) = 0$

$2\sin\theta(2\cos^2\theta + \cos\theta - 1) = 0$

$2\sin\theta(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\therefore \sin\theta - 0 \Rightarrow \theta = 0, \pi, 2\pi$

or $\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

or $\cos\theta = -1 \Rightarrow \theta = \pi$

$\therefore \theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ or 2π.

(c) (i) Consider the expansion of $(1+x)^{p+q}$ in ascending powers of x:

$$(1+x)^{p+q} = \binom{p+q}{0}x^0 + \binom{p+q}{1}x^1 + \binom{p+q}{2}x^2 + \ldots + \binom{p+q}{r}x^r + \ldots + \binom{p+q}{p+q}x^{p+q}$$

To find the term independent of x in the expansion of $\frac{(1+x)^{p+q}}{x^q}$ we must look for the term with x^0.

$$\therefore \text{Term with } x^0 = \frac{\binom{p+q}{q}x^q}{x^q} = \binom{p+q}{q}.$$

(ii) $$\frac{(1+x)^{p+q}}{x^q} = (1+x)^p\left(1+\frac{1}{x}\right)^q$$

$$= \left[\binom{p}{0}x^0 + \binom{p}{1}x^1 + \binom{p}{2}x^2 + \ldots + \binom{p}{q}x^p\right] \times \left[\binom{q}{0}\left(\frac{1}{x}\right)^0 + \binom{q}{1}\left(\frac{1}{x}\right)^1 + \binom{q}{2}\left(\frac{1}{x}\right)^2 + \ldots + \binom{q}{q}\left(\frac{1}{x}\right)^q\right]$$

The term independent of x on the RHS is

$$\binom{p}{0}(x^0)\binom{q}{0}\left(\frac{1}{x}\right)^0 + \binom{p}{1}x^1\binom{q}{1}\left(\frac{1}{x}\right)^1 + \binom{p}{2}x^2\binom{q}{2}\left(\frac{1}{x}\right)^2 + \ldots + \binom{p}{p}x^p\binom{q}{p}\left(\frac{1}{x}\right)^p$$

since $p \le q$

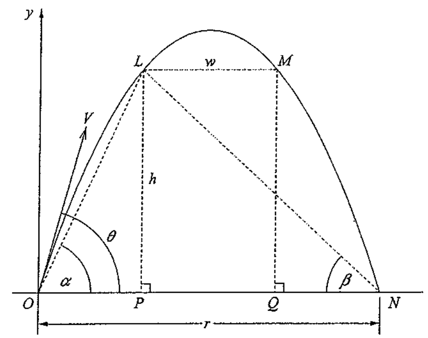
$$= \binom{p}{0}\binom{q}{0} + \binom{p}{1}\binom{q}{1} + \binom{p}{2}\binom{q}{2} + \ldots + \binom{p}{p}\binom{q}{p}$$

$$= 1 + \binom{p}{1}\binom{q}{1} + \binom{p}{2}\binom{q}{2} + \ldots + \binom{p}{p}\binom{q}{p}$$

$$= \binom{p+q}{q}$$

(which is the term independent of x on the LHS using part (i)).

Question 7



(a) $y = h$ at $t = t_1$ and $t = t_2$

$\therefore h = Vt\sin\theta - \frac{1}{2}gt^2$

$gt^2 - 2Vt\sin\theta + 2h = 0$

This quadratic equation has roots t_1 and

$\therefore$ Sum of roots $t_1 + t_2 = \frac{2V\sin\theta}{g}$

Product of roots $t_1t_2 = \frac{2h}{g}$.

(b)

$$\tan\alpha = \frac{h}{Vt_1\cos\theta}$$

$$\tan\beta = \frac{h}{Vt_2\cos\theta}$$

$$\tan\alpha + \tan\beta = \frac{h}{V\cos\theta}\left(\frac{1}{t_1} + \frac{1}{t_2}\right)$$

$$= \frac{h}{V\cos\theta}\left(\frac{t_1+t_2}{t_1t_2}\right)$$

$$= \frac{h}{V\cos\theta}\left(2V\frac{\sin\theta}{g}\times\frac{g}{2h}\right)$$

from (i)

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta.$$

(c) $\tan\alpha\tan\beta$

$$= \frac{h}{Vt_1\cos\theta} \times \frac{h}{Vt_2\cos\theta}$$

$$= \frac{h^2}{V^2\cos^2\theta}\left(\frac{1}{t_1t_2}\right)$$

$$= \frac{h^2}{V^2\cos^2\theta}\left(\frac{g}{2h}\right) \text{ from (i)}$$

$$= \frac{gh}{2V^2\cos^2\theta}.$$

(d) In ΔOLP $\tan\alpha = \frac{LP}{OP}$

$\therefore OP = LP\cot\alpha$

In ΔNLP $\tan\beta = \frac{LP}{NP}$

$\therefore NP = LP\cot\beta$

$r = OP + NP$

$= LP\cot\alpha + LP\cot\beta$

$= LP(\cot\alpha + \cot\beta)$

$= h(\cot\alpha + \cot\beta)$

By symmetry in the parabola,

$OP = QN$

$\therefore w = r - 2OP$

$= h(\cot\alpha + \cot\beta) - 2h\cot\alpha$

$= h(\cot\beta - \cot\alpha).$

(e) $\frac{dx}{dt} = V\cos\theta$

$\frac{dy}{dt} = V\sin\theta - gt$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= (V\sin\theta - gt) \times \frac{1}{V\cos\theta}$$

$$= \frac{V\sin\theta - gt}{V\cos\theta}$$

$$= \tan\theta - \frac{gt}{V\cos\theta}$$

When $t = t_1$, $\frac{dy}{dx} = \tan\phi$

$$\therefore \tan\phi = \tan\theta - \frac{gt}{V\cos\theta} \quad \text{——①}$$

Now

$$\tan\alpha\tan\beta = \frac{gh}{2V^2\cos^2\theta} \quad \text{from (c)}$$

$$\therefore \tan\beta = \frac{gh}{2V^2\cos^2\theta} \times \frac{1}{\tan\alpha}$$

$$= \frac{gh}{2V^2\cos^2\theta} \times \frac{Vt_1\cos\theta}{h}$$

$$= \frac{gt_1}{2V\cos\theta}$$

$$\therefore t_1 = \frac{2V\cos\theta\tan\beta}{g}$$

Substituting t_1 into ①,

$$\tan\phi = \tan\theta - \frac{g}{V\cos\theta} \times \frac{2V\cos\theta\tan\beta}{g}$$

$$= \tan\theta - 2\tan\beta$$

$$\tan\theta = \tan\alpha + \tan\beta \quad \text{from (b)}$$

$$= (\tan\alpha + \tan\beta) - 2\tan\beta$$

$$\therefore \tan\phi = \tan\alpha - \tan\beta.$$

(f) $r = h(\cot\alpha + \cot\beta)$ from (d)

$w = h(\cot\beta - \cot\alpha)$ from (d)

$$\therefore \frac{w}{r} = \frac{\cot\beta - \cot\alpha}{\cot\alpha + \cot\beta}$$

$$= \frac{\frac{1}{\tan\beta} - \frac{1}{\tan\alpha}}{\frac{1}{\tan\alpha} + \frac{1}{\tan\beta}}$$

$$= \frac{\frac{\tan\alpha - \tan\beta}{\tan\alpha\tan\beta}}{\frac{\tan\beta + \tan\alpha}{\tan\alpha\tan\beta}}$$

$$= \frac{\tan\alpha - \tan\beta}{\tan\alpha + \tan\beta}$$

Now

$\tan\alpha - \tan\beta = \tan\phi$ from (e)

$\tan\alpha + \tan\beta = \tan\theta$ from (b)

$$\therefore \frac{w}{r} = \frac{\tan\phi}{\tan\theta}$$

$$\therefore \frac{w}{\tan\phi} = \frac{r}{\tan\theta}.$$

End of Mathematics Extension 1 solutions