BOARD OF STUDIES

NEW SOUTH WALES

2008

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{x^2}{\left(5+x^3\right)^2}\,dx.$ **2**

(b) Find $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}.$ **2**

(c) Evaluate $\displaystyle\int_0^1 \tan^{-1}x\,dx.$ **3**

(d) Evaluate $\displaystyle\int_1^2 \frac{dx}{x\sqrt{2x-1}}.$ **4**

(e) It can be shown that **4**

$$\frac{8(1-x)}{\left(2-x^2\right)\left(2-2x+x^2\right)} = \frac{4-2x}{2-2x+x^2} - \frac{2x}{2-x^2}.\text{ (Do NOT prove this.)}$$

Use this result to evaluate $\displaystyle\int_0^1 \frac{8(1-x)}{\left(2-x^2\right)\left(2-2x+x^2\right)}\,dx.$

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Find real numbers a and b such that $(1+2i)(1-3i) = a+ib$. **2**

(b) (i) Write $\dfrac{1+i\sqrt{3}}{1+i}$ in the form $x+iy$, where x and y are real. **2**

(ii) By expressing both $1+i\sqrt{3}$ and $1+i$ in modulus-argument form, write $\dfrac{1+i\sqrt{3}}{1+i}$ in modulus-argument form. **3**

(iii) Hence find $\cos\dfrac{\pi}{12}$ in surd form. **1**

(iv) By using the result of part (ii), or otherwise, calculate $\left(\dfrac{1+i\sqrt{3}}{1+i}\right)^{12}$. **1**

(c) The point P on the Argand diagram represents the complex number $z = x+iy$ which satisfies **2**

$$z^2 + \overline{z}^2 = 8.$$

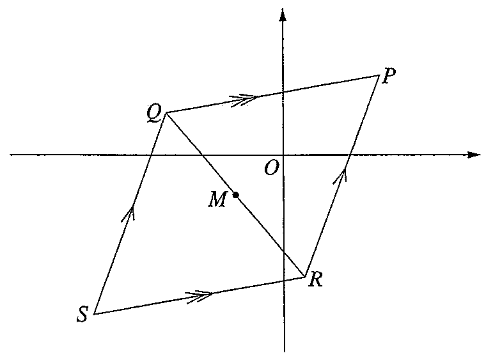
Find the equation of the locus of P in terms of x and y. What type of curve is the locus?

Question 2 continues on page 5

Marks

Question 2 (continued)

(d)



The point P on the Argand diagram represents the complex number z. The points Q and R represent the points ωz and $\bar{\omega} z$ respectively, where $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. The point M is the midpoint of QR.

(i) Find the complex number representing M in terms of z. **2**

(ii) The point S is chosen so that $PQSR$ is a parallelogram. **2**

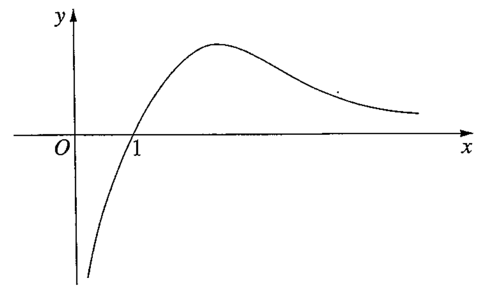
Find the complex number represented by S.

End of Question 2

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The following diagram shows the graph of $y = g(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = |g(x)|$ **1**

(ii) $y = \frac{1}{g(x)}$ **2**

(iii) $y = f(x)$, where **2**

$$f(x) = \begin{cases} g(x) & \text{for } x \geq 1 \\ g(2-x) & \text{for } x < 1. \end{cases}$$

(b) Let $p(z) = 1 + z^2 + z^4$.

(i) Show that $p(z)$ has no real zeros. **1**

Let α be a zero of $p(z)$.

(ii) Show that $\alpha^6 = 1$. **1**

(iii) Show that α^2 is also a zero of $p(z)$. **1**

Question 3 continues on page 7

Question 3 (continued) **Marks**

(c) For $n \geq 0$, let

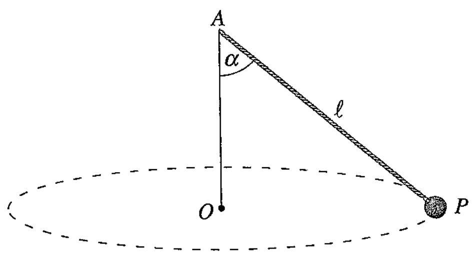
$$I_n = \int_0^{\frac{\pi}{4}} \tan^{2n}\theta \, d\theta.$$

(i) Show that for $n \geq 1$, **2**

$$I_n = \frac{1}{2n-1} - I_{n-1}.$$

(ii) Hence, or otherwise, calculate I_3. **2**

(d) **3**



A particle P of mass m is attached by a string of length ℓ to a point A. The particle moves with constant angular velocity ω in a horizontal circle with centre O which lies directly below A. The angle the string makes with OA is α.

The forces acting on the particle are the tension, T, in the string and the force due to gravity, mg.

By resolving the forces acting on the particle in the horizontal and vertical directions, show that

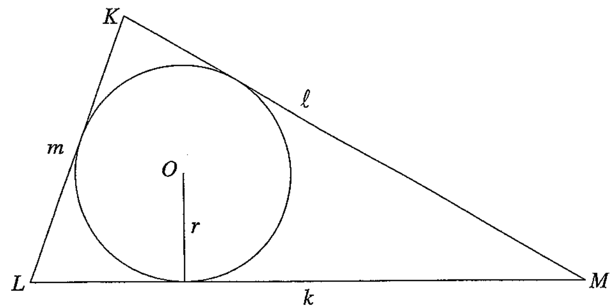
$$\omega^2 = \frac{g}{\ell \cos\alpha}.$$

End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a)



The diagram shows a circle, centre O and radius r, which touches all three sides of ΔKLM.

Let $LM = k$, $MK = \ell$, and $KL = m$.

(i) Write down an expression for the area of ΔLOM. **1**

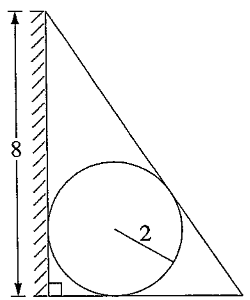
(ii) Let P be the perimeter of ΔKLM. Show that A, the area of ΔKLM, is given by **1**

$$A = \frac{1}{2}Pr.$$

Question 4 continues on page 9

Marks

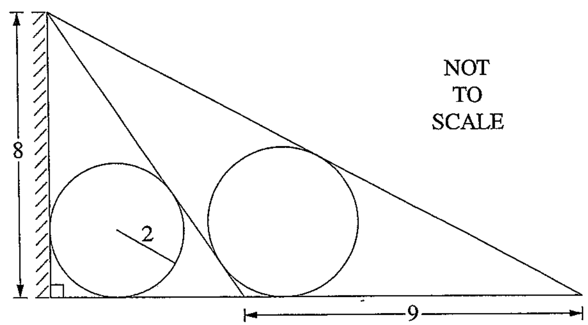
(iii) **3**



A wheel of radius 2 units rests against a fence of height 8 units. A thin straight board leans against the wheel with one end at the top of the fence and the other on the ground.

Using the result of part (ii), or otherwise, find how far from the foot of the fence the board touches the ground.

(iv) **3**



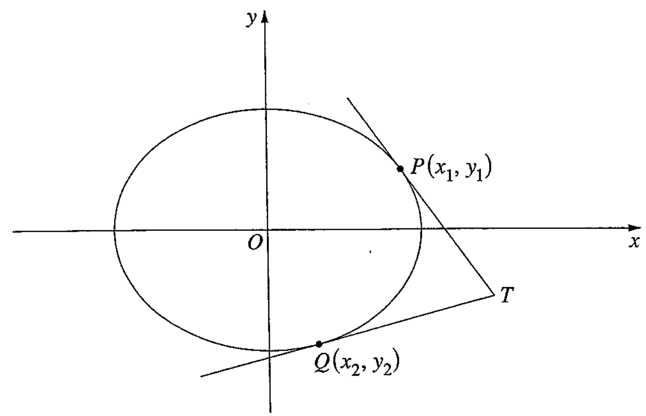
A second wheel rests on the ground, touching the board. A second thin straight board leans against the top of the fence and this second wheel. This board touches the ground 9 units further from the foot of the fence than the first board.

Find the radius of the second wheel.

Question 4 continues on page 10

Marks

(b)



The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

The tangents at P and Q meet at T.

(i) Show that the equation of the tangent at P is $\dfrac{x_1}{a^2}x + \dfrac{y_1}{b^2}y = 1$. **2**

(ii) Show that T lies on the line $\dfrac{(x_1 - x_2)}{a^2}x + \dfrac{(y_1 - y_2)}{b^2}y = 0$. **2**

(iii) Let M be the midpoint of PQ. **3**

Show that O, M and T are collinear.

End of Question 4

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Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) A model for the population, P, of elephants in Serengeti National Park is

$$P = \frac{21\,000}{7 + 3e^{-\frac{t}{3}}}$$

where t is the time in years from today.

(i) Show that P satisfies the differential equation **2**

$$\frac{dP}{dt} = \frac{1}{3}\left(1 - \frac{P}{3000}\right)P.$$

(ii) What is the population today? **1**

(iii) What does the model predict that the eventual population will be? **1**

(iv) What is the annual percentage rate of growth today? **1**

(b) Let $p(x) = x^{n+1} - (n+1)x + n$ where n is a positive integer.

(i) Show that $p(x)$ has a double zero at $x = 1$. **2**

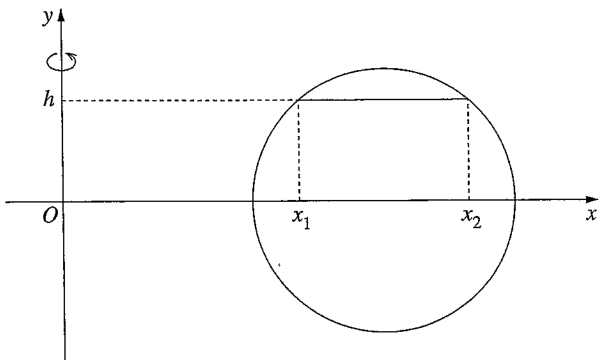
(ii) By considering concavity, or otherwise, show that $p(x) \geq 0$ for $x \geq 0$. **1**

(iii) Factorise $p(x)$ when $n = 3$. **2**

Question 5 continues on page 13

Question 5 (continued)

(c) Let a and b be constants, with $a > b > 0$. A torus is formed by rotating the circle $(x-a)^2 + y^2 = b^2$ about the y-axis.



The cross-section at $y = h$, where $-b \le h \le b$, is an annulus. The annulus has inner radius x_1 and outer radius x_2 where x_1 and x_2 are the roots of

$$(x-a)^2 = b^2 - h^2.$$

(i) Find x_1 and x_2 in terms of h. **1**

(ii) Find the area of the cross-section at height h, in terms of h. **2**

(iii) Find the volume of the torus. **2**

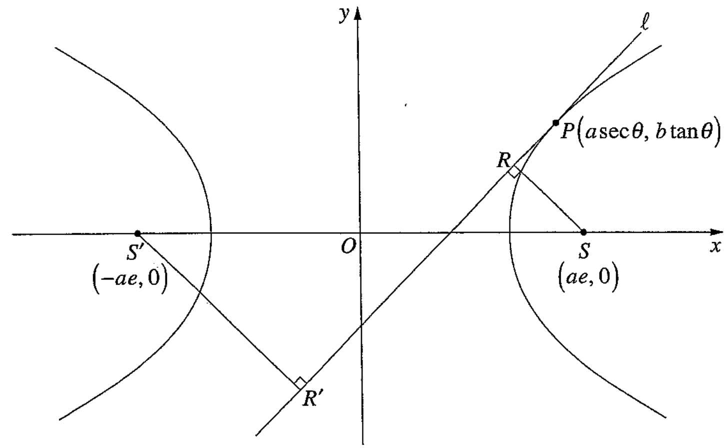
End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) Let ω be the complex number satisfying $\omega^3 = 1$ and $\text{Im}(\omega) > 0$. The cubic polynomial, $p(z) = z^3 + az^2 + bz + c$, has zeros 1, $-\omega$ and $-\bar{\omega}$. **3**

Find $p(z)$.

(b)



Let $P(a\sec\theta, b\tan\theta)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a > 0$ and $b > 0$ as shown in the diagram. The foci of the hyperbola are S and S', and ℓ is the tangent at the point P.

The points R and R' lie on ℓ so that SR and $S'R'$ are perpendicular to ℓ.

(i) Show that the line ℓ has equation **2**

$$bx\sec\theta - ay\tan\theta - ab = 0.$$

(ii) Show that $SR = \dfrac{ab(e\sec\theta - 1)}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}}$. **1**

(iii) Show that $SR \times S'R' = b^2$. **3**

Question 6 continues on page 15

Question 6 (continued) **Marks**

(c) Suppose n and r are integers with $1 < r \le n$.

(i) Show that **3**

$$\frac{1}{\binom{n}{r}} = \frac{r}{r-1}\left[\frac{1}{\binom{n-1}{r-1}} - \frac{1}{\binom{n}{r-1}}\right].$$

(ii) Hence show that, if m is an integer with $m \ge r$, then **2**

$$\frac{1}{\binom{r}{r}} + \frac{1}{\binom{r+1}{r}} + \cdots + \frac{1}{\binom{m}{r}} = \frac{r}{r-1}\left[1 - \frac{1}{\binom{m}{r-1}}\right].$$

(iii) What is the limiting value of the sum **1**

$$\sum_{n=r}^{m} \frac{1}{\binom{n}{r}}$$

as m increases without bound?

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) An urn contains n red balls, n white balls and n blue balls. Three balls are drawn at random from the urn, one at a time, without replacement.

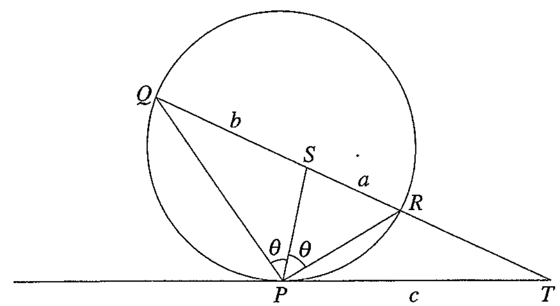
(i) What is the probability, p_s, that the three balls are all the same colour? **2**

(ii) What is the probability, p_d, that the three balls are all of different colours? **1**

(iii) What is the probability, p_m, that two balls are of one colour and the third is of a different colour? **1**

(iv) If n is large, what is the approximate ratio $p_s : p_d : p_m$? **1**

(b)



In the diagram, the points P, Q and R lie on a circle. The tangent at P and the secant QR intersect at T. The bisector of $\angle QPR$ meets QR at S so that $\angle QPS = \angle RPS = \theta$. The intervals RS, SQ and PT have lengths a, b and c respectively.

(i) Show that $\angle TSP = \angle TPS$. **2**

(ii) Hence show that $\dfrac{1}{a} = \dfrac{1}{b} + \dfrac{1}{c}$. **2**

Question 7 continues on page 17

Question 7 (continued)

(c) A fishing boat drifts with a current in a straight line across a fishing ground.

The boat's velocity v, at time t after the start of this drift is given by

$$v = b - (b - v_0)e^{-\alpha t},$$

where v_0, b and α are positive constants, and $v_0 < b$.

(i) Show that $\dfrac{dv}{dt} = \alpha(b - v)$. **1**

(ii) The physical significance of v_0 is that it represents the initial velocity of the boat. **1**

What is the physical significance of b?

(iii) Let x be the distance travelled by the boat from the start of the drift. **3**

Find x as a function of t. Hence show that

$$x = \frac{b}{\alpha}\log_e\left(\frac{b - v_0}{b - v}\right) + \frac{v_0 - v}{\alpha}.$$

(iv) The initial velocity of the boat is $\dfrac{b}{10}$. **1**

How far has the boat drifted when $v = \dfrac{b}{2}$?

End of Question 7

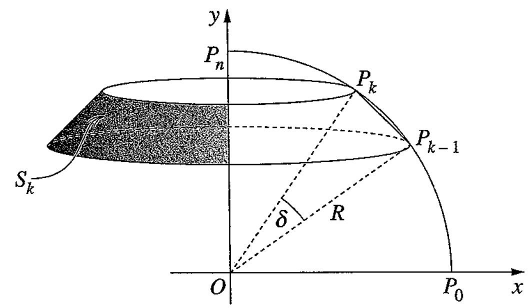
Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) It is given that $2\cos A\sin B = \sin(A + B) - \sin(A - B)$. (Do NOT prove this.) **3**

Prove by induction that, for integers $n \geq 1$,

$$\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}.$$

(b)



In the diagram, the points $P_0, P_1, \cdots, P_n$, are equally spaced points in the first quadrant on the circular arc of radius R and centre O. The point P_0 is $(R, 0)$, P_n is $(0, R)$ and $\angle P_{k-1}OP_k = \delta$ for $k = 1, \cdots, n$.

Each of the intervals $P_{k-1}P_k$ is rotated about the y-axis to form S_k, a part of a cone.

The area, A_k, of S_k is given by

$$A_k = 2\pi R^2 \sin\delta\cos\frac{(2k-1)\delta}{2}. \text{ (Do NOT prove this.)}$$

Let S be the surface formed by all of the S_k.

(i) Write down an expression for the area, A, of S. **4**

By using the result of part (a), or otherwise, find an expression for A in terms of n and R only.

(ii) Find the limiting value of A as n increases without bound. **1**

Question 8 continues on page 19

Marks

Question 8 (continued)

(c) Let $f(t) = \sin(a + nt)\sin b - \sin a \sin(b - nt)$, where a, b and n are constants with $a > 0$, $b > 0$, $a + b < \pi$ and $n \neq 0$.

(i) Show that **3**

$$f''(t) = -n^2 f(t) \text{ and } f(0) = 0.$$

(ii) Hence, or otherwise, show that **2**

$$f(t) = \sin(a + b)\sin nt.$$

(iii) Find all values of t for which **2**

$$\frac{\sin(a + nt)}{\sin(b - nt)} = \frac{\sin a}{\sin b}.$$

End of paper

2008 Higher School Certificate Solutions
Mathematics Extension 2

Question 1

(a) Let $u = 5 + x^3$, $du = 3x^2\,dx$

$$\int \frac{x^2}{(5+x^3)^2}\,dx = \int \frac{1}{u^2} \times \frac{1}{3}\,du$$
$$= \frac{1}{3}\int u^{-2}\,du$$
$$= -\frac{1}{3}u^{-1} + c$$
$$= \frac{-1}{3(5+x^3)} + c.$$

(b) *METHOD 1*

$$\int \frac{dx}{\sqrt{4x^2+1}} = \int \frac{dx}{2\sqrt{x^2 + \left(\frac{1}{2}\right)^2}}$$
$$= \frac{1}{2}\ln\left(x + \sqrt{x^2 + \frac{1}{4}}\right) + c$$
$$= \frac{1}{2}\ln\left(x + \frac{1}{2}\sqrt{4x^2+1}\right) + c$$

METHOD 2

Let $u = 2x$, $du = 2dx$

$$\int \frac{dx}{\sqrt{4x^2+1}} = \int \frac{dx}{\sqrt{(2x)^2 + 1^2}}$$
$$= \frac{1}{2}\int \frac{du}{\sqrt{u^2+1^2}}$$
$$= \frac{1}{2}\ln\left(u + \sqrt{u^2+1^2}\right) + c_1$$
$$= \frac{1}{2}\ln\left(2x + \sqrt{4x^2+1}\right) + c_1$$

which is equivalent $\left(c_1 + \frac{1}{2}\ln 2 = c\right)$.

(c) Let $u = \tan^{-1} x$, $du = \frac{1}{1+x^2}\,dx$

$v = x$, $dv = dx$

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\frac{du}{dx}\,dx$$
$$\int_0^1 \tan^{-1} x\,dx = \left[x\tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$
$$= \left[x\tan^{-1} x\right]_0^1 - \left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^1$$
$$= \left(\tan^{-1}1 - 0\right) - \left(\frac{1}{2}\ln 2 - \frac{1}{2}\ln 1\right)$$
$$= \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

(d) Let $u = \sqrt{2x-1}$, $du = \frac{dx}{\sqrt{2x-1}}$

or $x = \frac{1}{2}\left(u^2+1\right)$, $dx = u\,du$

$x = 1$, $u = 1$ and $x = 2$, $u = \sqrt{3}$

$$\int_1^2 \frac{dx}{x\sqrt{2x-1}} = \int_1^{\sqrt{3}} \frac{\not{u}\,du}{\frac{1}{2}\left(1+u^2\right)\not{u}}$$
$$= 2\left[\tan^{-1} u\right]_1^{\sqrt{3}}$$
$$= 2\left(\tan^{-1}\sqrt{3} - \tan^{-1}1\right)$$
$$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$
$$= \frac{\pi}{6}.$$

(e)
$$\int_0^1 \frac{8(1-x)}{\left(2-x^2\right)\left(2-2x+x^2\right)}\,dx$$
$$= \int_0^1 \left(\frac{4-2x}{2-2x+x^2} - \frac{2x}{2-x^2}\right)dx$$
$$= \int_0^1 \left(\frac{2}{(x-1)^2+1} - \frac{2x-2}{x^2-2x+2} - \frac{2x}{2-x^2}\right)dx$$
$$= \left[2\tan^{-1}(x-1) - \ln\left(x^2-2x+2\right) + \ln\left(2-x^2\right)\right]_0^1$$
$$= \left(2\tan^{-1}0 - \ln 1 + \ln 1\right) - \left(2\tan^{-1}(-1) - \ln 2 + \ln 2\right)$$
$$= 0 - 2\left(-\frac{\pi}{4}\right)$$
$$= \frac{\pi}{2}.$$

Note: In the fourth line we need $\ln\left(2-x^2\right)$ or possibly $\ln\left|x^2-2\right|$, but not $\ln\left(x^2-2\right)$ as this logarithm is not defined for $0 \le x \le 1$.

Question 2

(a)
$$(1+2i)(1-3i) = a + ib$$
$$1 - 3i + 2i + 6 = a + ib$$
$$7 - i = a + ib$$

Equating real and imaginary parts,

$$a = 7$$
$$b = -1.$$

(b) (i)
$$\frac{1+i\sqrt{3}}{1+i} = \frac{1+i\sqrt{3}}{1+i} \times \frac{1-i}{1-i}$$
$$= \frac{1 - i + i\sqrt{3} - i^2\sqrt{3}}{1-i^2}$$
$$= \frac{\left(1+\sqrt{3}\right) + \left(\sqrt{3}-1\right)i}{2}$$
$$= \frac{1+\sqrt{3}}{2} + \frac{\sqrt{3}-1}{2}i.$$

(ii)
$$1 + i\sqrt{3} = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$= 2\operatorname{cis}\frac{\pi}{3}$$
$$1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$
$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$= \sqrt{2}\operatorname{cis}\frac{\pi}{3}$$
$$\therefore \frac{1+i\sqrt{3}}{1+i} = \frac{2\operatorname{cis}\frac{\pi}{4}}{\sqrt{2}\operatorname{cis}\frac{\pi}{4}}$$
$$= \sqrt{2}\operatorname{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$
$$= \sqrt{2}\operatorname{cis}\frac{\pi}{12}.$$

(iii) By results of (i) and (ii),

$$\sqrt{2}\operatorname{cis}\frac{\pi}{12} = \frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}i$$
$$\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) = \frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}i$$

Equating real parts,

$$\sqrt{2}\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2}$$
$$\therefore \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$
$$= \frac{\left(\sqrt{3}+1\right)\sqrt{2}}{4}$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}.$$

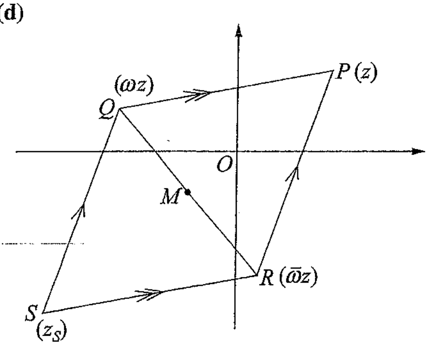
(iv) $\dfrac{1+i\sqrt{3}}{1+i}=\sqrt{2}\,\text{cis}\,\dfrac{\pi}{12}$ from (ii)

$$\therefore \left(\frac{1+i\sqrt{3}}{1+i}\right)^{12}=\left(\sqrt{2}\,\text{cis}\,\frac{\pi}{12}\right)^{12}$$
$$=\sqrt{2}^{12}\,\text{cis}\left(\frac{12\pi}{12}\right)$$
$$=2^6\,\text{cis}\,\pi$$
$$=-64.$$

(c)
$$z^2+\bar{z}^2=8$$
$$(x+iy)^2+(x-iy)^2=8$$
$$x^2+2ixy-y^2+x^2-2ixy-y^2=8$$
$$2\left(x^2-y^2\right)=8$$
$$x^2-y^2=4$$

$\therefore$ The locus of P is a rectangular hyperbola.

(d)



(i) M corresponds to the complex number

$$z_M=\frac{1}{2}(\omega z+\bar{\omega}z)$$
$$=\frac{z}{2}(\omega+\bar{\omega})$$
$$=\frac{z}{2}(2\,\mathrm{R_e}(\omega))$$
$$=z\,\mathrm{R_e}(\omega)$$
$$=z\cos\frac{2\pi}{3}$$
$$=-\frac{z}{2}.$$

(ii) *METHOD 1*

Since $PQSR$ is a parallelogram, the diagonals bisect each other.

$\therefore$ M is the midpoint of PS.

$$\therefore z_M=\frac{1}{2}(z_S+z)$$
$$2z_M=z_S+z$$
$$z_S=2z_M-z$$
$$=2\left(-\frac{z}{2}\right)-z \quad \text{from (i)}$$
$$=-z-z$$
$$=-2z.$$

METHOD 2

Since $PQSR$ is a parallelogram,

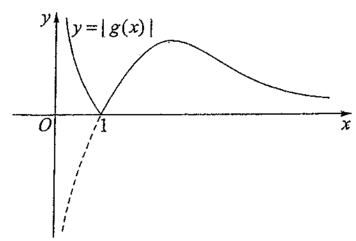
$$\overrightarrow{SR}=\overrightarrow{QP}$$
$$\bar{\omega}z-z_S=z-\omega z$$
$$z_S=\bar{\omega}z+\omega z-z$$
$$=z(\bar{\omega}+\omega-1)$$

Using

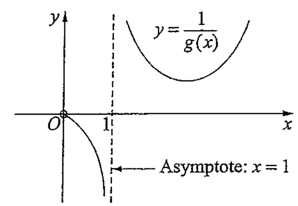
$$\omega=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},$$
$$\therefore z_S=z\left(2\cos\frac{2\pi}{3}-1\right)$$
$$=z(-1-1)$$
$$=-2z.$$

Question 3

(a) (i)



(ii)



Note: Since there is no scale on the y-axis it is not possible to say where the y-values are in relation to the original graph.

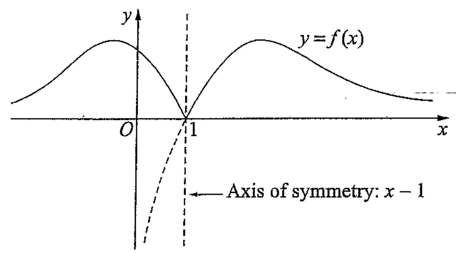
As $x\to 1^+, g(x)\to 0^+, \dfrac{1}{g(x)}\to+\infty$

As $x\to 1^-, g(x)\to 0^-, \dfrac{1}{g(x)}\to-\infty$

As $x\to 0^+, g(x)\to-\infty, \dfrac{1}{g(x)}\to 0^-$

As $x\to+\infty, g(x)\to 0^+, \dfrac{1}{g(x)}\to+\infty.$

(iii) $g(a-x)$ is a reflection of $g(x)$ about $x=\dfrac{a}{2}$, so $g(2-x)$ is the reflection of $g(x)$ about $x=1$.



(b) (i) $p(z)=1+z^2+z^4$

For all real z, $z^2+z^4\geq 0$,

so $p(z)\geq 1$.

Hence $p(z)\neq 0$ for any real z,

so $p(z)$ has no real zeros.

(ii) If α is a zero,

$$p(\alpha)=1+\alpha^2+\alpha^4=0$$

Hence
$$\alpha^6-1=\left(\alpha^2-1\right)\left(\alpha^4+\alpha^2+1\right)$$
$$=\left(\alpha^2-1\right)\times 0$$
$$=0$$
$$\therefore \alpha^6=1.$$

(iii)
$$p\left(\alpha^2\right)=1+\left(\alpha^2\right)^2+\left(\alpha^2\right)^4$$
$$=1+\alpha^4+\alpha^8$$
$$=1+\alpha^4+\alpha^2\times\alpha^6$$
$$=1+\alpha^4+\alpha^2 \quad \text{since } \alpha^6=1$$
$$=0$$

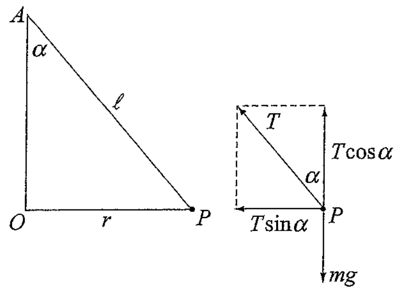
Hence α^2 is also a zero.

(c) (i)
$$I_n=\int_0^{\frac{\pi}{4}}\tan^{2n}\theta\,d\theta \quad \text{for } n\geq 0$$
$$=\int_0^{\frac{\pi}{4}}\tan^{2n-2}\theta\tan^2\theta\,d\theta$$
$$=\int_0^{\frac{\pi}{4}}\tan^{2n-2}\theta\left(\sec^2\theta-1\right)d\theta$$
$$=\int_0^{\frac{\pi}{4}}\tan^{2n-2}\theta\sec^2\theta\,d\theta-\int_0^{\frac{\pi}{4}}\tan^{2n-2}\theta\,d\theta$$
$$=\left[\frac{\tan^{2n-1}\theta}{2n-1}\right]_0^{\frac{\pi}{4}}-I_{n-1} \quad \text{for } n\geq 1$$
$$=\left(\frac{\left(\tan\frac{\pi}{4}\right)^{2n-1}}{2n-1}-0\right)-I_{n-1}$$
$$=\frac{1}{2n-1}-I_{n-1}.$$

(ii) $I_0 = \int_0^{\frac{\pi}{4}} d\theta = \frac{\pi}{4}$

$$I_3 = \frac{1}{5} - I_2 \quad \text{from (i)}$$
$$= \frac{1}{5} - \left(\frac{1}{3} - I_1\right)$$
$$= \frac{1}{5} - \frac{1}{3} + (1 - I_0)$$
$$= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}$$
$$= \frac{13}{15} - \frac{\pi}{4}.$$

(d)



Vertical: $T\cos\alpha - mg = 0$

$$T\cos\alpha = mg \quad \text{—①}$$

Horizontal: $T\sin\alpha = m\omega^2 r$ —②

and $\sin\alpha = \frac{r}{\ell}$ —③

Substituting ③ into ②,

$$\frac{Tr}{\ell} = m\omega^2 r$$
$$T = m\omega^2 \ell \quad \text{—④}$$

Substituting ④ into ①,

$$m\omega^2 \ell \cos\alpha = mg$$
$$\omega^2 = \frac{g}{\ell\cos\alpha}.$$

Question 4

(a) (i) $\Delta LOM = \frac{1}{2}kr.$

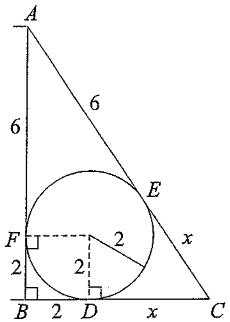
(ii) Similarly to (i),

$\Delta KOM = \frac{1}{2}\ell r$ and $\Delta KOL = \frac{1}{2}mr$

$\therefore$ Area of KLM

$$= \Delta LOM + \Delta KOM + \Delta KOL$$
$$= \frac{1}{2}kr + \frac{1}{2}\ell r + \frac{1}{2}mr$$
$$= \frac{1}{2}(k + \ell + m)r$$
$$= \frac{1}{2}Pr.$$

(iii) *METHOD 1*



Let CD be x.
Tangents from an external point to a circle are equal.

$\therefore CE = CD = x$
$BF = BD = 2$
$AF = AE = 6$

Perimeter of ΔABC is

$$P = 8 + (x+2) + (x+6)$$
$$= 2x + 16$$

Now $\Delta ABC = \frac{1}{2}(8)(x+2)$

$$= 4(x+2)$$

By result of (ii),

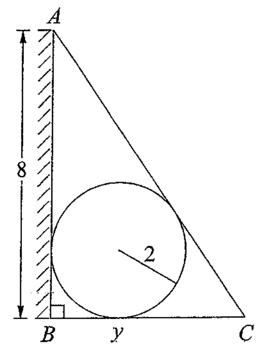
$$4(x+2) = \frac{1}{2}(2x+16)2$$
$$4x + 8 = 2x + 16$$
$$2x = 8$$
$$x = 4$$
$$BC = 6 \text{ units.}$$

METHOD 2

From Method 1,
$CE = CD = x$
$BF = BD = 2$
$AF = AE = 6$

$$AC^2 = AB^2 + BC^2$$
$$(x+6)^2 = 8^2 + (x+2)^2$$
$$x^2 + 12x + 36 = 64 + x^2 + 4x + 4$$
$$8x = 32$$
$$x = 4$$
$$\therefore BC = 6 \text{ units.}$$

METHOD 3



Let BC be y.

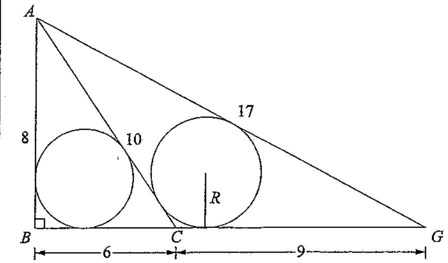
$$AC = \sqrt{y^2 + 8^2}$$
$$\therefore P = 8 + y + \sqrt{y^2 + 8^2}$$

By result of (ii),

Now $\Delta ABC = \frac{1}{2} \times 8y = 4y$

$$4y = \frac{1}{2}\left(8 + y + \sqrt{y^2 + 8^2}\right)2$$
$$4y = 8 + y + \sqrt{y^2 + 8^2}$$
$$3y - 8 = \sqrt{y^2 + 8^2}$$
$$(3y-8)^2 = y^2 + 8^2$$
$$9y^2 - 48y + 64 = y^2 + 64$$
$$8y^2 - 48y = 0$$
$$8y(y-6) = 0$$
$$y = 6 \quad (y = 0 \text{ rejected})$$
$$\therefore BC = 6 \text{ units.}$$

(iv)



$$AC = \sqrt{8^2 + 6^2}$$
$$= 10$$
$$AG = \sqrt{8^2 + 15^2}$$
$$= 17$$

$\therefore$ Perimeter of ΔACG is

$$P = 10 + 9 + 17$$
$$= 36$$

Now $\Delta ACG = \frac{1}{2}(8)(9)$

$$= 36$$

By result of (ii)

$$\frac{1}{2}(36)R = 36$$
$$R = 2 \text{ units.}$$

(b) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

By implicit differentiation,

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$
$$\therefore \frac{dy}{dx} = -\frac{2x}{a^2} \times \frac{b^2}{2y}$$
$$= -\frac{b^2 x}{a^2 y}$$

At $P(x_1, y_1)$,

$$\frac{dy}{dx} = -\frac{b^2 x_1}{a^2 y_1}$$
$$= \text{gradient at } P$$

Hence equation of the tangent at P is

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$a^2y_1y-a^2y_1^2=-b^2x_1x+b^2x_1^2$$

$$b^2x_1x+a^2y_1y=b^2x_1^2+a^2y_1^2$$

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1 \quad \text{—①}$$

(since P lies on the ellipse).

(ii) Similarly to (i), the tangent at Q is

$$\frac{x_2x}{a^2}+\frac{y_2y}{b^2}=1 \quad \text{—②}$$

Since T lies on the tangents at P and Q, T must lie on the line

① = ②

① − ② = 0

$$\frac{(x_1-x_2)x}{a^2}+\frac{(y_1-y_2)y}{b^2}=0$$

i.e. $$\frac{(x_1-x_2)x}{a^2}+\frac{(y_1-y_2)y}{b^2}=0. \text{—③}$$

(iii) M is the point $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$.

At M, substituting into ③,

$$\frac{(x_1-x_2)x}{a^2}+\frac{(y_1-y_2)y}{b^2}$$

$$=\frac{x_1-x_2}{a^2}\cdot\frac{x_1+x_2}{2}+\frac{y_1-y_2}{b^2}\cdot\frac{y_1+y_2}{2}$$

$$=\frac{x_1^2-x_2^2}{2a^2}+\frac{y_1^2-y_2^2}{2b^2}$$

$$=\frac{1}{2}\left[\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)-\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}\right)\right]$$

$$=\frac{1}{2}[1-1]$$

$$=0$$

$\therefore M$ lies on the line

$$\frac{(x_1-x_2)x}{a^2}+\frac{(y_1-y_2)y}{b^2}=0$$

Clearly, line ③ passes through origin O, and T lies on line ③ by result of (ii).

Hence, O, M and T are collinear.

Question 5

(a) $$P=\frac{21\,000}{7+3e^{-\frac{t}{3}}}$$

(i) For the differential equation given,

$$\text{LHS}=\frac{dP}{dt}$$

$$=\frac{-21000}{\left(7+3e^{-\frac{t}{3}}\right)^2}\times\left(-\frac{1}{3}\times 3e^{-\frac{t}{3}}\right)$$

$$=\frac{21\,000e^{-\frac{t}{3}}}{\left(7+3e^{-\frac{t}{3}}\right)^2}$$

$$\text{RHS}=\frac{1}{3}\left(1-\frac{P}{3000}\right)P$$

$$=\frac{1}{3}\left(1-\frac{1}{7+3e^{-\frac{t}{3}}}\right)\times\frac{21\,000}{7+3e^{-\frac{t}{3}}}$$

$$=\frac{1}{3}\cdot\frac{\left(7+3e^{-\frac{t}{3}}-7\right)\times 21\,000}{\left(7+3e^{-\frac{t}{3}}\right)^2}$$

$$=\frac{e^{-\frac{t}{3}}\times 21\,000}{\left(7+3e^{-\frac{t}{3}}\right)^2}$$

LHS = RHS as required.

(ii) At $t=0$,

$$P=\frac{21\,000}{7+2\times 1}$$

$$=2100$$

$\therefore$ The population today is 2100.

(iii) As $t\to\infty$, $e^{-\frac{t}{3}}\to 0$

So $$P\to\frac{21\,000}{7+3\times 0}$$

$$=3000$$

$\therefore$ The population will eventually approach 3000.

(iv) When $t=0$,

$$\frac{dP}{dt}=\frac{1}{3}\left(1-\frac{2100}{3000}\right)2100$$

$$=210$$

$\therefore$ Annual percentage growth rate today

$$=\frac{210}{2100}\times 100\%$$

$$=10\%.$$

(b) (i) $p(x)=x^{n+1}-(n+1)x+n$

$$p(1)=1-(n+1)+n$$

$$=0$$

$$p'(x)=(n+1)x^n-(n+1)$$

$$p'(1)=(n+1)-(n+1)$$

$$=0$$

Hence $p(x)$ has a double zero at $x=1$.

(ii) $p''(x)=n(n+1)x^{n-1}$

For $x>0$, $p''(x)>0$, so the curve is concave up for $x>0$.

From (i), $p(x)$ has a stationary point at $(1,0)$, so this is a minimum.

Hence $p(x)\geq 0$ for $x\geq 0$.

(iii) For $n=3$,

$$p(x)=x^4-4x+3$$

$$=(x-1)^2\left(ax^2+bx+c\right)$$

$$=\left(x^2-2x+1\right)\left(ax^2+bx+c\right)$$

Coefficient of x^4: $1=a$

Constant term: $3=c$

Coefficient of x: $-4=b-2c$

$\therefore b=2$

$$\therefore p(x)=(x-1)^2\left(x^2+2x+3\right).$$

(c) (i) $(x-a)^2=b^2-h^2$

$$x=a\pm\sqrt{b^2-h^2}$$

$$\therefore x_1=a-\sqrt{b^2-h^2}$$

$$x_2=a+\sqrt{b^2-h^2}.$$

(ii) $A=\pi\left(x_2^2-x_1^2\right)$

$$=\pi(x_2+x_1)(x_2-x_1)$$

$$=\pi\times 2a\times\left(2\sqrt{b^2-h^2}\right)$$

$$=4\pi a\sqrt{b^2-h^2}.$$

(iii) $\Delta V=A\Delta h$

$$V=\lim_{\Delta h\to 0}\sum_{h=-b}^{b}4\pi a\sqrt{b^2-h^2}\,\Delta h$$

$$=\int_{-b}^{b}4\pi a\sqrt{b^2-h^2}\,dh$$

$$=4\pi a\int_{-b}^{b}\sqrt{b^2-h^2}\,dh$$

The integral is the area of a semicircle of radius b.

$$\therefore V=4\pi a\times\left(\frac{1}{2}\pi b^2\right)$$

$$=2\pi^2ab^2.$$

Question 6

(a) $p(z) = z3 + az2 + bz + c$

METHOD 1

Since ω is the complex cube root of unity and $\text{Im}(\omega) > 0$,

$$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$
$$\bar{\omega} = \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$$

Hence $\omega + \bar{\omega} = 2\cos\frac{2\pi}{3}$

$= -1$

and $\omega\bar{\omega} = |\omega|^2$

$= 1$

$$\therefore p(z) = (z-1)(z+\omega)(z+\bar{\omega})$$
$$= (z-1)\left[z^2 + (\omega+\bar{\omega})z + \omega\bar{\omega}\right]$$
$$= (z-1)\left(z^2 - z + 1\right)$$
$$= z^3 - 2z^2 + 2z - 1.$$

METHOD 2

From Method 1, $\omega + \bar{\omega} = -1$ and $\omega\bar{\omega} = 1$

Sum of roots

$$-a = 1 + (-\omega) + (-\bar{\omega})$$
$$= 1 - (\omega + \bar{\omega})$$
$$= 1 - (-1)$$
$$= 2$$
$$\therefore a = -2$$

Sum of roots taken two at a time

$$b = 1\times(-\omega) + 1\times(-\bar{\omega}) + (-\omega)(-\bar{\omega})$$
$$= -(\omega + \bar{\omega}) + \omega\bar{\omega}$$
$$= -(-1) + 1$$
$$= 2$$

Product of roots

$$-c = 1\times(-\omega)\times(-\bar{\omega})$$
$$= \omega\bar{\omega}$$
$$= 1$$
$$\therefore c = -1$$

Hence $p(z) = z^3 - 2z^2 + 2z - 1.$

(b) (i) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Differentiating implicity,

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{2x}{a^2}\times\frac{b^2}{2y}$$
$$= \frac{b^2x}{a^2y}$$

At $P(a\sec\theta, b\tan\theta)$,

$$\frac{dy}{dx} = \frac{ab^2\sec\theta}{a^2b\tan\theta}$$
$$= \frac{b\sec\theta}{a\tan\theta}$$

$\therefore$ Equation of tangent at P is

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$
$$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$$
$$bx\sec\theta - ay\tan\theta = ab\sec^2\theta - ab\tan^2\theta$$
$$= ab\left(\sec^2\theta - \tan^2\theta\right)$$
$$= ab$$

i.e. $bx\sec\theta - ay\tan\theta - ab = 0.$

(ii) Perpendicular distance

$$SR = \frac{|b.ae\sec\theta - 0 - ab|}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}$$
$$= \frac{ab|e\sec\theta - 1|}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}}$$

as $a > 0,\ b > 0$

$$= \frac{ab(e\sec\theta - 1)}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}}$$

as $e > 0,\ \sec\theta > 1.$

(iii) $$S'R' = \frac{|b(-ae)\sec\theta - 0 - ab|}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}$$
$$= \frac{|-ab(e\sec\theta + 1)|}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}}$$

as $a > 0,\ b > 0$

$$= \frac{ab(e\sec\theta + 1)}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}}$$

as $e > 0,\ \sec\theta > 1$

$$\therefore SR\times S'R' = \frac{a^2b^2(e\sec\theta - 1)(e\sec\theta + 1)}{a^2\tan^2\theta + b^2\sec^2\theta}$$
$$= \frac{a^2b^2\left(e^2\sec^2\theta - 1\right)}{a^2\tan^2\theta + b^2\sec^2\theta}$$
$$= \frac{a^2b^2\left(e^2\sec^2\theta - 1\right)}{a^2\tan^2\theta + a^2\left(e^2 - 1\right)\sec^2\theta}$$

as $b^2 = a^2\left(e^2 - 1\right)$

$$= \frac{a^2b^2\left(e^2\sec^2\theta - 1\right)}{a^2\left[e^2\sec^2\theta - \left(\sec^2\theta - \tan^2\right.\right.}$$
$$= \frac{b^2\left(e^2\sec^2\theta - 1\right)}{e^2\sec^2\theta - 1}$$
$$= b^2.$$

(c) (i) For the equation given,

$$\text{LHS} = \frac{1}{\binom{n}{r}}$$
$$= \frac{r!(n-r)!}{n!}$$

RHS

$$= \frac{r}{r-1}\left[\frac{1}{\binom{n-1}{r-1}} - \frac{1}{\binom{n}{r-1}}\right]$$
$$= \frac{r}{r-1}\left[\frac{(r-1)!(n-r)!}{(n-1)!} - \frac{(r-1)!(n-r+1)!}{n!}\right]$$
$$= \frac{r(r-1)!(n-r)!}{r-1}\left[\frac{1}{(n-1)!} - \frac{n-r+1}{n!}\right]$$
$$= \frac{r!(n-r)!}{r-1}\times\frac{n-(n-r+1)}{n!}$$
$$= \frac{r!(n-r)!}{r-1}\times\frac{r-1}{n!}$$
$$= \frac{r!(n-r)!}{n!}$$
$$\therefore \frac{1}{\binom{n}{r}} = \frac{r}{r-1}\left[\frac{1}{\binom{n-1}{r-1}} - \frac{1}{\binom{n}{r-1}}\right].$$

(ii) By result of (i),

$$\frac{1}{\binom{r}{r}} = \frac{r}{r-1}\left[\frac{1}{\binom{r-1}{r-1}} - \frac{1}{\binom{r}{r-1}}\right]$$
$$\frac{1}{\binom{r+1}{r}} = \frac{r}{r-1}\left[\frac{1}{\binom{r}{r-1}} - \frac{1}{\binom{r+1}{r-1}}\right]$$
$$\frac{1}{\binom{r+2}{r}} = \frac{r}{r-1}\left[\frac{1}{\binom{r+1}{r-1}} - \frac{1}{\binom{r+2}{r-1}}\right]$$

$\vdots$ $\qquad$ $\vdots$

$$\frac{1}{\binom{m}{r}} = \frac{r}{r-1}\left[\frac{1}{\binom{m-1}{r-1}} - \frac{1}{\binom{m}{r-1}}\right]$$

Adding the identities,

$$\frac{1}{\binom{r}{r}}+\frac{1}{\binom{r+1}{r}}+\ldots+\frac{1}{\binom{m}{r}}$$

$$=\frac{r}{r-1}\left[\frac{1}{\binom{r-1}{r-1}}-\frac{1}{\binom{m}{r-1}}\right]$$

$$=\frac{r}{r-1}\left[1-\frac{1}{\binom{m}{r-1}}\right].$$

(iii) $\lim_{n\to\infty}\sum_{n=r}^{m}\frac{1}{\binom{n}{r}}$

$=\lim_{n\to\infty}\frac{r}{r-1}\left[1-\frac{1}{\binom{m}{r-1}}\right]$ from (ii)

$=\frac{r}{r-1}$ as $\frac{1}{\binom{m}{r-1}}\to 0$ as $m\to\infty$.

Question 7

(a) (i) $p_s=\frac{n-1}{3n-1}\times\frac{n-2}{3n-2}$.

(ii) $p_d=\frac{2n}{3n-1}\times\frac{n}{3n-2}$.

(iii) $p_s+p_d+p_m=1$

$\therefore\ p_m=1-(p_s+p_d)$

$$=1-\left(\frac{(n-1)(n-2)}{(3n-1)(3n-2)}+\frac{2n^2}{(3n-1)(3n-2)}\right)$$

$$=\frac{(3n-1)(3n-2)-(n-1)(n-2)-2n^2}{(3n-1)(3n-2)}$$

$$=\frac{(9n^2-9n+2)-(n^2-3n+2)-2n^2}{(3n-1)(3n-2)}$$

$$=\frac{6n^2-6n}{(3n-1)(3n-2)}$$

$$=\frac{6n(n-1)}{(3n-1)(3n-2)}.$$

(iv) Using results from (i), (ii) and (iii),

$$\lim_{n\to\infty}p_s=\lim_{n\to\infty}\frac{1-\frac{1}{n}}{3-\frac{1}{n}}\times\frac{1-\frac{2}{n}}{3-\frac{2}{n}}$$

$$=\frac{1}{9}$$

$$\lim_{n\to\infty}p_d=\lim_{n\to\infty}\frac{2}{3-\frac{1}{n}}\times\frac{1}{3-\frac{2}{n}}$$

$$=\frac{2}{9}$$

$$\lim_{n\to\infty}p_m=\frac{6}{3-\frac{1}{n}}\times\frac{1-\frac{1}{n}}{3-\frac{2}{n}}$$

$$=\frac{6}{9}$$

So for large n,

$p_s:p_d:p_m \doteqdot 1:2:6.$

(b) (i) $\angle TSP=\angle SQP+\angle SPQ$
(exterior $\angle$ of a triangle equals sum of interior opposite $\angle$'s).
$=\angle SQP+\angle RPS$ (both θ)
$=\angle TPR+\angle RPS$
$=\angle TPS$
($\angle$ between a chord and tangent is equal to the $\angle$ in the alternate segment).

(ii) The square of the tangent is equal to the product of the secants, so

$$TR\times TQ=TP^2$$

From (i), ΔTSP is isosceles, so

$$TR=c-a$$
$$TQ=c+b$$
$$\therefore\ (c-a)(c+b)=c^2$$
$$c^2-ac+bc-ab=c^2$$
$$bc=ac+ab$$
$$\frac{bc}{abc}=\frac{ac}{abc}+\frac{ab}{abc}$$
$$\therefore \frac{1}{a}=\frac{1}{b}+\frac{1}{c}.$$

(c) (i) $v=b-(b-v_0)e^{-\alpha t}$

$$\frac{dv}{dt}=-(b-v_0)e^{-\alpha t}\times(-\alpha)$$
$$=\alpha(b-v_0)e^{-\alpha t}$$

But $b-v=(b-v_0)e^{-\alpha t}$

$$\therefore\ \frac{dv}{dt}=\alpha(b-v).$$

(ii) If $v<b$, $\frac{dv}{dt}>0$,

but if $v=b$, $\frac{dv}{dt}=0$.

So v is the limiting or terminal velocity of the boat.

As $t\to\infty$, $e^{-\alpha t}\to 0$ and $v\to b$.

(iii) $\frac{dx}{dt}=b-(b-v_0)e^{-\alpha t}$

$$x=bt-\frac{(b-v_0)e^{-\alpha t}}{-\alpha}+c$$
$$=bt+\frac{b-v_0}{\alpha}e^{-\alpha t}+c$$

At $t=0$, $x=0$

$$\therefore\ 0=0+\frac{b-v_0}{\alpha}+c$$
$$\therefore\ c=-\frac{b-v_0}{\alpha}$$
$$x=bt+\frac{b-v_0}{\alpha}\left(e^{-\alpha t}-1\right)$$
$$=bt-\frac{b-v_0}{\alpha}\left(1-e^{-\alpha t}\right)$$
$$e^{-\alpha t}=\frac{v-b}{-(b-v_0)}$$
$$=\frac{b-v}{b-v_0}$$
$$-\alpha t=\ln\left(\frac{b-v}{b-v_0}\right)$$
$$t=-\frac{1}{\alpha}\ln\left(\frac{b-v}{b-v_0}\right)$$
$$=-\frac{1}{\alpha}\ln\left(\frac{b-v_0}{b-v}\right)$$

So

$$x=\frac{b}{\alpha}\ln\left(\frac{b-v_0}{b-v}\right)-\frac{b-v_0}{\alpha}\left(1-\frac{b-v}{b-v_0}\right)$$
$$=\frac{b}{\alpha}\ln\left(\frac{b-v_0}{b-v}\right)-\frac{1}{\alpha}(b-v_0-b+v)$$
$$=\frac{b}{\alpha}\ln\left(\frac{b-v_0}{b-v}\right)+\frac{v_0-v}{\alpha}.$$

(iv) $v_0=\frac{b}{10}$ $\quad v=\frac{b}{2}$

$$x=\frac{b}{\alpha}\ln\left(\frac{b-\frac{b}{10}}{b-\frac{b}{2}}\right)+\frac{\frac{b}{10}-\frac{b}{2}}{\alpha}$$
$$=\frac{b}{\alpha}\ln\left(\frac{9}{10}\div\frac{1}{2}\right)+\frac{b}{\alpha}\times\left(-\frac{4}{10}\right)$$
$$=\frac{b}{\alpha}\left(\ln\left(\frac{9}{5}\right)-\frac{2}{5}\right).$$

Question 8

(a) When $n=1$,

LHS $=\cos\theta$

$$\text{RHS}=\frac{\sin 2\theta}{2\sin\theta}$$
$$=\frac{2\sin\theta\cos\theta}{2\sin\theta}$$
$$=\cos\theta$$

LHS = RHS

$\therefore$ It is true for $n=1$.

Assume the proposition is true for some positive integer $k\geq 1$,

i.e. $\cos\theta+\cos 3\theta+\ldots+\cos(2k-1)\theta$

$$=\frac{\sin 2k\theta}{2\sin\theta}$$

Then

$\cos\theta+\cos 3\theta+\ldots+\cos(2k-1)\theta$

$+\cos(2k+1)\theta$

$=\dfrac{\sin 2k\theta}{2\sin\theta}+\cos(2k+1)\theta$ by assumption

$=\dfrac{1}{2\sin\theta}\left[\sin 2k\theta+2\sin\theta\cos(2k+1)\theta\right]$

$=\dfrac{1}{2\sin\theta}\left[\sin 2k\theta+\sin(\theta+(2k+1)\theta)\theta\right.$

$\left.-\sin((2k+1)\theta-\theta)\right]$

$=\dfrac{1}{2\sin\theta}\left[\sin 2k\theta+\sin(2k+2)\theta-\sin 2k\theta\right]$

$=\dfrac{1}{2\sin\theta}\sin 2(k+1)\theta$

$=\dfrac{\sin 2(k+1)\theta}{2\sin\theta}$

$\therefore$ The proposition will be true for $n=k+1$ if it is true for $n=k$. Since it is proved true for $n=1$, it will be true for $n=2,3,4,\ldots$, i.e. true for all positive integers $n\geq 1$.

(b) (i) Given $A_k=2\pi R^2\sin\delta\cos(2k-1)\dfrac{\delta}{2}$,

$A=A_1+A_2+\ldots+A_n$

$=2\pi R^2\sin\delta\cos\dfrac{\delta}{2}+2\pi R^2\sin\delta\cos\dfrac{3\delta}{2}$

$+\ldots+2\pi R^2\sin\delta\cos(2n-1)\dfrac{\delta}{2}$

$=2\pi R^2\sin\delta\left[\cos\dfrac{\delta}{2}+\cos\dfrac{3\delta}{2}\right.$

$\left.+\ldots+\cos(2n-1)\dfrac{\delta}{2}\right]$

$=2\pi R^2\sin\delta\times\dfrac{\sin\dfrac{2n\delta}{2}}{2\sin\dfrac{\delta}{2}}$

$=2\pi R^2\sin\delta\,\dfrac{\sin(n\delta)}{2\sin\dfrac{\delta}{2}}$

$=\pi R^2\left(2\sin\dfrac{\delta}{2}\cos\dfrac{\delta}{2}\right)\dfrac{\sin(n\delta)}{\sin\left(\dfrac{\delta}{2}\right)}$

$=2\pi R^2\cos\dfrac{\delta}{2}\sin(n\delta)$

$=2\pi R^2\cos\dfrac{\pi}{2n}\sin\dfrac{\pi}{2}$ since $\delta=\dfrac{\pi}{2n}$

$=2\pi R^2\cos\dfrac{\pi}{2n}.$

(ii) $\lim\limits_{n\to\infty}A=\lim\limits_{n\to\infty}2\pi R^2\cos\dfrac{\pi}{2n}$

$=2\pi R^2\lim\limits_{n\to\infty}\cos\dfrac{\pi}{2n}$

$=2\pi R^2.$

(c) (i) $f(t)=\sin(a+nt)\sin b$

$-\sin a\sin(b-nt)$

$f'(t)=n\cos(a+nt)\sin b$

$+n\sin a\cos(b-nt)$

$f''(t)=n^2\sin(a+nt)\sin b$

$+n^2\sin a\sin(b-nt)$

$=-n^2f(t)$

and $f(0)=\sin a\sin b-\sin a\sin b$

$=0.$

(ii) *METHOD 1*

$f(t)=\sin(a+nt)\sin b-\sin a\sin(b-nt)$

$=(\sin a\cos nt+\sin nt\cos a)\sin b$

$-\sin a(\sin b\cos nt-\sin nt\cos b)$

$=\cos a\sin b\sin nt+\sin a\cos b\sin nt$

$=(\cos a\sin b+\sin a\cos b)\sin nt$

$=\sin(a+b)\sin nt.$

METHOD 2

From (i), $f''(t)=-n^2f(t)$

$\therefore$ It executes simple harmonic motion.

Hence $f(t)=A\sin(nt+\alpha)$

as $f(0)=0$

$\therefore 0=A\sin\alpha$

$\alpha=0$

$\therefore f(t)=A\sin nt$

$T=\dfrac{2\pi}{n}$

At the first extreme point,

$t=\dfrac{\frac{2\pi}{n}}{4}$

$=\dfrac{\pi}{2n}$

and $f\left(\dfrac{\pi}{2n}\right)=A$

Now $f\left(\dfrac{\pi}{2n}\right)=\sin\left(a+n.\dfrac{\pi}{2n}\right)\sin b$

$-\sin a\sin\left(b-n.\dfrac{\pi}{2n}\right)$

$=\sin\left(a+\dfrac{\pi}{2}\right)\sin b$

$-\sin a\sin\left(b-\dfrac{\pi}{2}\right)$

$=\cos a\sin b+\sin a\cos b$

$=\sin(a+b)$

$\therefore\quad A=\sin(a+b)$

$\therefore f(t)=\sin(a+b)\sin nt.$

(iii) $\dfrac{\sin(a+nt)}{\sin(b-nt)}=\dfrac{\sin a}{\sin b}$

$\sin b\sin(a+nt)=\sin a\sin(b-nt)$

$\sin(a+nt)\sin b-\sin a\sin(b-nt)=0$

$\sin(a+b)\sin nt=0$ from (ii)

$\sin nt=0$ $\quad\sin(a+b)\neq 0$

$nt=k\pi$ $\quad k=0,1,2,3,\ldots$

$t=\dfrac{k\pi}{n}$ $\quad k=0,1,2,3,\ldots$

End Mathematics Extension 2 solutions