BOARD OF STUDIES
NEW SOUTH WALES

2008
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–10
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n\neq -1;\ x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\ a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a\neq 0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x>0$

Total marks – 120
Attempt Questions 1–10
All questions are of equal value

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use the Question 1 Writing Booklet.

(a) Evaluate $2\cos\frac{\pi}{5}$ correct to three significant figures. **2**

(b) Factorise $3x^2 + x - 2$. **2**

(c) Simplify $\frac{2}{n} - \frac{1}{n+1}$. **2**

(d) Solve $|4x - 3| = 7$. **2**

(e) Expand and simplify $(\sqrt{3} - 1)(2\sqrt{3} + 5)$. **2**

(f) Find the sum of the first 21 terms of the arithmetic series $3 + 7 + 11 + \cdots$. **2**

Marks

Question 2 (12 marks) Use the Question 2 Writing Booklet.

(a) Differentiate with respect to x:

(i) $(x^2 + 3)^9$ **2**

(ii) $x^2 \log_e x$ **2**

(iii) $\frac{\sin x}{x+4}$. **2**

(b) Let M be the midpoint of $(-1, 4)$ and $(5, 8)$. **2**

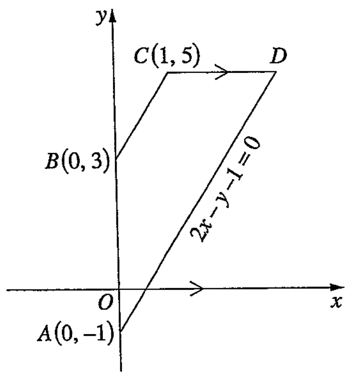
Find the equation of the line through M with gradient $-\frac{1}{2}$.

(c) (i) Find $\int \frac{dx}{x+5}$. **1**

(ii) Evaluate $\int_0^{\frac{\pi}{12}} \sec^2 3x \, dx$. **3**

Question 3 (12 marks) Use the Question 3 Writing Booklet. **Marks**

(a)



In the diagram, $ABCD$ is a quadrilateral. The equation of the line AD is $2x - y - 1 = 0$.

(i) Show that $ABCD$ is a trapezium by showing that BC is parallel to AD. **2**

(ii) The line CD is parallel to the x-axis. Find the coordinates of D. **1**

(iii) Find the length of BC. **1**

(iv) Show that the perpendicular distance from B to AD is $\frac{4}{\sqrt{5}}$. **2**

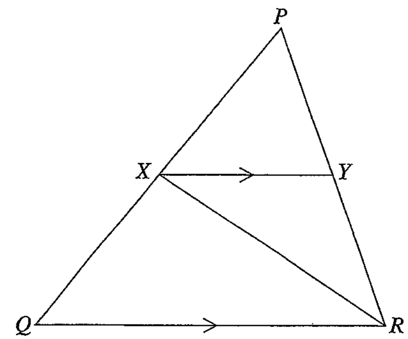
(v) Hence, or otherwise, find the area of the trapezium $ABCD$. **2**

(b) (i) Differentiate $\log_e(\cos x)$ with respect to x. **2**

(ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{4}} \tan x \, dx$. **2**

Question 4 (12 marks) Use the Question 4 Writing Booklet. **Marks**

(a) **2**



In the diagram, XR bisects $\angle PRQ$ and $XY \parallel QR$.

Copy or trace the diagram into your writing booklet.

Prove that ΔXYR is an isosceles triangle.

(b) The zoom function in a software package multiplies the dimensions of an image by 1.2. In an image, the height of a building is 50 mm. After the zoom function is applied once, the height of the building in the image is 60 mm. After a second application, its height is 72 mm.

(i) Calculate the height of the building in the image after the zoom function has been applied eight times. Give your answer to the nearest mm. **2**

(ii) The height of the building in the image is required to be more than 400 mm. Starting from the original image, what is the least number of times the zoom function must be applied? **2**

(c) Consider the parabola $x^2 = 8(y - 3)$.

(i) Write down the coordinates of the vertex. **1**

(ii) Find the coordinates of the focus. **1**

(iii) Sketch the parabola. **1**

(iv) Calculate the area bounded by the parabola and the line $y = 5$. **3**

Marks

Question 5 (12 marks) Use the Question 5 Writing Booklet.

(a) The gradient of a curve is given by $\frac{dy}{dx} = 1 - 6\sin 3x$. The curve passes through the point $(0, 7)$. **3**

What is the equation of the curve?

(b) Consider the geometric series

$$5 + 10x + 20x^2 + 40x^3 + \cdots .$$

(i) For what values of x does this series have a limiting sum? **2**

(ii) The limiting sum of this series is 100. **2**

Find the value of x.

(c) Light intensity is measured in lux. The light intensity at the surface of a lake is 6000 lux. The light intensity, I lux, a distance s metres below the surface of the lake is given by

$$I = Ae^{-ks}$$

where A and k are constants.

(i) Write down the value of A. **1**

(ii) The light intensity 6 metres below the surface of the lake is 1000 lux. **2**

Find the value of k.

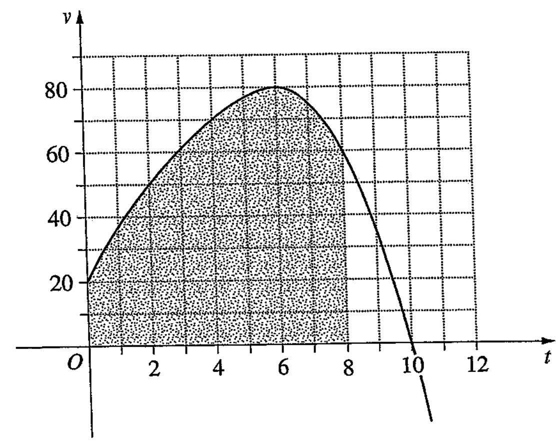
(iii) At what rate, in lux per metre, is the light intensity decreasing 6 metres below the surface of the lake? **2**

Marks

Question 6 (12 marks) Use the Question 6 Writing Booklet.

(a) Solve $2\sin^2 \frac{x}{3} = 1$ for $-\pi \le x \le \pi$. **3**

(b) The graph shows the velocity of a particle, v metres per second, as a function of time, t seconds.



(i) What is the initial velocity of the particle? **1**

(ii) When is the velocity of the particle equal to zero? **1**

(iii) When is the acceleration of the particle equal to zero? **1**

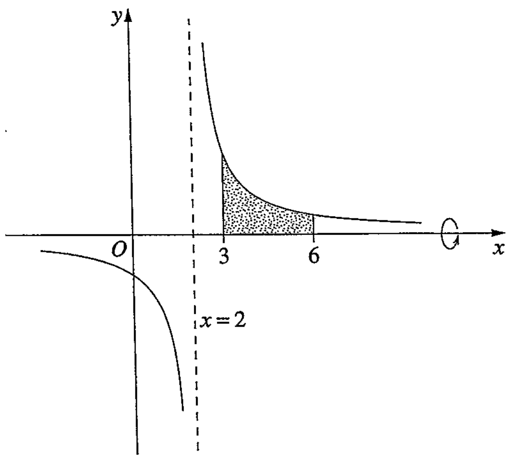
(iv) By using Simpson's Rule with five function values, estimate the distance travelled by the particle between $t = 0$ and $t = 8$. **3**

Question 6 continues on page 9

Marks

Question 6 (continued)

(c) The graph of $y = \dfrac{5}{x-2}$ is shown below. **3**



The shaded region in the diagram is bounded by the curve $y = \dfrac{5}{x-2}$, the x-axis and the lines $x = 3$ and $x = 6$.

Find the volume of the solid of revolution formed when the shaded region is rotated about the x-axis.

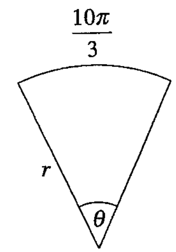
End of Question 6

Marks

Question 7 (12 marks) Use the Question 7 Writing Booklet.

(a) Solve $\log_e x - \dfrac{3}{\log_e x} = 2$. **3**

(b)



The diagram shows a sector with radius r and angle θ where $0 < \theta \leq 2\pi$.

The arc length is $\dfrac{10\pi}{3}$.

(i) Show that $r \geq \dfrac{5}{3}$. **2**

(ii) Calculate the area of the sector when $r = 4$. **2**

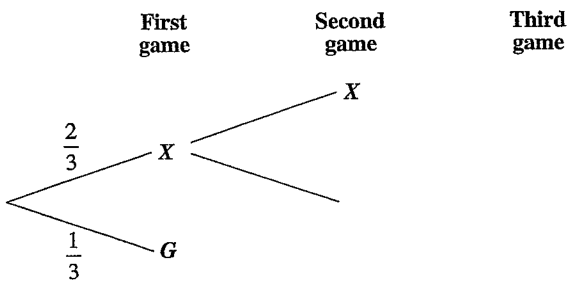
Question 7 continues on page 11

Question 7 (continued)

Marks

(c) Xena and Gabrielle compete in a series of games. The series finishes when one player has won two games. In any game, the probability that Xena wins is $\frac{2}{3}$ and the probability that Gabrielle wins is $\frac{1}{3}$.

Part of the tree diagram for this series of games is shown.



(i) Copy and complete the tree diagram showing the possible outcomes. **1**

(ii) What is the probability that Gabrielle wins the series? **2**

(iii) What is the probability that three games are played in the series? **2**

End of Question 7

Marks

Question 8 (12 marks) Use the Question 8 Writing Booklet.

(a) Let $f(x) = x^4 - 8x^2$.

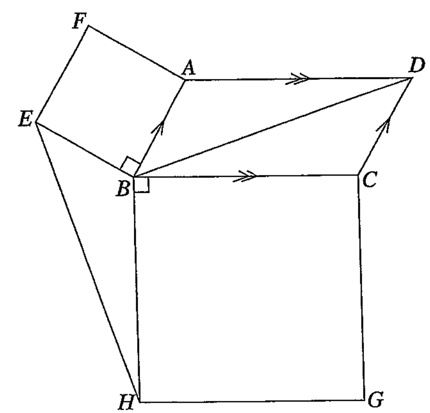
(i) Find the coordinates of the points where the graph of $y = f(x)$ crosses the axes. **2**

(ii) Show that $f(x)$ is an even function. **1**

(iii) Find the coordinates of the stationary points of $f(x)$ and determine their nature. **4**

(iv) Sketch the graph of $y = f(x)$. **1**

(b)



In the diagram, $ABCD$ is a parallelogram and $ABEF$ and $BCGH$ are both squares.

Copy or trace the diagram into your writing booklet.

(i) Prove that $CD = BE$. **1**

(ii) Prove that $BD = EH$. **3**

Marks

Question 9 (12 marks) Use the Question 9 Writing Booklet.

(a) It is estimated that 85% of students in Australia own a mobile phone.

(i) Two students are selected at random. What is the probability that neither of them owns a mobile phone? **2**

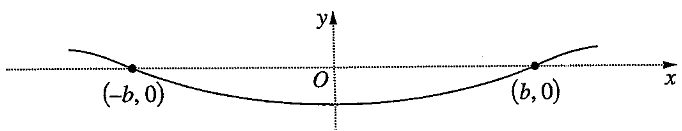
(ii) Based on a recent survey, 20% of the students who own a mobile phone have used their mobile phone during class time. A student is selected at random. What is the probability that the student owns a mobile phone and has used it during class time? **1**

(b) Peter retires with a lump sum of \$100 000. The money is invested in a fund which pays interest each month at a rate of 6% per annum, and Peter receives a fixed monthly payment of \$$M$ from the fund. Thus, the amount left in the fund after the first monthly payment is $\$(100\,500 - M)$.

(i) Find a formula for the amount, $\$A_n$, left in the fund after n monthly payments. **2**

(ii) Peter chooses the value of M so that there will be nothing left in the fund at the end of the 12th year (after 144 payments). Find the value of M. **3**

(c) A beam is supported at $(-b, 0)$ and $(b, 0)$ as shown in the diagram.



It is known that the shape formed by the beam has equation $y = f(x)$, where $f(x)$ satisfies

$$f''(x) = k\left(b^2 - x^2\right) \quad (k \text{ is a positive constant})$$

and $$f'(-b) = -f'(b).$$

(i) Show that **2**

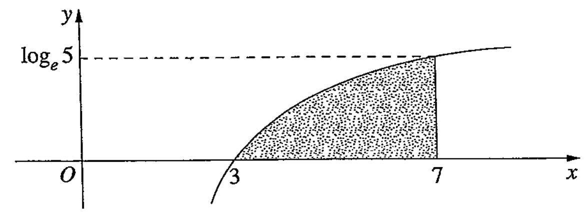
$$f'(x) = k\left(b^2 x - \frac{x^3}{3}\right).$$

(ii) How far is the beam below the x-axis at $x = 0$? **2**

Marks

Question 10 (12 marks) Use the Question 10 Writing Booklet.

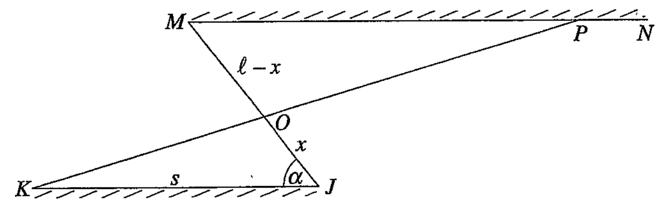
(a) **5**



In the diagram, the shaded region is bounded by $y = \log_e(x-2)$, the x-axis and the line $x = 7$.

Find the exact value of the area of the shaded region.

(b)



The diagram shows two parallel brick walls KJ and MN joined by a fence from J to M. The wall KJ is s metres long and $\angle KJM = \alpha$. The fence JM is ℓ metres long.

A new fence is to be built from K to a point P somewhere on MN. The new fence KP will cross the original fence JM at O.

Let $OJ = x$ metres, where $0 < x < \ell$.

(i) Show that the total area, A square metres, enclosed by ΔOKJ and ΔOMP is given by **3**

$$A = s\left(x - \ell + \frac{\ell^2}{2x}\right)\sin\alpha.$$

(ii) Find the value of x that makes A as small as possible. Justify the fact that this value of x gives the minimum value for A. **3**

(iii) Hence, find the length of MP when A is as small as possible. **1**

End of paper

2008 Higher School Certificate Solutions
Mathematics

Question 1

(a) $2\cos\frac{\pi}{5} = 1.9998\ldots$

$= 2.00$ (to 3 sig figs)

(b) *METHOD 1*

$3x^2 + x - 2$

$= (3x-2)(x+1)$

3x −2
x 1

METHOD 2

$AB = -6 \quad A + B = 1$

$\therefore A = 3 \quad B = -2$

$\therefore 3x^2 + x - 2 = 3x^2 + 3x - 2x - 2$

$= 3x(x+1) - 2(x+1)$

$= (3x-2)(x+1)$

(c) $\frac{2}{n} - \frac{1}{n+1} = \frac{2(n+1)-n}{n(n+1)}$

$= \frac{2n+2-n}{n(n+1)}$

$= \frac{n+2}{n(n+1)}.$

(d) $|4x-3| = 7$

$4x - 3 = 7$ or $4x - 3 = -7$

$4x = 10 \qquad 4x = -4$

$x = \frac{5}{2} \qquad x = -1$

$\therefore x = \frac{5}{2}$ or -1

(e) $(\sqrt{3}-1)(2\sqrt{3}+5)$

$= 6 + 5\sqrt{3} - 2\sqrt{3} - 5$

$= 1 + 3\sqrt{3}.$

(f) $3 + 7 + 11 + \ldots$

is an arithmetic series with

$a = 3,\ d = 4,\ n = 21$

Using the formula for sum,

$S_n = \frac{n}{2}[2a + (n-1)d]$

$\therefore S_{21} = \frac{21}{2}[2\times 3 + (21-1)\times 4]$

$= 903$

Question 2

(a) (i) $\frac{d}{dx}\left[(x^2+3)^9\right]$

$= 9\times(x^2+3)^8 \times 2x$

$= 18x(x^2+3)^8$

(ii) Product rule: $\frac{d}{dx}(uv) = uv^1 + vu^1$

$\therefore \frac{d}{dx}(x^2 \log_e x)$

$= x^2 \times \frac{1}{x} + \log_e x \times 2x$

$= x + 2x\log_e x$

(iii) Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu^1 - uv^1}{v^2}$

$\therefore \frac{d}{dx}\left(\frac{\sin x}{x+4}\right)$

$= \frac{(x+4)\times\cos x - \sin x \times 1}{(x+4)^2}$

$= \frac{(x+4)\cos x - \sin x}{(x+4)^2}.$

(b) $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$= \left(\frac{-1+5}{2}, \frac{4+8}{2}\right)$

$= \left(\frac{4}{2}, \frac{12}{2}\right)$

$= (2, 6)$

$\therefore$ Using the point–gradient formula,

$y - y_1 = m(x - x_1)$

$y - 6 = -\frac{1}{2}(x-2).$

METHOD 1 $\quad y - 6 = -\frac{1}{2}x + 1$

$y = -\frac{1}{2}x + 7.$

METHOD 2 $\quad 2y - 12 = -x + 2$

$x + 2y - 14 = 0.$

(c) (i) $\int \frac{dx}{x+5} = \ln(x+5) + c.$

(ii) $\int_0^{\frac{\pi}{12}} \sec^2 3x\, dx$

$= \left[\frac{1}{3}\tan 3x\right]_0^{\frac{\pi}{12}}$

$= \frac{1}{3}\tan\frac{3\pi}{12} - \frac{1}{3}\tan 0$

$= \frac{1}{3}\tan\frac{\pi}{4} - 0$

$= \frac{1}{3}.$

Question 3

(a) (i) Gradient of $BC = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{5-3}{1-0}$

$= 2$

Rearranging, $2x - y - 1 = 0$

$y = 2x - 1$

$\therefore$ Gradient of $AD = 2$

$\therefore BC \parallel AD\ (m_{AD} = m_{BC} = 2)$

$\therefore ABCD$ is a trapezium $(BC \parallel AD)$.

(ii) $D(x, 5)$ as $CD \parallel x$-axis.

Substitute $D(x,5)$ into

$2x - y - 1 = 0$

$\therefore 2x - 5 - 1 = 0$

$2x = 6$

$x = 3$

$\therefore D$ is $(3, 5)$.

(iii) Using the distance formula with $B(0,3)$ and $C(1,5)$,

$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$\therefore BC = \sqrt{(1-0)^2 + (5-3)^2}$

$= \sqrt{1+4}$

$= \sqrt{5}.$

(iv) Using the perpendicular distance formula with $2x - y - 1 = 0$ and $B(0,3)$,

$\perp$ distance $= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$= \frac{|2\times 0 - 1\times 3 - 1|}{\sqrt{2^2 + (-1)^2}}$

$= \frac{4}{\sqrt{5}}$ as required.

(v) *METHOD 1*

Area of trapezium $ABCD$

$= \frac{1}{2}h(a+b)$

$= \frac{1}{2} \times$ perpendicular height $\times$ (length of BC + length of AD)

Using the distance formula with $A(0,-1)$ and $D(3,5)$,

$AD = \sqrt{(3-0)^2 + (5-(-1))^2}$

$= \sqrt{9+36}$

$= \sqrt{45}$

$= 3\sqrt{5}$

$BC = \sqrt{5}$ and height $= \frac{4}{\sqrt{5}}$

$\therefore$ Area of trapezium $ABCD$

$= \frac{1}{2} \times \frac{4}{\sqrt{5}} \times (\sqrt{5} + 3\sqrt{5})$

$= \frac{1}{2} \times \frac{4}{\sqrt{5}} \times 4\sqrt{5}$

$= 8 \text{ units}^2$

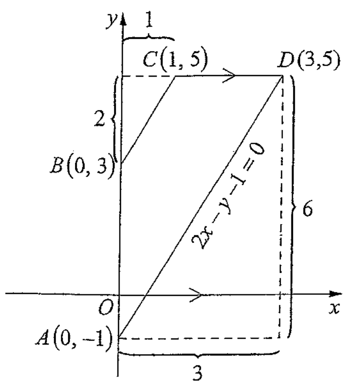
METHOD 2

Area of trapezium $ABCD$

$= A_{\text{rectangle}} - A_{\text{small}\Delta} - A_{\text{large}\Delta}$

$= 6 \times 3 - \frac{1}{2} \times 1 \times 2 - \frac{1}{2} \times 3 \times 6$

$= 8 \text{ units}^2$



(b) (i) $\frac{d}{dx}[\log_e(\cos x)] = \frac{1}{\cos x} \times -\sin x$

$= -\tan x$

(ii) $\int_0^{\frac{\pi}{4}} \tan x \, dx$

$= -\left[\log_e(\cos x)\right]_0^{\frac{\pi}{4}}$

$= -\left[\log_e\left(\cos\frac{\pi}{4}\right) - \log_e(\cos 0)\right]$

$= -\left[\log_e\left(\frac{1}{\sqrt{2}}\right) - \log_e 1\right]$

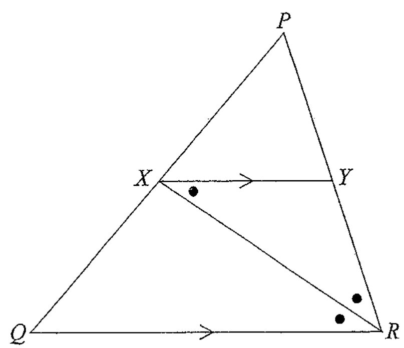
$= -\left[\log_e\left(\frac{1}{\sqrt{2}}\right) - 0\right]$

$= -\log_e(2)^{-\frac{1}{2}}$

$= \frac{1}{2}\log_e 2$

Question 4

(a)



$\angle QRX = \angle YRX$ (XR bisects $\angle PRQ$)

$\angle YXR = \angle QRX$ (alternate angles, $XY \parallel QR$)

$\therefore \angle YXR = \angle YRX$

$\therefore \Delta XYR$ is isosceles (2 angles equal)

(b) (i) 50, 60, 72, …

is a geometric sequence with

$a = 50, \quad r = 1.2$.

If the zoom is applied 8 times, we require the 9th term.

$T_n = ar^{n-1}$

$\therefore T_9 = 50 \times 1.2^{9-1}$

$= 50 \times 1.2^8$

$= 214.9908\ldots$

$= 215$ mm (to nearest mm).

(ii) We require $T_{n+1} > 400$ (since the first term is not an application of the zoom function).

$\therefore 50 \times 1.2^n > 400$

$1.2^n > 8$

$\ln 1.2^n > \ln 8$

$n \ln 1.2 > \ln 8$

$n > \frac{\ln 8}{\ln 1.2}$

$n > 11.4053\ldots$

$\therefore$ The zoom function must be applied 12 times.

(c) (i) Compare $x^2 = 8(y-3)$ with $(x-h)^2 = 4a(y-k)$

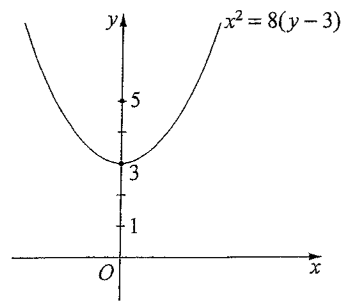
Vertex (h, k) is $(0, 3)$.

(ii) $4a = 8$

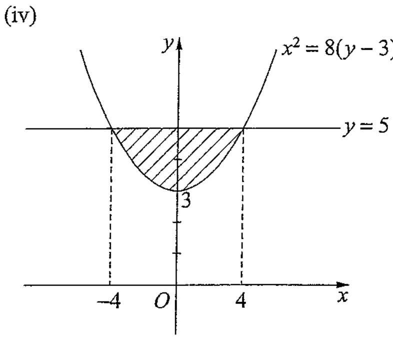
$\therefore a = 2$ which is the focal length.

$\therefore$ Focus is $(0, 5)$.

(iii)



(iv)



When $y = 5$,

$x^2 = 8(5-3)$

$x^2 = 8 \times 2$

$x^2 = 16$

$\therefore x = \pm 4$

rearranging,

$x^2 = 8(y-3)$

$\frac{x^2}{8} = y - 3$

$\therefore y = \frac{x^2}{8} + 3$

$\therefore$ Required area

$= 8 \times 5 - \int_{-4}^{4}\left(\frac{x^2}{8} + 3\right)dx$

$= 40 - 2\int_0^4\left(\frac{x^2}{8} + 3\right)dx$

(as the function is even)

$= 40 - 2\left[\frac{x^3}{24} + 3x\right]_0^4$

$= 40 - 2\left[\left(\frac{4^3}{24} + 3 \times 4\right) - 0\right]$

$= 40 - 2\left(\frac{64}{24} + 12\right)$

$= 40 - \frac{64}{12} - 24$

$= \frac{32}{3}$ or $10\frac{2}{3}$ units.

Question 5

(a) $\frac{dy}{dx}=1-6\sin 3x$

$\therefore y=x+\frac{6\cos 3x}{3}+c$

$y=x+2\cos 3x+c$

When $x=0$, $y=7$.

$\therefore 7=0+2\cos(3\times 0)+c$

$7=2+c$

$c=5$

$\therefore y=x+2\cos 3x+5.$

(b) $5+10x+20x^2+40x^3+\ldots$ is a geometric series with $a=5$ and $r=\frac{10x}{5}=2x$.

(i) For a limiting sum,

$|r|<1$

$|2x|<1$

$-1<2x<1$

$\therefore -\frac{1}{2}<x<\frac{1}{2}.$

(ii) Using the limiting sum formula,

$S_\infty=\frac{a}{1-r}$

$\therefore 100=\frac{5}{1-2x}$

$100(1-2x)=5$

$1-2x=\frac{5}{100}$

$-2x=-\frac{95}{100}$

$\therefore x=\frac{19}{40}.$

(c) (i) When $s=0$, $I=6000$.

$I=Ae^{-ks}$

$\therefore 6000=Ae^0$

$\therefore A=6000.$

(ii) When $s=6$, $I=1000$.

$\therefore 1000=6000e^{-6k}$

$e^{-6k}=\frac{1}{6}$

$e^{-6k}=6^{-1}$

$e^{6k}=6$

$6k=\log_e 6$

$\therefore k=\frac{1}{6}\log_e 6$

$=0.2986$ (to 4 sig. figs)

(iii) $\frac{dI}{ds}=-kAe^{-ks}$

When $s=6$,

$\frac{dI}{ds}=-k\times 6000\times e^{-6k}$

$=-k\times 6000\times e^{-6\times\frac{1}{6}\log_e 6}$

$=-k\times 6000\times\frac{1}{6}$

$=-1000k$

$=298.6$ lux/m (to 4 sig. fig

$\therefore$ Decreasing at 298.6 lux/m.

Question 6

(a) $2\sin^2\frac{x}{3}=1$

$\sin^2\frac{x}{3}=\frac{1}{2}$

$\sin\frac{x}{3}=\pm\frac{1}{\sqrt{2}}$

Since $-\pi\le x\le\pi$,

$-\frac{\pi}{3}\le\frac{x}{3}\le\frac{\pi}{3}$

$\therefore \frac{x}{3}=\pm\frac{\pi}{4}$

$\therefore x=\pm\frac{3\pi}{4}.$

(b) (i) Initial velocity is when $t=0$.

$\therefore v=20$ m/s.

(ii) Velocity $=0$ when $t=10$ s.

(iii) Acceleration $=0$ when $\frac{dv}{dt}=0$, i.e. at the stationary point, when $t=6$ s.

(iv) From the graph,

t	0	2	4	6	8
v	20	50	70	80	60
wts	1	4	2	4	1
wts × v	20	200	140	320	60

METHOD 1

$\int_a^b f(x)\,dx$

$=\frac{\text{total width}}{\sum\text{weights}}\times\sum(\text{weights}\times v)$

Distance

$=\frac{8}{12}[20+200+140+320+60]$

$=\frac{2}{3}\times 740$

$=493$ m.

METHOD 2

$\int_a^b f(x)\,dx$

$=\frac{b-a}{6}\left[f(a)\times 4f\left(\frac{a+b}{2}\right)+f(b)\right]$

Using two applications,

distance $\doteqdot \frac{4-0}{6}[20+4\times 50+70]$

$+\frac{8-4}{6}[70+4\times 80+60]$

$=\frac{2}{3}\times 290+\frac{2}{3}\times 450$

$=493$ m (to nearest m).

METHOD 3

$\int_a^b f(x)\,dx=\frac{h}{3}[f(a)+4f(\text{evens})+2f(\text{odds})+f(b)]$

Distance

$=\frac{2}{3}[20+4(50+80)+2\times 70+60]$

$=\frac{2}{3}[20+520+140+60]$

$=\frac{2}{3}\times 740$

$=493$ m (to nearest m).

(c) $V=\pi\int_a^b y^2\,dx$

$=\pi\int_3^6\left(\frac{5}{x-2}\right)^2 dx$

$=\pi\int_3^6\frac{25}{(x-2)^2}\,dx$

$=25\pi\int_3^6 (x-2)^{-2}\,dx$

$=25\pi\left[\frac{(x-2)^{-1}}{-1\times 1}\right]_3^6$

$=25\pi\left[\frac{-1}{(x-2)}\right]_3^6$

$=25\pi\left[\frac{-1}{(6-2)}-\frac{-1}{(3-2)}\right]$

$=25\pi\left[-\frac{1}{4}+1\right]$

$=25\pi\times\frac{3}{4}$

$=\frac{75\pi}{4}$ units^3.

Question 7

(a) $\log_e x - \dfrac{3}{\log_e x} = 2$

Let $u = \log_e x$.

Then $u - \dfrac{3}{u} = 2$

$u^2 - 3 = 2u$

$u^2 - 2u - 3 = 0$

$(u-3)(u+1) = 0$

$\therefore u = 3$ or $u = -1$

$\log_e x = 3$ $\log_e x = -1$

$x = e^3$ $x = e^{-1}$

$\therefore x = e^3$ or $\dfrac{1}{e}$.

(b) (i) Using the arc length formula,

$\ell = r\theta$

$\therefore \dfrac{10\pi}{3} = r\theta$

$\theta = \dfrac{10\pi}{3r}$

But $0 < \theta \le 2\pi$

$\therefore 0 < \dfrac{10\pi}{3r} \le 2\pi$

$0 < \dfrac{10}{3r} \le 2$

$0 < \dfrac{5}{3} \le r \quad (r > 0)$

$\therefore r \ge \dfrac{5}{3}$.

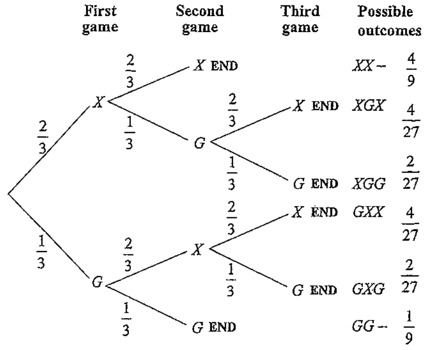
(ii) Using the formula for area of a sector,

$A = \dfrac{1}{2} r^2 \theta$

$= \dfrac{1}{2} \times 4^2 \times \dfrac{10\pi}{3 \times 4}$

$= \dfrac{20\pi}{3}$ units^2.

(c) (i)



(ii) $P(G \text{ wins})$

$= P(XGG) + P(GXG) + P(GG)$

$= \dfrac{2}{3} \times \dfrac{1}{3} \times \dfrac{1}{3} + \dfrac{1}{3} \times \dfrac{2}{3} \times \dfrac{1}{3} + \dfrac{1}{3} \times \dfrac{1}{3}$

$= \dfrac{2}{27} + \dfrac{2}{27} + \dfrac{1}{9}$

$= \dfrac{7}{27}$.

(iii) $P(3 \text{ games}) = 1 - P(2 \text{ games})$

$= 1 - [P(XX) + P(GG)]$

$= 1 - \left[\dfrac{2}{3} \times \dfrac{2}{3} + \dfrac{1}{3} \times \dfrac{1}{3}\right]$

$= 1 - \left[\dfrac{4}{9} + \dfrac{1}{9}\right]$

$= \dfrac{4}{9}$.

Question 8

(a) (i) x-axis: $f(x) = 0$

$\therefore x^4 - 8x^2 = 0$

$x^2(x^2 - 8) = 0$

$\therefore x^2 = 0$ or $x^2 - 8 = 0$

$x = 0$ $x^2 = 8$

$x = \pm 2\sqrt{2}$

y-axis: $x = 0$, $\therefore f(x) = 0$

$\therefore$ Graph crosses the axes at $(0,0)$ $(2\sqrt{2}, 0)$ and $(-2\sqrt{2}, 0)$.

(ii) $f(x)$ is even if $f(a) = f(-a)$.

$f(a) = a^4 - 8a^2$

$f(-a) = (-a)^4 - 8(-a)^2$

$= a^4 - 8a^2$

$= f(a)$

$\therefore f(x)$ is an even function.

(iii) Stationary points occur when $f'(x) = 0$.

$f'(x) = 4x^3 - 16x$

$\therefore 4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$\therefore 4x = 0$ or $x^2 - 4 = 0$

$x = 0$ $x^2 = 4$

$x = \pm 2$

When $x = 2$, $y = 2^4 - 8 \times 2^2$

$= -16$

When $x = -2$, $y = (-2)^4 - 8 \times (-2)^2$

$= -16$

To determine the nature of the points the second derivative can be used.

$f''(x) = 12x^2 - 16$

$f''(0) = 12 \times 0^2 - 16$

$= -16$

$< 0 \Rightarrow$ concave down

$\therefore$ Maximum at $(0, 0)$

$f''(2) = 12 \times 2^2 - 16$

$= 32$

$> 0 \Rightarrow$ concave up

$\therefore$ Minimum at $(2, -16)$

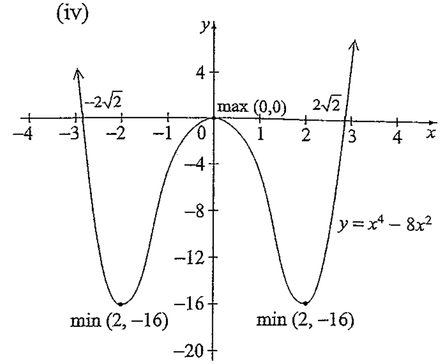
$f''(-2) = 12 \times (-2)^2 - 16$

$= 32$

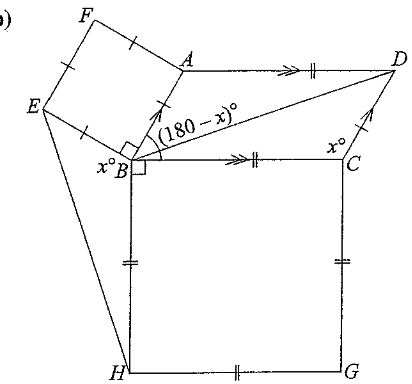
$> 0 \Rightarrow$ concave up

$\therefore$ Minimum at $(-2, -16)$.

(iv)



(b)



(i) $CD = AB$ (opposite sides of parallelogram $ABCD$ are equal)

$AB = BE$ (sides of square $ABEF$ are equal)

$\therefore CD = BE$.

(ii) Let $\angle BCD = x°$

Then $\angle ABC = (180 - x°)$ (cointerior angles, $AB \parallel DC$)

$\therefore \angle HBE = 360° - 90° - 90° - (180 - x)°$ (angles at a point add to 360°)

$= x°$

Now, in ΔHBE and ΔBCD

- $CD = BE$ (proven)
- $BC = HB$ (sides of square $BCGH$ are equal)
- $\angle BCD = \angle HBE$ (proven)

$\therefore \Delta BCD \equiv \Delta HBE$ (SAS)

$\therefore BD = EH$ (corresponding sides in congruent triangles).

Question 9

(a) (i) P(doesn't own a mobile)
$= 1 - 0.85$
$= 0.15$
P(neither owns a mobile)
$= 0.15 \times 0.15$
$= 0.0225.$

(ii) P(owns a mobile and used in class)
$= 0.85 \times 0.2$
$= 0.17.$

(b) $P = \$100\,000$
$r = 6\%$ p.a. $= 0.5\%$ per month
$\$M =$ monthly payment.

(i) $A_1 = 100\,000(1.005) - M$
$A_2 = A_1(1.005) - M$
$= [100\,000(1.005) - M](1.005) - M$
$= 100\,000(1.005)^2 - M(1.005 + 1)$
$A_3 = A_2(1.005) - M$
$= [100\,000(1.005)^2 - M(1.005 + 1)](1.005) - M$
$= 100\,000(1.005)^3 - M(1.005^2 + 1.005 + 1)$
$\vdots$
$\therefore A_n = 100\,000(1.005)^n - M(1.005^{n-1} + 1.005^{n-2} + \ldots + 1)$
$= 100\,000(1.005)^n - M(1 + 1.005 + \ldots + 1.005^{n-1}).$

(ii) $A_{144} = 100\,000(1.005)^{144} - M(1 + 1.005 + \ldots + 1.005^{143})$

Series is geometric with
$a = 1,\ r = 1.005,\ n = 144.$
Using the formula for sum,

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\therefore S_{144} = \frac{1(1.005^{144} - 1)}{1.005 - 1}$$
$$= \frac{1.005^{144} - 1}{0.005}$$

When $A_{144} = 0$,

$$0 = 100\,000(1.005)^{144} - \frac{M(1.005^{144} - 1)}{0.005}$$

$$\therefore M = \frac{100\,000(1.005)^{144}}{\left(\dfrac{1.005^{144} - 1}{0.005}\right)}$$
$$= \frac{100\,000(1.005)^{144} \times 0.005}{1.005^{144} - 1}$$

(c) (i) $f'(x)\int f''(x) = k\int (b^2 - x^2)\,dx$
$$= k\left(b^2x - \frac{x^3}{3} + c\right)$$
$$f'(-b) = k\left(-b^3 + \frac{b^3}{3} + c\right)$$
$$-f'(-b) = -k\left(-b^3 + \frac{b^3}{3} + c\right)$$

Since $f'(-b) = -f(b)$,

$$k\left(-b^3 + \frac{b^3}{3} + c\right) = -k\left(b^3 - \frac{b^3}{3} + 3\right)$$
$$-b^3 + \frac{b^3}{3} + c = -b^3 + \frac{b^3}{3} - c$$
$$\therefore 2c = 0$$
$$c = 0$$
$$\therefore f'(x) = k\left(b^2x - \frac{x^3}{3}\right).$$

(ii) $f(x) = \int f''(x) = k\int \left(b^2x - \frac{x^3}{3}\right)dx$
$$= k\left(\frac{b^2x^2}{2} - \frac{x^4}{12} + c\right)$$

At $(b, 0)$, $f(b) = 0$

$$\therefore k\left(\frac{b^4}{2} - \frac{b^4}{12} + c\right) = 0$$
$$\frac{5b^4}{12} + c = 0$$
$$\therefore c = -\frac{5b^4}{12}$$

$$\therefore f(x) = k\left(\frac{b^2x^2}{2} - \frac{x^4}{12} - \frac{5b^4}{12}\right)$$

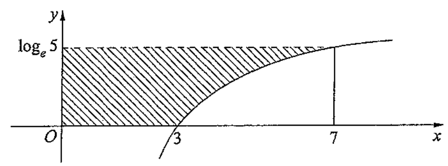
When $x = 0$,

$$f(0) = k\left(0 - 0 - \frac{5b^4}{12}\right)$$
$$= -\frac{5b^4}{12}$$

$\therefore$ At $x = 0$, the beam is $\dfrac{5b^4k}{12}$ below the x-axis.

Question 10

(a) By subtraction: required area
= area of rectangle – shaded region below.



To find the shaded region above,
$y = \log_e(x - 2)$
$e^y = x - 2$
$\therefore x = e^y + 2$

$$\therefore A = \int_0^{\log_e 5} (e^y + 2)\,dy$$
$$= \left[e^y + 2y\right]_0^{\log_e 5}$$
$$= \left(e^{\log_e 5} + 2\log_e 5\right) - \left(e^0 + 2 \times 0\right)$$
$$= 5 + 2\log_e 5 - 1$$
$$= 4 + 2\log_e 5$$

Area of rectangle $= 7\log_e 5$
$\therefore$ Required area
$= 7\log_e 5 - (4 + 2\log_e 5)$
$= 5\log_e 5 - 4$ units^2.

(b) (i)
$\angle OMP = \angle OJK$ and
$\angle OPM = \angle OJK$ (alternate angles $MP \parallel KJ$)
$\angle MOP = JOK$ (vertically opposite angles)
$\therefore \Delta MOP \,|||\, \Delta JOK$ (equiangular)

$\therefore \dfrac{MP}{JK} = \dfrac{OM}{OJ}$ (corresponding sides of similar triangles in the same ratio)

$$\therefore \frac{MP}{s} = \frac{\ell - x}{x}$$
$$MP = \frac{s(\ell - x)}{x}$$

$\therefore A =$ area ΔOKJ + area ΔOMP

$$= \frac{1}{2\ln} \times s \times x \times \sin\alpha + \frac{1}{2} \times \frac{s(\ell - x)}{x} \times (\ell - x) \times \sin\alpha$$
$$= \frac{1}{2}sx\sin\alpha + \frac{1}{2x}s(\ell - x)^2\sin\alpha$$
$$= s\sin\alpha\left(\frac{x}{2} + \frac{\ell^2 - 2\ell x + x^2}{2x}\right)$$
$$= s\left(\frac{x^2 + \ell^2 - 2\ell x + x^2}{2x}\right)\sin\alpha$$
$$= s\left(\frac{\ell^2 - 2\ell x + 2x^2}{2x}\right)\sin\alpha$$
$$= s\left(\frac{\ell^2}{2x} - \ell + x\right)\sin\alpha$$
$$= s\left(x - \ell + \frac{\ell^2}{2x}\right)\sin\alpha.$$

(ii) Minimum value occurs when $\frac{dA}{dx}=0$.

$$\frac{dA}{dx}=s\left(1-\frac{\ell^2}{2x^2}\right)\sin\alpha$$

$$\therefore s\left(1-\frac{\ell^2}{2x^2}\right)\sin\alpha=0$$

$$1-\frac{\ell^2}{2x^2}=0$$

$$1=\frac{\ell^2}{2x^2}$$

$$x^2=\frac{\ell^2}{2}$$

$$x=\frac{\ell}{\sqrt{2}}\quad(\ell>0).$$

METHOD 1

Use the second derivative test.

$$\frac{d^2A}{dx}=s\left(\frac{\ell^2}{x^3}\right)\sin\alpha$$

When $x=\frac{\ell}{\sqrt{2}}$,

$$\frac{d^2A}{dx^2}=s\left(\ell^2\div\frac{\ell^3}{2\sqrt{2}}\right)\sin\alpha$$

$$=s\left(\frac{2\sqrt{2}}{\ell^3}\right)\sin\alpha$$

>0 since s,ℓ,α are all positive.

$\therefore$ Minimum at $x=\frac{\ell}{\sqrt{2}}$.

METHOD 2

Test values on either side of $\frac{\ell}{\sqrt{2}}$ using the first derivative.

For example,

x	$\frac{\ell}{2}$	$\frac{\ell}{\sqrt{2}}$	ℓ
$\frac{dA}{dx}$	①	0	②

Value for ①: when $x=\frac{\ell}{2}$,

$$\frac{dA}{dx}=s\left(1-\ell^2\div\frac{2\ell^2}{2^2}\right)\sin\alpha$$

$$=s(1-2)\sin\alpha$$

$$=-s\sin\alpha$$

Value for ②: when $x=\ell$,

$$\frac{dA}{dx}=s\left(1-\frac{\ell^2}{2\ell^2}\right)\sin\alpha$$

$$=s\left(1-\frac{1}{2}\right)\sin\alpha$$

$$=\frac{s}{2}\sin\alpha$$

$\therefore$ We have

x	$\frac{\ell}{2}$	$\frac{\ell}{\sqrt{2}}$	ℓ
$\frac{dA}{dx}$	$-$	0	$+$

$\therefore$ Minimum at $x=\frac{\ell}{\sqrt{2}}$.

$$MP=\frac{s(\ell-x)}{x}\quad\text{from (i)}$$

(iii) when $x=\frac{\ell}{\sqrt{2}}$,

$$MP=\frac{s\left(\ell-\frac{\ell}{\sqrt{2}}\right)}{\frac{\ell}{\sqrt{2}}}$$

$$=s\left(\frac{\sqrt{2}\ell-\ell}{\sqrt{2}}\right)\div\frac{\ell}{\sqrt{2}}$$

$$=s\left(\frac{\sqrt{2}\ell-\ell}{\sqrt{2}}\right)\times\frac{\sqrt{2}}{\ell}$$

$$=s\left(\sqrt{2}-1\right)\text{ metres.}$$

End of Mathematics solutions