BOARD OF STUDIES
NEW SOUTH WALES

2012

HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–8

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 9–19

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Let $z = 5 - i$ and $w = 2 + 3i$.

What is the value of $2z + \overline{w}$?

(A) $12 + i$

(B) $12 + 2i$

(C) $12 - 4i$

(D) $12 - 5i$

2 The equation $x^3 - y^3 + 3xy + 1 = 0$ defines y implicitly as a function of x.

What is the value of $\frac{dy}{dx}$ at the point $(1, 2)$?

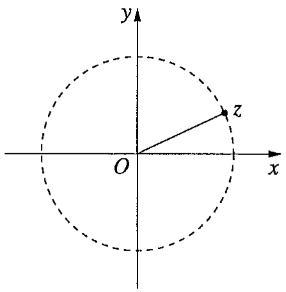
(A) $\frac{1}{3}$

(B) $\frac{1}{2}$

(C) $\frac{3}{4}$

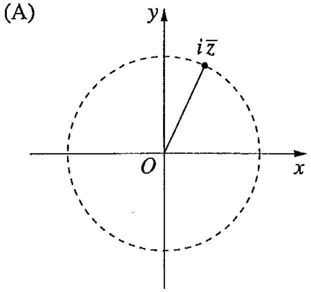
(D) 1

3 The complex number z is shown on the Argand diagram below.

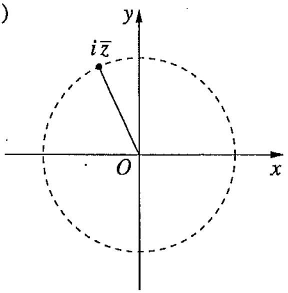


Which of the following best represents $i\overline{z}$?

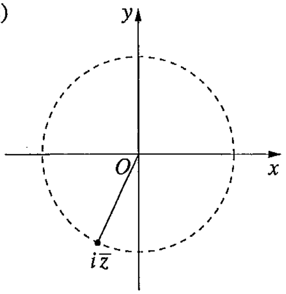
(A)



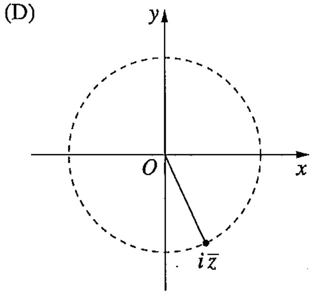
(B)



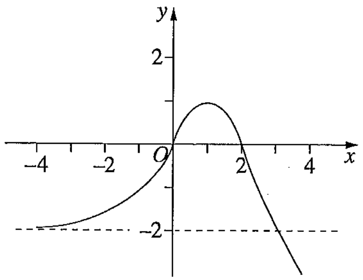
(C)



(D)

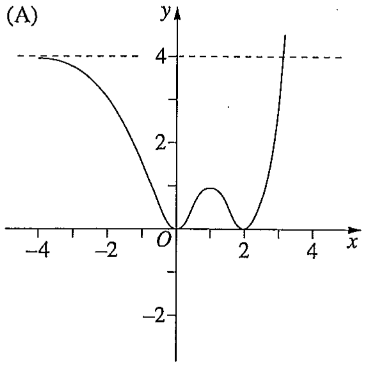


4 The graph $y = f(x)$ is shown below.

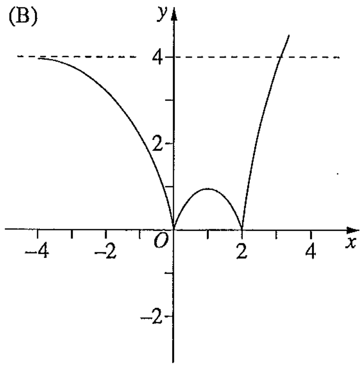


Which of the following graphs best represents $y = \left[f(x)\right]^2$?

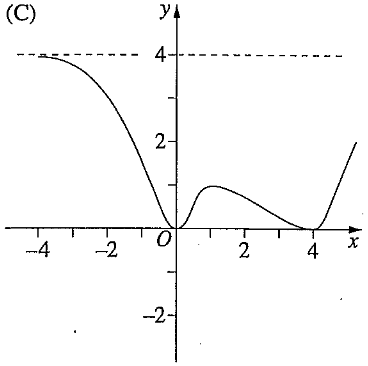
(A)



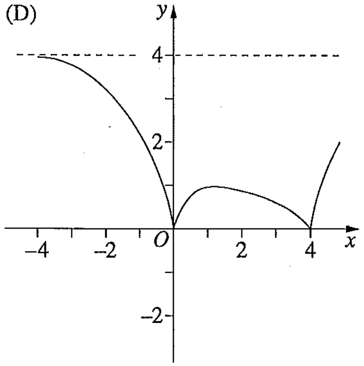
(B)



(C)



(D)



5 The equation $2x^3 - 3x^2 - 5x - 1 = 0$ has roots α, β and γ.

What is the value of $\dfrac{1}{\alpha^3\beta^3\gamma^3}$?

(A) $\dfrac{1}{8}$

(B) $-\dfrac{1}{8}$

(C) 8

(D) −8

6 What is the eccentricity of the hyperbola $\dfrac{x^2}{6} - \dfrac{y^2}{4} = 1$?

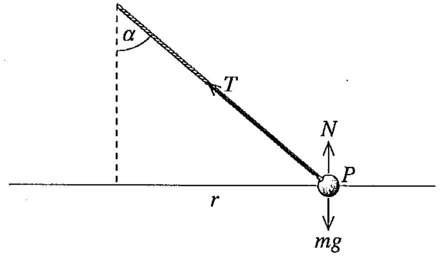
(A) $\dfrac{\sqrt{10}}{2}$

(B) $\dfrac{\sqrt{15}}{3}$

(C) $\dfrac{\sqrt{3}}{3}$

(D) $\dfrac{\sqrt{13}}{3}$

7 A particle P of mass m attached to a string is rotating in a circle of radius r on a smooth horizontal surface. The particle is moving with constant angular velocity ω. The string makes an angle α with the vertical. The forces acting on P are the tension T in the string, a reaction force N normal to the surface and the gravitational force mg.



Which of the following is the correct resolution of the forces on P in the vertical and horizontal directions?

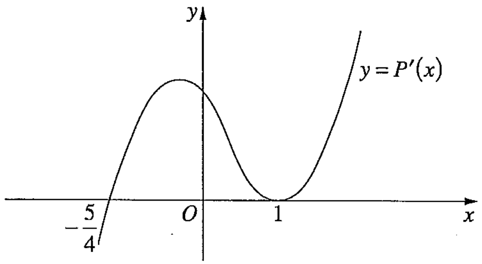
(A) $T\cos\alpha + N = mg$ and $T\sin\alpha = mr\omega^2$

(B) $T\cos\alpha - N = mg$ and $T\sin\alpha = mr\omega^2$

(C) $T\sin\alpha + N = mg$ and $T\cos\alpha = mr\omega^2$

(D) $T\sin\alpha - N = mg$ and $T\cos\alpha = mr\omega^2$

8 The following diagram shows the graph $y = P'(x)$, the derivative of a polynomial $P(x)$.



Which of the following expressions could be $P(x)$?

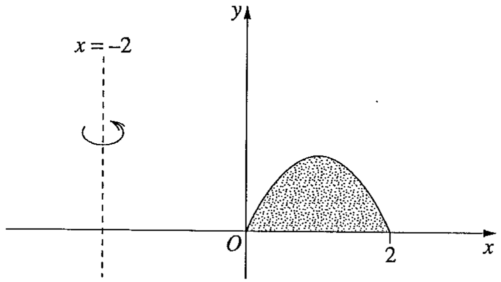
(A) $(x-2)(x-1)^3$

(B) $(x+2)(x-1)^3$

(C) $(x-2)(x+1)^3$

(D) $(x+2)(x+1)^3$

9 The diagram shows the graph $y = x(2-x)$ for $0 \le x \le 2$. The region bounded by the graph and the x-axis is rotated about the line $x = -2$ to form a solid.



Which integral represents the volume of the solid?

(A) $2\pi\int_0^2 x(2-x)^2\,dx$

(B) $2\pi\int_0^2 x^2(2-x)\,dx$

(C) $2\pi\int_0^2 x(2-x)(2+x)\,dx$

(D) $2\pi\int_0^2 x(2-x)(x-2)\,dx$

10 Without evaluating the integrals, which one of the following integrals is greater than zero?

(A) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x}{2+\cos x}\,dx$

(B) $\displaystyle\int_{-\pi}^{\pi} x^3 \sin x\,dx$

(C) $\displaystyle\int_{-1}^{1} \left(e^{-x^2}-1\right)dx$

(D) $\displaystyle\int_{-2}^{2} \tan^{-1}\left(x^3\right)dx$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Express $\dfrac{2\sqrt{5}+i}{\sqrt{5}-i}$ in the form $x+iy$, where x and y are real. **2**

(b) Shade the region on the Argand diagram where the two inequalities **2**

$$|z+2|\geq 2 \quad\text{and}\quad |z-i|\leq 1$$

both hold.

(c) By completing the square, find $\displaystyle\int \frac{dx}{x^2+4x+5}$. **2**

(d) (i) Write $z=\sqrt{3}-i$ in modulus–argument form. **2**

(ii) Hence express z^9 in the form $x+iy$, where x and y are real. **1**

(e) Evaluate $\displaystyle\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx$. **3**

(f) Sketch the following graphs, showing the x- and y-intercepts.

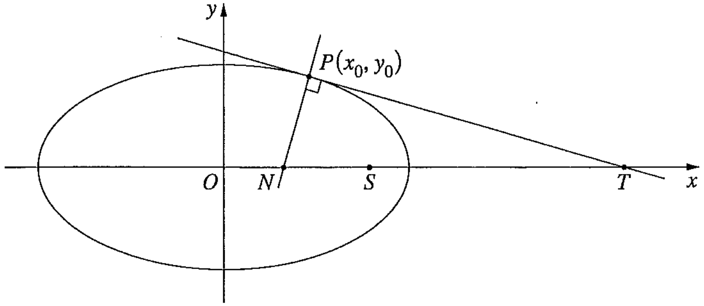
(i) $y=|x|-1$ **1**

(ii) $y=x(|x|-1)$ **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, find $\int \frac{d\theta}{1-\cos\theta}$. **3**

(b) The diagram shows the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a > b$. The ellipse has focus S and eccentricity e. The tangent to the ellipse at $P(x_0, y_0)$ meets the x-axis at T. The normal at P meets the x-axis at N.



(i) Show that the tangent to the ellipse at P is given by the equation **2**

$$y - y_0 = -\frac{b^2x_0}{a^2y_0}\left(x - x_0\right).$$

(ii) Show that the x-coordinate of N is x_0e^2. **2**

(iii) Show that $ON \times OT = OS^2$. **2**

Question 12 continues on page 11

(c) For every integer $n \geq 0$ let **3**

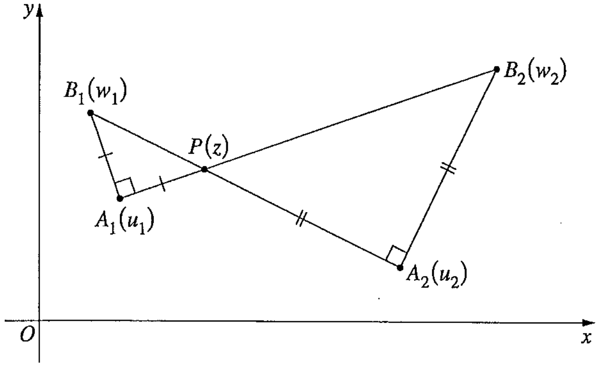
$$I_n = \int_1^{e^2} \left(\log_e x\right)^n \, dx.$$

Show that for $n \geq 1$

$$I_n = e^2 2^n - nI_{n-1}.$$

(d) On the Argand diagram the points A_1 and A_2 correspond to the distinct complex numbers u_1 and u_2 respectively. Let P be a point corresponding to a third complex number z.

Points B_1 and B_2 are positioned so that $\triangle A_1PB_1$ and $\triangle A_2B_2P$, labelled in an anti-clockwise direction, are right-angled and isosceles with right angles at A_1 and A_2, respectively. The complex numbers w_1 and w_2 correspond to B_1 and B_2, respectively.



(i) Explain why $w_1 = u_1 + i\left(z - u_1\right)$. **1**

(ii) Find the locus of the midpoint of B_1B_2 as P varies. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) An object on the surface of a liquid is released at time $t = 0$ and immediately sinks. Let x be its displacement in metres in a downward direction from the surface at time t seconds.

The equation of motion is given by

$$\frac{dv}{dt} = 10 - \frac{v^2}{40},$$

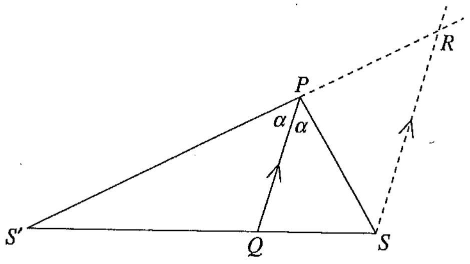
where v is the velocity of the object.

(i) Show that $v = \dfrac{20\left(e^t - 1\right)}{e^t + 1}$. **4**

(ii) Use $\dfrac{dv}{dt} = v\dfrac{dv}{dx}$ to show that $x = 20\log_e\left(\dfrac{400}{400 - v^2}\right)$. **2**

(iii) How far does the object sink in the first 4 seconds? **2**

(b) The diagram shows $\triangle S'SP$. The point Q is on $S'S$ so that PQ bisects $\angle S'PS$. The point R is on $S'P$ produced so that $PQ \parallel RS$.

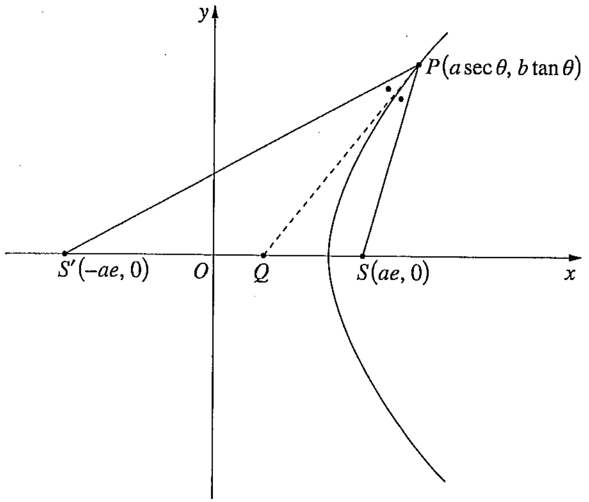


(i) Show that $PS = PR$. **1**

(ii) Show that $\dfrac{PS}{QS} = \dfrac{PS'}{QS'}$. **2**

Question 13 continues on page 13

(c) Let P be a point on the hyperbola given parametrically by $x = a\sec\theta$ and $y = b\tan\theta$, where a and b are positive. The foci of the hyperbola are $S(ae, 0)$ and $S'(-ae, 0)$ where e is the eccentricity. The point Q is on the x-axis so that PQ bisects $\angle SPS'$.



(i) Show that $SP = a(e\sec\theta - 1)$. **1**

(ii) It is given that $S'P = a(e\sec\theta + 1)$. Using part (b), or otherwise, show that the x-coordinate of Q is $\dfrac{a}{\sec\theta}$. **2**

(iii) The slope of the tangent to the hyperbola at P is $\dfrac{b\sec\theta}{a\tan\theta}$. (Do NOT prove this.) **1**

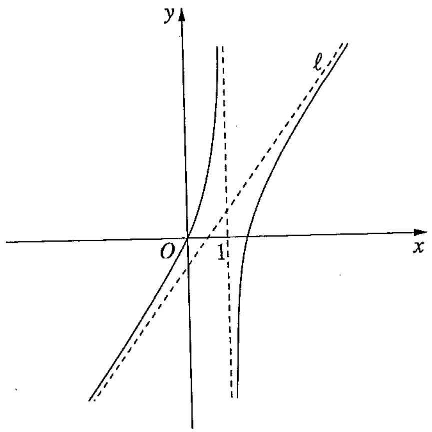
Show that the tangent at P is the line PQ.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{3x^2+8}{x\left(x^2+4\right)}\,dx$. **3**

(b) The diagram shows the graph $y = \dfrac{x(2x-3)}{x-1}$. The line ℓ is an asymptote.



(i) Use the above graph to draw a one-third page sketch of the graph **2**

$$y = \frac{x-1}{x(2x-3)}$$

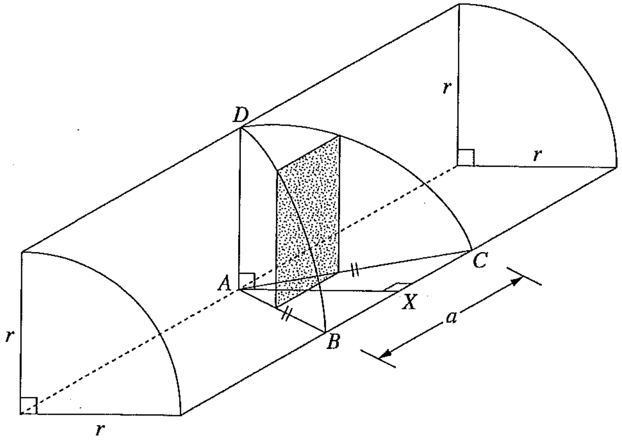
indicating all asymptotes and all x- and y-intercepts.

(ii) By writing $\dfrac{x(2x-3)}{x-1}$ in the form $mx+b+\dfrac{a}{x-1}$, find the equation of the line ℓ. **2**

Question 14 continues on page 15

(c) The solid $ABCD$ is cut from a quarter cylinder of radius r as shown. Its base is an isosceles triangle ABC with $AB = AC$. The length of BC is a and the midpoint of BC is X. **4**

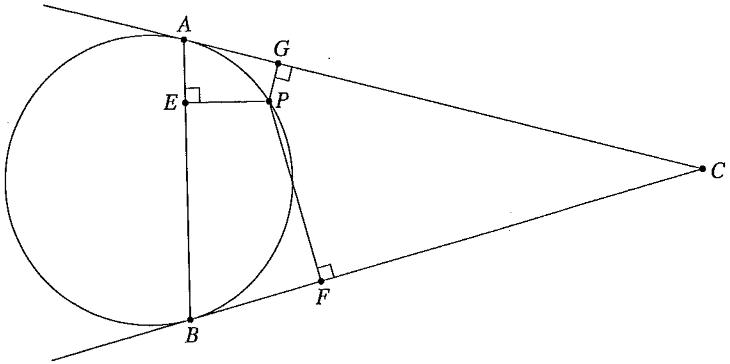
The cross-sections perpendicular to AX are rectangles. A typical cross-section is shown shaded in the diagram.



Find the volume of the solid $ABCD$.

Question 14 continues on page 16

(d) The diagram shows points A and B on a circle. The tangents to the circle at A and B meet at the point C. The point P is on the circle inside $\triangle ABC$. The point E lies on AB so that $AB \perp EP$. The points F and G lie on BC and AC respectively so that $FP \perp BC$ and $GP \perp AC$.



Copy or trace the diagram into your writing booklet.

(i) Show that $\triangle APG$ and $\triangle BPE$ are similar. **2**

(ii) Show that $EP^2 = FP \times GP$. **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove that $\sqrt{ab} \le \dfrac{a+b}{2}$, where $a \ge 0$ and $b \ge 0$. **1**

(ii) If $1 \le x \le y$, show that $x(y - x + 1) \ge y$. **2**

(iii) Let n and j be positive integers with $1 \le j \le n$. Prove that **2**

$$\sqrt{n} \le \sqrt{j(n-j+1)} \le \frac{n+1}{2}.$$

(iv) For integers $n \ge 1$, prove that **1**

$$\left(\sqrt{n}\right)^n \le n! \le \left(\frac{n+1}{2}\right)^n.$$

(b) Let $P(z) = z^4 - 2kz^3 + 2k^2z^2 - 2kz + 1$, where k is real.

Let $\alpha = x + iy$, where x and y are real.

Suppose that α and $i\alpha$ are zeros of $P(z)$, where $\bar{\alpha} \ne i\alpha$.

(i) Explain why $\bar{\alpha}$ and $-i\bar{\alpha}$ are zeros of $P(z)$. **1**

(ii) Show that $P(z) = z^2(z-k)^2 + (kz-1)^2$. **1**

(iii) Hence show that if $P(z)$ has a real zero then **2**

$$P(z) = \left(z^2+1\right)(z+1)^2 \text{ or } P(z) = \left(z^2+1\right)(z-1)^2.$$

(iv) Show that all zeros of $P(z)$ have modulus 1. **2**

(v) Show that $k = x - y$. **1**

(vi) Hence show that $-\sqrt{2} \le k \le \sqrt{2}$. **2**

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) In how many ways can m identical yellow discs and n identical black discs be arranged in a row? **1**

(ii) In how many ways can 10 identical coins be allocated to 4 different boxes? **1**

(b) (i) Show that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ for $|x|<1$ and $|y|<1$. **1**

(ii) Use mathematical induction to prove **3**

$$\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{2j^2}\right) = \tan^{-1}\left(\frac{n}{n+1}\right)$$

for all positive integers n.

(iii) Find $\displaystyle\lim_{n\to\infty}\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{2j^2}\right)$. **1**

Question 16 continues on page 19

(c) Let n be an integer where $n>1$. Integers from 1 to n inclusive are selected randomly one by one with repetition being possible. Let $P(k)$ be the probability that exactly k different integers are selected before one of them is selected for the second time, where $1 \le k \le n$.

(i) Explain why $P(k) = \dfrac{(n-1)!\,k}{n^k (n-k)!}$. **2**

(ii) Suppose $P(k) \ge P(k-1)$. Show that $k^2 - k - n \le 0$. **2**

(iii) Show that if $\sqrt{n+\frac{1}{4}} > k - \frac{1}{2}$ then the integers n and k satisfy $\sqrt{n} > k - \frac{1}{2}$. **2**

(iv) Hence show that if $4n+1$ is not a perfect square, then $P(k)$ is greatest when k is the closest integer to $\sqrt{n}$. **2**

You may use part (iii) and also that $k^2 - k - n > 0$ if $P(k) < P(k-1)$.

End of paper

2012 Higher School Certificate
Solutions
Mathematics Extension 2

SECTION I

Summary

1	D	4	A	7	A	9	C
2	D	5	C	8	B	10	B
3	A	6	B				

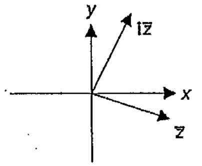
1 (D) $2z+\overline{w}=2(5-i)+2-3i$

$$=10-2i+2-3i$$
$$=12-5i.$$

2 (D) $3x^2-3y^2y'+3xy'+3y=0$

$$y'\left(3x-3y^2\right)=-3x^2-3y$$
$$y'=\frac{-3x^2-3y}{3x-3y^2}$$
$$\therefore \quad \frac{dy}{dx}=\frac{x^2+y}{y^2-x}$$
$$=\frac{1+2}{4-1} \quad \text{at } (1,2)$$
$$=1.$$

3 (A) $\overline{z}$ is the reflection of z in the x-axis.
$i\overline{z}$ is $\overline{z}$ rotated 90° anticlockwise.



4 (A) By elimination, answers C and D have the x-intercepts changed. The intercepts on the x axis become minimum turning points and not cusps as in B.

5 (C) $\alpha\beta\gamma=\frac{1}{2}$,

$$\text{so } \frac{1}{\alpha^3\beta^3\gamma^3}=\frac{1}{(\alpha\beta\gamma)^3}$$
$$=8.$$

6 (B) $b^2=a^2\left(e^2-1\right), \quad a^2=6, \quad b^2=4$

$$e^2=1+\frac{b^2}{a^2}$$
$$=1+\frac{4}{6}$$
$$=\frac{5}{3}$$
$$e=\frac{\sqrt{5}}{\sqrt{3}}=\frac{\sqrt{15}}{3}.$$

7 (A) Vertically, $T\cos\alpha+N=mg$.
Horizontally, $T\sin\alpha=mr\omega^2$.

8 (B) $P'(x)$ has a double root at $x=1$ thus $P(x)$ will have a triple root at $x=1$.
$P'\left(-\frac{5}{4}\right)=0$ with the gradients going from negative to positive indicates a local minimum on $P(x)$. This could c the x-axis at $x=-2$ which would allov an $(x+2)$ factor for $P(x)$.

9 (C) $\delta V=2\pi r\,y\,\delta x$

$$=2\pi(x+2)x(2-x)\delta x$$

where $r=x+2$ and $y=x(2-x)$.

$$V=2\pi\int_0^2 x(2-x)(x+2)\,dx.$$

10 (B) (A) and (D) are odd functions
$\therefore \int_{-a}^{a} f(x)\,dx=0$. (C) is an even function but is below the x-axis. Only (B) is an even function with a positive value.

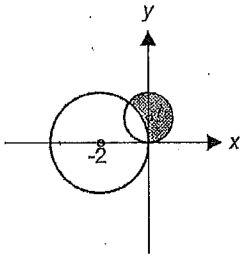
SECTION II

Question 11

(a)

$$\frac{2\sqrt{5}+i}{\sqrt{5}-i}=\frac{2\sqrt{5}+i}{\sqrt{5}-i}\times\frac{\sqrt{5}+i}{\sqrt{5}+i}$$
$$=\frac{10+2\sqrt{5}i+\sqrt{5}i-1}{5+1}$$
$$=\frac{9+3\sqrt{5}i}{6}$$
$$=\frac{3}{2}+i\frac{\sqrt{5}}{2}.$$

(b)



(c) Completing the square gives:

$$x^2+4x+5=x^2+4x+4+1$$
$$=(x+2)^2+1$$

Thus

$$\int\frac{dx}{x^2+4x+5}=\int\frac{dx}{(x+2)+1}$$
$$=\tan^{-1}(x+2)+C.$$

(d) (i)

$$z=\sqrt{3}-i$$
$$|z|=\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}=2$$
$$\arg(z)=\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)=-\frac{\pi}{6}$$
$$\therefore \; z=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$$
$$=2cis\left(-\frac{\pi}{6}\right).$$

(ii)

$$z^9=(\sqrt{3}-i)^9$$
$$=\left(2\text{cis}\left(-\frac{\pi}{6}\right)\right)^9$$
$$=2^9\text{cis}\left(-\frac{9\pi}{6}\right) \quad \text{[by DeMoivre]}$$
$$=512cis\left(-\frac{3\pi}{2}\right)$$
$$=0+i512$$
$$=512i.$$

(e) *Method 1:*

$$\int_0^1\frac{e^{2x}}{e^{2x}+1}\,dx=\frac{1}{2}\int_0^1\frac{2e^{2x}}{e^{2x}+1}\,dx$$
$$=\frac{1}{2}\left[\ln\left(e^{2x}+1\right)\right]_0^1$$
$$=\frac{1}{2}\left[\ln\left(e^{2x}+1\right)-\ln 2\right]$$
$$=\frac{1}{2}\ln\frac{e^{2x}+1}{2}.$$

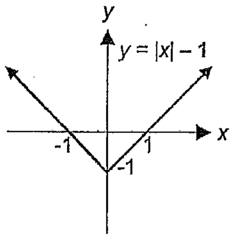
OR

Method 2:

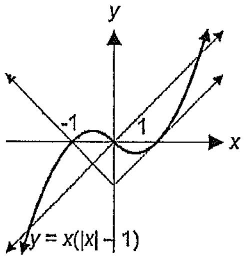
Let $u=e^{2x}$, $du=2e^{2x}dx$
when $x=0$, $u=1$
$x=1$, $u=e^2$

$$\int_0^1\frac{e^{2x}}{e^{2x}+1}\,dx=\int_1^{e^2}\frac{\frac{1}{2}du}{u+1}$$
$$=\left[\frac{1}{2}\ln(u+1)\right]_1^{e^2}$$
$$=\frac{1}{2}\left[\ln\left(e^2+1\right)-\ln 2\right]$$
$$=\frac{1}{2}\ln\frac{e^2+1}{2}.$$

(f) (i)



(ii)



The graph is obtained by multiplying the graphs of $y=|x|-1$ and $y=x$

Question 12

(a) *Method 1:*

Using the substitution $t=\tan\frac{\theta}{2}$

$$\cos\theta=\frac{1-t^2}{1+t^2} \qquad d\theta=\frac{2}{1+t^2}dt$$

$$\int\frac{d\theta}{1-\cos\theta}=\int\frac{1}{\left(1-\left(\frac{1-t^2}{1+t^2}\right)\right)}\frac{2}{1+t^2}dt$$

$$=\int\frac{2}{(1+t^2)-(1-t^2)}dt$$

$$=\int\frac{2}{2t^2}dt$$

$$=\int\frac{dt}{t^2}$$

$$=-\frac{1}{t}+C$$

$$=-\frac{1}{\tan\left(\frac{\theta}{2}\right)}+C$$

$$=-\cot\frac{\theta}{2}+C.$$

OR

Method 2:

$$\int\frac{d\theta}{1-\cos\theta}=\int\frac{d\theta}{1-\left(1-2\sin^2\frac{\theta}{2}\right)}$$

$$=\int\frac{d\theta}{2\sin^2\frac{\theta}{2}}$$

$$=\frac{1}{2}\int\operatorname{cosec}^2\frac{\theta}{2}\,d\theta$$

$$=-\cot\frac{\theta}{2}+C.$$

using $\int\operatorname{cosec}^2x\,dx=-\cot x$

(b) (i) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{2x}{a^2}\cdot\frac{b^2}{2y}$$

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

The gradient at $P(x_0,y_0)$ is $-\frac{b^2x_0}{a^2y_0}$

The equation of the tangent at $P(x_0,y_0)$ is $y-y_0=-\frac{b^2x_0}{a^2y_0}(x-x_0)$ as required.

(ii) For the normal at P

$$m_N=\frac{a^2y_0}{b^2x_0}$$

$$y-y_0=\frac{a^2y_0}{b^2x_0}(x-x_0)$$

when $y=0$,

$$-b^2x_0y_0=a^2y_0(x-x_0)$$

$$-b^2x_0=a^2(x-x_0)$$

$$-b^2x_0=a^2x-a^2x_0$$

$$a^2x=a^2x_0-b^2x_0$$

$$x=\frac{a^2x_0-b^2x_0}{a^2}$$

$$=\frac{x_0\left(a^2-b^2\right)}{a^2} \quad \text{but } a^2e^2=a^2-b^2$$

$$=x_0e^2 \quad \text{as required.}$$

(iii) At T put $y=0$ so

$$0-y_0=-\frac{b^2x_0}{a^2y_0}(x-x_0)$$

$$a^2{y_0}^2=b^2x_0x-b^2{x_0}^2$$

$$b^2x_0x=b^2{x_0}^2+a^2{y_0}^2$$

$$x=\frac{b^2{x_0}^2+a^2{y_0}^2}{b^2x_0}$$

But since $P(x_0,y_0)$ lies on the tangent

$$\frac{{x_0}^2}{a^2}+\frac{{y_0}^2}{b^2}=1\Rightarrow b^2{x_0}^2+a^2{y_0}^2=a^2b^2$$

So $OT=x=\frac{a^2b^2}{b^2x_0}=\frac{a^2}{x_0}$

Now $ON\times OT=x_0e^2\times\frac{a^2}{x_0}$

$$=a^2e^2$$

$$=OS^2$$

Noting that the coordinates of S are $(ae,0)$.

(c) Using integration by parts:

Let $u=(\ln x)^n$ and $v'=1$

$u'=n(\ln x)^{n-1}$ $\qquad v=x$

$$I_n=\int_1^{e^2}1.(\ln x)^n\,dx$$

$$=\left[x(\ln x)^n\right]_1^{e^2}-n\int_1^{e^2}(\ln x)^{n-1}\,dx$$

$$=\left[e^2\left(\ln e^2\right)^n-1.\ln 1\right]-nI_{n-1}$$

$$=\left[e^2(2)^n-1\times0\right]-nI_{n-1}$$

$$=e^22^n-nI_{n-1}.$$

(d) (i) $\overrightarrow{A_1B_1}=\overrightarrow{A_1P}$ rotate 90^0 anticlockwise

$$w_1-u_1=(z-u_1)i$$

$$w_1=u_1+i(z-u_1) \quad \text{as required.}$$

(ii) $\overrightarrow{A_2P}=\overrightarrow{A_2B_2}$ rotate 90^0 anticlockwise

$$z-u_2=(w_2-u_2)i$$

$$z-u_2+iu_2=iw_2$$

$$-w_2=iz-iu_2-u_2$$

$$w_2=u_2+iu_2-iz$$

$$w_2=u_2+i(u_2-z)$$

The midpoint of B_1B_2 as P varies is:

$$\frac{w_1+w_2}{2}=\frac{u_1+i(z-u_1)+u_2+i(u_2-z)}{2}$$

$$=\frac{u_1+u_2}{2}+\frac{(u_2-u_1)}{2}i$$

which is a fixed point.

Question 13

(a) (i) $$\frac{dv}{dt}=10-\frac{v^2}{40}=\frac{400-v^2}{40}$$

$$\int\frac{40\,dv}{400-v^2}=\int dt$$

$$t=\int\frac{40}{400-v^2}dv$$

$$=\int\frac{40}{(20-v)(20+v)}dv$$

$$=\int\frac{1}{20+v}dv-\int\frac{-1}{20-v}dv$$

$$=\left[\ln(20+v)-\ln(20-v)\right]+C$$

$$=\ln\left(\frac{20+v}{20-v}\right)+C$$

But $t=0,v=0$

$$0=\ln\left(\frac{20+0}{20-0}\right)+C$$

$$C=\ln1=0$$

$$\therefore t=\ln\left(\frac{20+v}{20-v}\right)$$

$$e^t=\frac{20+v}{20-v}$$

$$20e^t-ve^t=20+v$$

$$ve^t+v=20e^t-20$$

$$v(e^t+1)=20(e^t-1)$$

$$v=\frac{20(e^t-1)}{e^t+1} \quad \text{as required.}$$

(ii) $$v\frac{dv}{dx}=\frac{400-v^2}{40}$$

$$\frac{40v\,dv}{400-v^2}=dx$$

$$\int\frac{40v\,dv}{400-v^2}=\int dx$$

$$x=-20\int\frac{-2v}{400-v^2}dv$$

$$=-20\ln(400-v^2)+C$$

But $x=0, v=0$

$0=-20\ln(400-0)+C$

$C=20\ln 400$

$x=-20\ln(400-v^2)+20\ln 400$

$=20\ln\left(\frac{400}{400-v^2}\right)$ as required.

(iii) Put $t=4$ in v

$$v=\frac{20(e^4-1)}{e^4+1}$$

Now put this in for v to find x

$$x=20\log_e\left(\frac{400}{400-v^2}\right)$$

$$=20\log_e\left(\frac{400}{400-\left[\frac{20(e^4-1)}{e^4+1}\right]^2}\right)$$

$$=20\log_e\left(\frac{(e^4+1)^2}{(e^4+1)^2-(e^4-1)^2}\right)$$

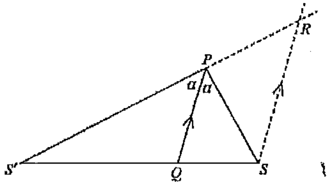
$$=20\log_e\left(\frac{(e^4+1)^2}{4e^4}\right)$$

$$=20\log_e\left(\frac{(e^4+1)^2}{\left(2e^2\right)^2}\right)$$

$=40\log_e\left(\frac{e^4+1}{2e^2}\right)$ m [exact answer]

≈ 53 m (nearest metre)

(b)



(i) $\angle S'PQ=\angle PRS$ (corresponding $\angle$'s $PQ \parallel RS$)

$\angle QPS=\angle PSR$ (alternate $\angle$'s $PQ \parallel RS$)

Now $\angle PRS=\angle PSR=\alpha$

These are equal base angles of ΔPSR

$\therefore \Delta PSR$ is isosceles

$\therefore PS=PR$ as required.

(ii) $\frac{QS'}{QS}=\frac{PS'}{PR}$ (parallel lines cut off equal ratios on all transversals)

But $PS=PR$ from (i)

$$\frac{QS'}{QS}=\frac{PS'}{PS}$$

$\therefore \frac{PS}{QS}=\frac{PS'}{QS'}$ as required.

(c) (i) The equation of the directrix is $x=\frac{a}{e}$.

Let M be the point $\left(\frac{a}{e},b\tan\theta\right)$.

$$PM=a\sec\theta-\frac{a}{e}$$

$$=a\left(\sec\theta-\frac{1}{e}\right)$$

$$SP=ePM$$

$$=ea\left(\sec\theta-\frac{1}{e}\right)$$

$$=a\left(e\sec\theta-1\right).$$

(ii) $\frac{PS}{QS}=\frac{PS'}{QS'}$ from (b)(ii)

$$\frac{a(e\sec\theta-1)}{ae-x}=\frac{a(e\sec\theta+1)}{ae+x}$$

$$\frac{(e\sec\theta-1)}{ae-x}=\frac{(e\sec\theta+1)}{ae+x}$$

$$ae^2\sec\theta+xe\sec\theta-ae-x=$$

$$ae^2\sec\theta-xe\sec\theta+ae-x$$

$$2xe\sec\theta=2ae$$

$$x=\frac{a}{\sec\theta}.$$

(iii) Gradient of PQ is:

$$m_{PQ}=\frac{b\tan\theta}{a\sec\theta-\frac{a}{\sec\theta}}$$

$$=\frac{b\tan\theta\sec\theta}{a\left(\sec^2\theta-1\right)}$$

$$=\frac{b\tan\theta\sec\theta}{a\tan^2\theta}$$

$$=\frac{b\sec\theta}{a\tan\theta}$$

$\therefore PQ$ is the tangent at P.

Question 14

(a) Using partial fractions

$$\frac{3x^2+8}{x(x^2+4)}\equiv\frac{a}{x}+\frac{bx+c}{x^2+4}$$

$$3x^2+8\equiv a(x^2+4)+(bx+c)x$$

$$\equiv ax^2+4a+bx^2+cx$$

$$\equiv(a+b)x^2+cx+4a$$

Equating co-efficients.

$c=0$

and $4a=8$

$a=2$

also $a+b=0$

$2+b=3$

$b=1$

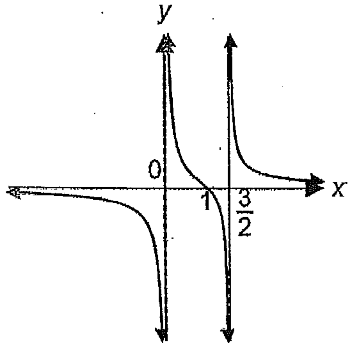
$$\int\frac{3x^2+8}{x(x^2+4)}dx=\int\frac{2}{x}dx+\int\frac{x}{x^2+4}dx$$

$$=2\ln x+\frac{1}{2}\ln(x^2+4)+C.$$

(b) (i) $y=\frac{x-1}{x(2x-3)}$

Asymptotes are $x=0, x=\frac{3}{2}, y=0$

x-intercept is 1.



(ii) Using long division:

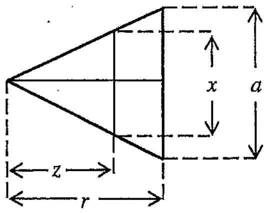
$$\begin{array}{r} 2x-1 \\ x-1\overline{)2x^2-3x+0} \\ \underline{2x^2-2x} \\ -x+0 \\ \underline{-x+1} \\ -1 \end{array}$$

Now $y=\frac{x(2x-3)}{x-1}$

$$=2x-1-\frac{1}{x-1}$$

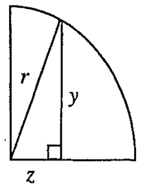
The equation of ℓ is $y=2x-1$.

(c) Let the dimensions of the rectangular slice be x and y. Let the distance from the back be z. From the top:



Similar triangles: $\frac{x}{a}=\frac{z}{r}\Rightarrow x=\frac{az}{r}$

From the front:



Using Pythagoras' Theorem:

$$y = \sqrt{r^2 - z^2}$$

Area of slice $= xy = \dfrac{az}{r}\sqrt{r^2 - z^2}$

$$V = \lim_{\delta z \to 0} \sum_{z=0}^{r} \frac{az}{r}\sqrt{r^2 - z^2}\,\delta z$$

$$= \int_0^r \frac{az}{r}\sqrt{r^2 - z^2}\, dz$$

$$= \frac{a}{r}\int_0^r z(r^2 - z^2)^{\frac{1}{2}}\, dz$$

Let $u^2 = r^2 - z^2$

$2u\, du = -2z\, dz$

$-u\, du = z\, dz$

If $z = 0$ then $u^2 = r^2 - 0^2 \Rightarrow u = r \quad (r > 0)$

If $z = r$ then $u^2 = r^2 - r^2 \Rightarrow u = 0$

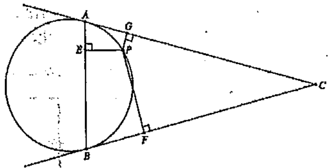
$$V = \frac{a}{r}\int_r^0 -u \times u du$$

$$= -\frac{a}{r}\left[\frac{u^3}{3}\right]_r^0$$

$$= \frac{ar^2}{3} \text{ units }^3$$

$\therefore$ Volume $= \dfrac{ar^2}{3}$ units^3.

(d) (i)



In ΔAPG and ΔBPE

$\angle AGP = \angle BEP$ (given 90^0)

$\angle GAP = \angle EBP$ (alternate segment thm)

$\Delta APG ||| \Delta BPE$ (equiangular)

(ii) Similarly $\Delta APE ||| \Delta BPF$

$\dfrac{GP}{EP} = \dfrac{AP}{BP}$ matching sides $\Delta APG ||| \Delta BPE$

$\dfrac{AP}{BP} = \dfrac{EP}{FP}$ matching sides $\Delta APE ||| \Delta BPF$

$\therefore \dfrac{GP}{EP} = \dfrac{EP}{FP}$

$\therefore EP^2 = FP \times GP$ as required.

Question 15

(a) (i) $(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 + 2ab + b^2 \geq 4ab$

$(a+b)^2 \geq 4ab$

$(a+b) \geq 2\sqrt{ab}$

$\sqrt{ab} \leq \dfrac{a+b}{2}.$

(ii) Given $1 \leq x \leq y$ then

$y \geq x$

$y(x-1) \geq x(x-1)$ since $x - 1 \geq 0$

$yx - y \geq x^2 - x$

$yx - x^2 + x \geq y$

$x(y - x + 1) \geq y$ as required.

(iii) Put $x = j$ and $y = n$ into (ii)

$j(n - j + 1) \geq n$

$\sqrt{n} \leq \sqrt{j(n-j+1)}$ since $1 \leq j \leq n$

$\sqrt{j(n-j+1)} \leq \dfrac{j+n-j+1}{2}$ using (i)

$\leq \dfrac{n+1}{2}$

$\therefore \sqrt{n} \leq \sqrt{j(n-j+1)} \leq \dfrac{n+1}{2}$ as required.

(iv) Using (iii):

Put $j = 1: \sqrt{n} \leq \sqrt{1(n)} \leq \dfrac{n+1}{2}$

Put $j = 2: \sqrt{n} \leq \sqrt{2(n-1)} \leq \dfrac{n+1}{2}$

Put $j = 3: \sqrt{n} \leq \sqrt{3(n-2)} \leq \dfrac{n+1}{2}$

and so on

Put $j = n: \sqrt{n} \leq \sqrt{n} \leq \dfrac{n+1}{2}$

Multiplying all these terms together

$$\left(\sqrt{n}\right)^n \leq \sqrt{n!n!} \leq \left(\frac{n+1}{2}\right)^n$$

$$\left(\sqrt{n}\right)^n \leq n! \leq \left(\frac{n+1}{2}\right)^n \quad \text{as required.}$$

(b) (i) Note that:

$\overline{i\alpha} = \overline{i(x+iy)}$

$= \overline{ix - y}$

$= -y - ix$

$= -i(x - iy)$

$= -i\overline{\alpha}$

Since the coefficients are real, the zeros must occur in conjugate pairs. Thus if α and $i\alpha$ are zeros then their conjugates, namely $\overline{\alpha}$ and $-i\overline{\alpha}$, must also be zeros.

(ii) $P(z) = z^4 - 2kz^3 + 2k^2z^2 - 2kz + 1$

$= z^4 - 2kz^3 + k^2z^2 + k^2z^2 - 2kz + 1$

$= z^2(z^2 - 2kz + k^2) + (kz)^2 - 2kz + 1$

$= z^2(z-k)^2 + (kz-1)^2.$

(iii) If $P(z)$ has a real zero then

$z^2(z-k)^2 + (kz-1)^2 = 0$ factorises.

Since $z \neq 0$, $(z-k)^2 = 0 \Rightarrow z = k$,

and $(kz-1)^2 = 0 \Rightarrow kz = 1 \Rightarrow k^2 = 1$

Hence $k = \pm 1$

$\therefore z^2(z-1)^2 + (z-1)^2 = 0$

$(z^2+1)(z-1)^2 = 0$

or

$z^2(z+1)^2 + (-z-1)^2 = 0$

$z^2(z+1)^2 + (z+1)^2 = 0$

$(z^2+1)(z+1)^2 = 0$ as required.

(iv) The roots are α, $i\alpha$, $\overline{\alpha}$ and $i\overline{\alpha}$.
The product of the roots is:

$\dfrac{e}{a} = \alpha \cdot \overline{\alpha} \cdot i\alpha \cdot -i\overline{\alpha}$

$1 = (\alpha \cdot \overline{\alpha})^2$

$= \left(|\alpha|^2\right)^2$

$= |\alpha|^4$

$|\alpha| = 1.$

Also from (iii), the zeros are $1, 1, i, -i$ or $-1, -1, i, -i$ and these also have a modulus of 1.

(v) Using the sum of the roots:

$\sum \alpha = \alpha + \overline{\alpha} + i\alpha - i\overline{\alpha}$

$-\dfrac{b}{a} = (x+iy) + (x-iy)$

$+ (-y+ix) + (-ix-y)$

$2k = 2x - 2y$

$\therefore k = x - y$ as required.

(vi) $(x-y)^2 = x^2 - 2xy + y^2$

$(x+y)^2 = x^2 + 2xy + y^2$

$|\alpha| = 1 \quad \Rightarrow x^2 + y^2 = 1$

$k^2 = (x-y)^2$ from (v)

$= x^2 + y^2 - 2xy$

$= x^2 + y^2 - \left((x+y)^2 - x^2 - y^2\right)$

$= 2\left(x^2 + y^2\right) - (x+y)^2$

$= 2 - (x+y)^2$ since $x^2 + y^2 = 1$

≤ 2 since $(x+y)^2 \geq 0$

$\therefore -\sqrt{2} \leq k \leq \sqrt{2}$ as required.

Question 16

(a) (i) By definition $\dfrac{(m+n)!}{m!n!}$.

(ii) □ ⊙ □ ⊙ □ ⊙ □

10 coins in 4 boxes means various groups of coins adding up to 10 placed inside the 4 boxes with spaces between each box.

Using part i) $m(\text{coins}) = 10$

$n(\text{spaces}) = 3$

$$\frac{(m+n)!}{m!n!} = \frac{13!}{10!3!} = 286.$$

(b) (i) *Method 1:*

Let $A = \tan^{-1} x, B = \tan^{-1} y$

$\therefore x = \tan A, \quad y = \tan B$