BOARD OF STUDIES
NEW SOUTH WALES

2012
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 7–18

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

1320

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 What is 4.097 84 correct to three significant figures?

(A) 4.09

(B) 4.10

(C) 4.097

(D) 4.098

2 Which of the following is equal to $\dfrac{1}{2\sqrt{5}-\sqrt{3}}$?

(A) $\dfrac{2\sqrt{5}-\sqrt{3}}{7}$

(B) $\dfrac{2\sqrt{5}+\sqrt{3}}{7}$

(C) $\dfrac{2\sqrt{5}-\sqrt{3}}{17}$

(D) $\dfrac{2\sqrt{5}+\sqrt{3}}{17}$

3 The quadratic equation $x^2+3x-1=0$ has roots α and β.

What is the value of $\alpha\beta+(\alpha+\beta)$?

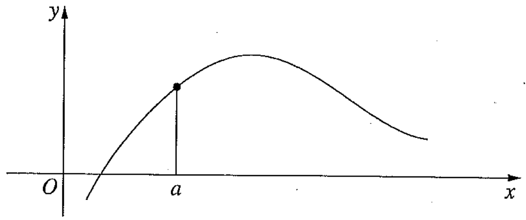
(A) 4

(B) 2

(C) –4

(D) –2

4 The diagram shows the graph $y=f(x)$.



Which of the following statements is true?

(A) $f'(a)>0$ and $f''(a)<0$

(B) $f'(a)>0$ and $f''(a)>0$

(C) $f'(a)<0$ and $f''(a)<0$

(D) $f'(a)<0$ and $f''(a)>0$

5 What is the perpendicular distance of the point $(2,-1)$ from the line $y=3x+1$?

(A) $\dfrac{6}{\sqrt{10}}$

(B) $\dfrac{6}{\sqrt{5}}$

(C) $\dfrac{8}{\sqrt{10}}$

(D) $\dfrac{8}{\sqrt{5}}$

6 What are the solutions of $\sqrt{3}\tan x = -1$ for $0 \le x \le 2\pi$?

(A) $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$

(B) $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$

(C) $\frac{5\pi}{6}$ and $\frac{7\pi}{6}$

(D) $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$

7 Let $a = e^x$.

Which expression is equal to $\log_e\left(a^2\right)$?

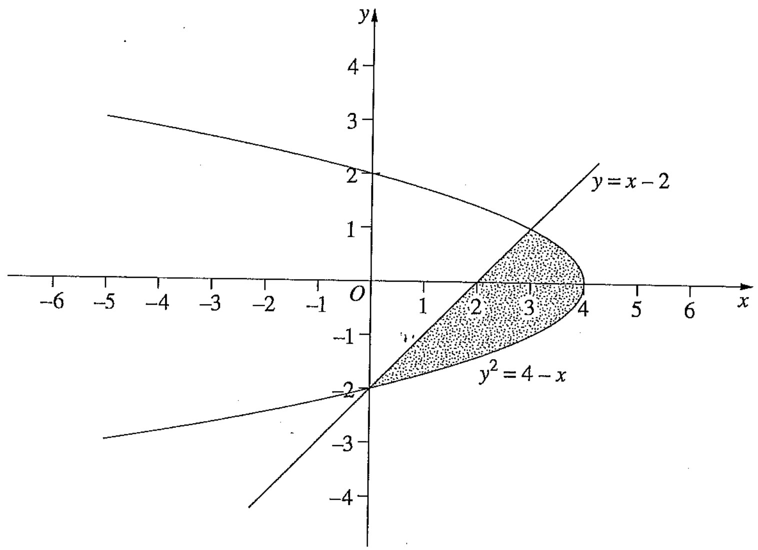
(A) e^{2x}

(B) e^{x^2}

(C) $2x$

(D) x^2

8 The diagram shows the region enclosed by $y = x - 2$ and $y^2 = 4 - x$.



Which of the following pairs of inequalities describes the shaded region in the diagram?

(A) $y^2 \le 4 - x$ and $y \le x - 2$

(B) $y^2 \le 4 - x$ and $y \ge x - 2$

(C) $y^2 \ge 4 - x$ and $y \le x - 2$

(D) $y^2 \ge 4 - x$ and $y \ge x - 2$

9 What is the value of $\int_1^4 \frac{1}{3x}\,dx$?

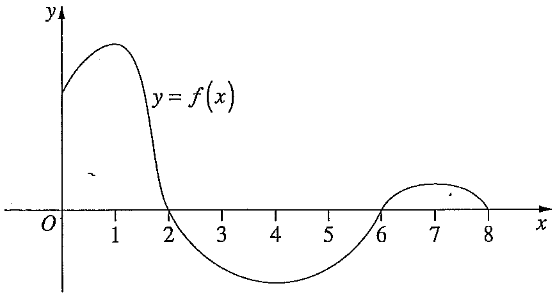
(A) $\frac{1}{3}\ln 3$

(B) $\frac{1}{3}\ln 4$

(C) $\ln 9$

(D) $\ln 12$

10 The graph of $y = f(x)$ has been drawn to scale for $0 \le x \le 8$.



Which of the following integrals has the greatest value?

(A) $\int_0^1 f(x)\,dx$

(B) $\int_0^2 f(x)\,dx$

(C) $\int_0^7 f(x)\,dx$

(D) $\int_0^8 f(x)\,dx$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Factorise $2x^2 - 7x + 3$. **2**

(b) Solve $|3x - 1| < 2$. **2**

(c) Find the equation of the tangent to the curve $y = x^2$ at the point where $x = 3$. **2**

(d) Differentiate $\left(3 + e^{2x}\right)^5$. **2**

(e) Find the coordinates of the focus of the parabola $x^2 = 16(y - 2)$. **2**

(f) The area of a sector of a circle of radius 6 cm is 50 cm^2. **2**

Find the length of the arc of the sector.

(g) Find $\int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2}\,dx$. **3**

Question 12 (15 marks) Use the Question 12 Writing Booklet.

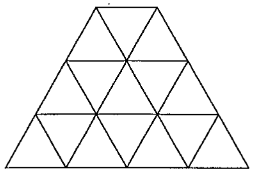
(a) Differentiate with respect to x.

(i) $(x-1)\log_e x$ **2**

(ii) $\dfrac{\cos x}{x^2}$ **2**

(b) Find $\displaystyle\int \frac{4x}{x^2+6}\,dx$. **2**

(c) Jay is making a pattern using triangular tiles. The pattern has 3 tiles in the first row, 5 tiles in the second row, and each successive row has 2 more tiles than the previous row.



(i) How many tiles would Jay use in row 20? **2**

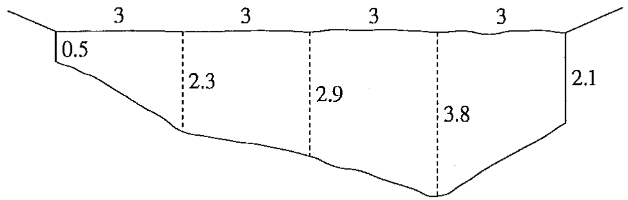
(ii) How many tiles would Jay use altogether to make the first 20 rows? **1**

(iii) Jay has only 200 tiles. **2**

How many complete rows of the pattern can Jay make?

Question 12 continues on page 9

(d) At a certain location a river is 12 metres wide. At this location the depth of the river, in metres, has been measured at 3 metre intervals. The cross-section is shown below.



(i) Use Simpson's rule with the five depth measurements to calculate the approximate area of the cross-section. **3**

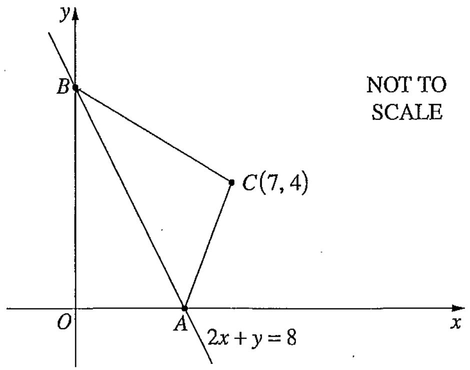
(ii) The river flows at 0.4 metres per second. **1**

Calculate the approximate volume of water flowing through the cross-section in 10 seconds.

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) The diagram shows a triangle ABC. The line $2x + y = 8$ meets the x and y axes at the points A and B respectively. The point C has coordinates $(7, 4)$.



(i) Calculate the distance AB. **2**

(ii) It is known that $AC = 5$ and $BC = \sqrt{65}$. (Do NOT prove this.) **2**

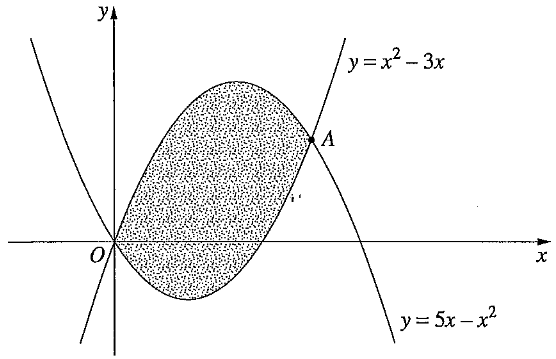
Calculate the size of $\angle ABC$ to the nearest degree.

(iii) The point N lies on AB such that CN is perpendicular to AB. **3**

Find the coordinates of N.

Question 13 continues on page 11

(b) The diagram shows the parabolas $y = 5x - x^2$ and $y = x^2 - 3x$. The parabolas intersect at the origin O and the point A. The region between the two parabolas is shaded.



(i) Find the x-coordinate of the point A. **1**

(ii) Find the area of the shaded region. **3**

(c) Two buckets each contain red marbles and white marbles. Bucket A contains 3 red and 2 white marbles. Bucket B contains 3 red and 4 white marbles.

Chris randomly chooses one marble from each bucket.

(i) What is the probability that both marbles are red? **1**

(ii) What is the probability that at least one of the marbles is white? **1**

(iii) What is the probability that both marbles are the same colour? **2**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) A function is given by $f(x) = 3x^4 + 4x^3 - 12x^2$.

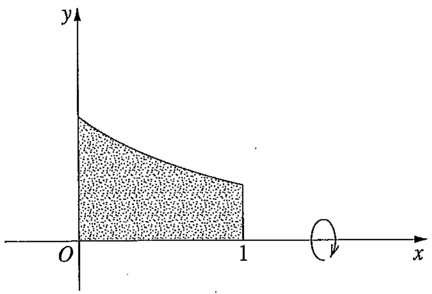
(i) Find the coordinates of the stationary points of $f(x)$ and determine their nature. **3**

(ii) Hence, sketch the graph $y = f(x)$ showing the stationary points. **2**

(iii) For what values of x is the function increasing? **1**

(iv) For what values of k will $3x^4 + 4x^3 - 12x^2 + k = 0$ have no solution? **1**

(b) The diagram shows the region bounded by $y = \dfrac{3}{(x+2)^2}$, the x-axis, the y-axis, and the line $x = 1$. **3**



The region is rotated about the x-axis to form a solid.

Find the volume of the solid.

Question 14 continues on page 13

(c) Professor Smith has a colony of bacteria. Initially there are 1000 bacteria. The number of bacteria, $N(t)$, after t minutes is given by

$$N(t) = 1000e^{kt}.$$

(i) After 20 minutes there are 2000 bacteria. **1**

Show that $k = 0.0347$ correct to four decimal places.

(ii) How many bacteria are there when $t = 120$? **1**

(iii) What is the rate of change of the number of bacteria per minute, when $t = 120$? **1**

(iv) How long does it take for the number of bacteria to increase from 1000 to 100 000? **2**

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) Rectangles of the same height are cut from a strip and arranged in a row. The first rectangle has width 10 cm. The width of each subsequent rectangle is 96% of the width of the previous rectangle.

10 cm

NOT TO SCALE

(i) Find the length of the strip required to make the first ten rectangles. **2**

(ii) Explain why a strip of length 3 m is sufficient to make any number of rectangles. **1**

(b) The velocity of a particle is given by

$$\dot{x} = 1 - 2\cos t,$$

where x is the displacement in metres and t is the time in seconds. Initially the particle is 3 m to the right of the origin.

(i) Find the initial velocity of the particle. **1**

(ii) Find the maximum velocity of the particle. **1**

(iii) Find the displacement, x, of the particle in terms of t. **2**

(iv) Find the position of the particle when it is at rest for the first time. **2**

Question 15 continues on page 15

(c) Ari takes out a loan of \$360 000. The loan is to be repaid in equal monthly repayments, \$$M$, at the end of each month, over 25 years (300 months). Reducible interest is charged at 6% per annum, calculated monthly.

Let \$$A_n$ be the amount owing after the nth repayment.

(i) Write down an expression for the amount owing after two months, \$$A_2$. **1**

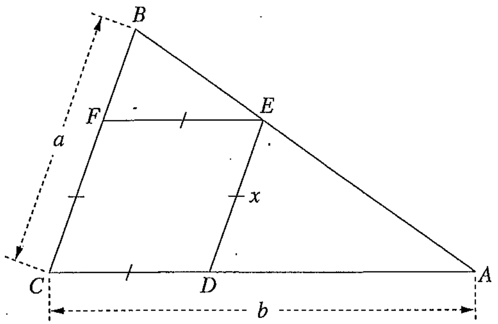
(ii) Show that the monthly repayment is approximately \$2319.50. **2**

(iii) After how many months will the amount owing, \$$A_n$, become less than \$180 000? **3**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet.

(a) The diagram shows a triangle ABC with sides $BC = a$ and $AC = b$. The points D, E and F lie on the sides AC, AB and BC, respectively, so that $CDEF$ is a rhombus with sides of length x.



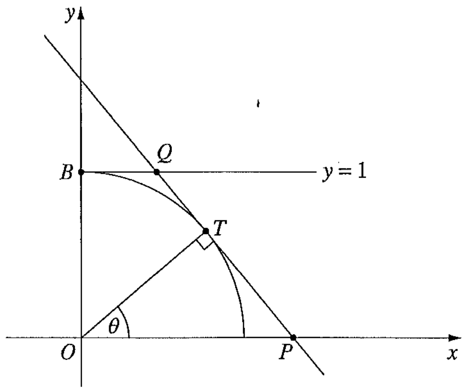
(i) Prove that $\triangle EBF$ is similar to $\triangle AED$. **2**

(ii) Find an expression for x in terms of a and b. **2**

Question 16 continues on page 17

(b) The diagram shows a point T on the unit circle $x^2 + y^2 = 1$ at angle θ from the positive x-axis, where $0 < \theta < \dfrac{\pi}{2}$.

The tangent to the circle at T is perpendicular to OT, and intersects the x-axis at P, and the line $y = 1$ at Q. The line $y = 1$ intersects the y-axis at B.



(i) Show that the equation of the line PT is **2**

$$x\cos\theta + y\sin\theta = 1.$$

(ii) Find the length of BQ in terms of θ. **1**

(iii) Show that the area, A, of the trapezium $OPQB$ is given by **2**

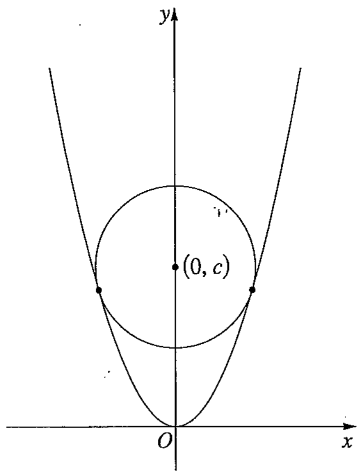
$$A = \frac{2 - \sin\theta}{2\cos\theta}.$$

(iv) Find the angle θ that gives the minimum area of the trapezium. **3**

Question 16 continues on page 18

Question 16 (continued)

(c) The circle $x^2 + (y - c)^2 = r^2$, where $c > 0$ and $r > 0$, lies inside the parabola $y = x^2$. The circle touches the parabola at exactly two points located symmetrically on opposite sides of the y-axis, as shown in the diagram.



(i) Show that $4c = 1 + 4r^2$. **2**

(ii) Deduce that $c > \frac{1}{2}$. **1**

End of paper

2012 Higher School Certificate Solutions Mathematics

SECTION I

Summary

1	B	4	A	7	C	9	B
2	D	5	C	8	A	10	B
3	C	6	D				

1 (B) Three significant figures means that there are only three digits used and the value must be rounded off correctly.

2 (D)
$$\frac{1}{2\sqrt{5}-\sqrt{3}}=\frac{1}{2\sqrt{5}-\sqrt{3}}\times\frac{2\sqrt{5}+\sqrt{3}}{2\sqrt{5}+\sqrt{3}}$$
$$=\frac{2\sqrt{5}+\sqrt{3}}{20-3}$$
$$=\frac{2\sqrt{5}+\sqrt{3}}{17}.$$

3 (C)
$$\alpha+\beta=-3,\quad \alpha\beta=-1$$
$$\alpha\beta+(\alpha+\beta)=-1+(-3)$$
$$=-4.$$

4 (A) At $(a, f(a))$, the curve has a positive gradient and the curve is concave down.

5 (C) $3x-y+1=0,\ (2,-1)$
$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$
$$=\frac{|3(2)-1(-1)+1|}{\sqrt{3^2+(-1)^2}}$$
$$=\frac{8}{\sqrt{10}}.$$

6 (D) $\sqrt{3}\tan x=-1$
$$\tan x=\frac{-1}{\sqrt{3}}$$
$\text{Tan}\,\theta$ is negative in the 2nd and 4th quadrants.

and $\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$,
$$\therefore x=\frac{5\pi}{6},\frac{11\pi}{6}.$$

7 (C)
$$\log_e\left(a^2\right)=2\log_e(a)$$
$$=2\log_e\left(e^x\right).$$
$$=2x.$$

8 (A) $(3,0)$ is in the required region.

Substitute into:

$y^2\le 4-x$ and $y\le x-2$

$0^2\le 4-3$ $\qquad$ $0\le 3-2$

$0\le 1$ $\qquad$ $0\le 1$

true $\qquad$ true

$\therefore$ the shaded region is $y^2\le 4-x$ and $y\le x-2$.

9 (B)
$$\int_1^4\frac{1}{3x}\,dx=\left[\frac{1}{3}\ln(x)\right]_1^4$$
$$=\frac{1}{3}(\ln 4-\ln 1)$$
$$=\frac{1}{3}\ln\left(\frac{4}{1}\right)$$
$$=\frac{1}{3}\ln 4.$$

10 (B) In evaluating integrals, areas above the x-axis are positive and areas below are negative.

$\therefore$ the greatest value is between $x=0$ and $x=2$.

SECTION II

Question 11

(a)
$$2x^2-7x+3=(2x-1)(x-3).$$

(b)
$$|3x-1|<2$$
$$-2<3x-1<2$$
$$-1<3x<3$$
$$-\frac{1}{3}<x<1.$$

(c) When $x=3$, $y=9$.
$$y=x^2$$
$$y'=2x$$
Gradient of tangent is 6.

Equation of tangent is:
$$y-y_1=m(x-x_1)$$
$$y-9=6(x-3)$$
$$y-9=6x-18$$
$$y=6x-9 \quad\text{or}\quad 6x-y-9=0.$$

(d)
$$\frac{d}{dx}\left(3+e^{2x}\right)^5=5\left(3+e^{2x}\right)^4.e^{2x}.2$$
$$=10e^{2x}\left(3+e^{2x}\right)^4.$$

(e)
$$x^2=16(y-2)$$
$$(x-0)^2=4a(y-2)$$
$(x-h)^2=4a(y-k)$ $\therefore$ vertex is $(0,2)$
$$4a=16$$
$a=4$ $\qquad$ $\therefore$ focal length is 4

The focus is $(0,6)$.

(f)
$$\text{Area}=\frac{1}{2}r^2\theta$$
$$50=\frac{1}{2}\times 6^2\times\theta$$
$$\theta=\frac{100}{36}.$$
$$\ell=r\theta$$
$$=6\times\frac{100}{36}$$
$$=16\frac{2}{3}\text{ cm}.$$

(g)
$$\int_0^{\frac{\pi}{2}}\sec^2\frac{x}{2}\,dx=\left[\frac{1}{\frac{1}{2}}\tan\frac{x}{2}\right]_0^{\frac{\pi}{2}}$$
$$=2\left[\tan\frac{\pi}{4}-\tan 0\right]$$
$$=2[1-0]$$
$$=2.$$

Question 12

(a) (i)
$$\frac{d}{dx}(x-1)\log_e x=1.\log_e x+(x-1).\frac{1}{x}$$
$$=\log_e x+1-\frac{1}{x}.$$

(ii)
$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\,{}^{du}\!/\!_{dx}-u\,{}^{dv}\!/\!_{dx}}{v^2}$$
$$\frac{d}{dx}\frac{\cos x}{x^2}=\frac{x^2(-\sin x)-(\cos x)(2x)}{\left(x^2\right)^2}$$
$$=\frac{-x(x\sin x+2\cos x)}{x^4}$$
$$=\frac{-(x\sin x+2\cos x)}{x^3}.$$

(b)
$$\int\frac{4x}{x^2+6}\,dx=2\int\frac{2x}{x^2+6}\,dx$$
$$=2\ln\left(x^2+6\right)+C.$$

(c) (i) $T_n=a+(n-1)d \qquad a=3,\ d=2.$
$$T_{20}=3+(19)2$$
$$=41.$$
$\therefore$ 41 tiles would be used.

(ii) $S_n=\frac{n}{2}\left(2a+(n-1)d\right).$

$S_{20}=\frac{20}{2}\left(2\times3+(20-1)\times2\right)$

$=10\left(6+19\times2\right)$

$=440.$

$\therefore$ 440 tiles are needed.

OR

$S_n=\frac{n}{2}(a+\ell).$

$S_{20}=\frac{20}{2}(3+41)$

$=10(44)$

$=440.$

$\therefore$ 440 tiles are needed.

(iii) $S_n=\frac{n}{2}\left(2a+(n-1)d\right)$

$200\geq\frac{n}{2}\left(2\times3+(n-1)\times2\right)$

$\geq\frac{n}{2}(6+2n-2)$

$\geq\frac{n}{2}(2n+4)$

$\geq n^2+2n.$

$n^2+2n-200\leq0.$

Solve this inequality to find n.

$n=\frac{-1\pm\sqrt{2^2-4\times1\times-200}}{2}$

$\approx13\cdot2$ or -15.2

Since $n>0$, there will be 13 complete rows.

(d) (i) $\text{Area}\approx\frac{h}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+y_5\right)$

$\approx\frac{3}{3}(0.5+4\times2.3+2\times2.9+4\times3.8+2.1)$

$\approx32.8\text{ m}^2.$

(ii) Distance = speed × time

$=0.4\times10$

$=4\text{m}.$

$\text{Volume}=32.8\times4$

$=131.2\text{ m}^3.$

Question 13

(a) (i) On the line:

$2x+y=8$

when $x=0,\quad y=8,\ OB=8$

when $y=0,\quad x=4,\ OA=4.$

$AB=\sqrt{4^2+8^2}$

$=\sqrt{16+64}$

$=\sqrt{80}$

$=4\sqrt{5}$ units.

(ii) Using the cosine rule:

$\cos B=\frac{a^2+c^2-b^2}{2ac}$

$\cos B=\frac{\left(\sqrt{65}\right)^2+\left(\sqrt{80}\right)^2-5^2}{2\times\sqrt{65}\times\sqrt{80}}$

$=\frac{65+80-25}{2\times\sqrt{5\times13}\times\sqrt{16\times5}}$

$=\frac{120}{2\times5\times4\times\sqrt{13}}$

$=\frac{3}{\sqrt{13}}.$

$\angle ABC=33\cdot69\ldots$

$=34°$ (nearest degree).

(iii) $m_{AB}=-2$ since $y=-2x+8$

$m_{CN}=\frac{1}{2}$ since $m_{AB}m_{CN}=-1$

Equation of CN:

$y-y_1=m(x-x_1)$

$y-4=\frac{1}{2}(x-7)$

$2y-8=x-7$

$x-2y+1=0.$

To find N, solve simultaneously:

$y=-2x+8$ ①

$x-2y+1=0$ ②

Substitute ① in ②

$x-2(-2x+8)+1=0$

$x+4x-16+1=0$

$5x=15$

$x=3$

$y=2.$

$\therefore$ N is $(3,2)$.

(b) (i) Intersection of the parabolas when:

$x^2-3x=5x-x^2$

$2x^2-8x=0$

$2x(x-4)=0$

$x=0,4.$

$\therefore$ the x-coordinate of A is 4.

(ii) $\text{Area}=\int_0^4\left(5x-x^2\right)-\left(x^2-3x\right)dx$

$=\int_0^4 8x-2x^2\,dx$

$=\left[4x^2-\frac{2}{3}x^3\right]_0^4$

$=\left(64-\frac{128}{3}\right)-(0)$

$=21\frac{1}{3}$ units².

(c) (i) $P\left(r_Ar_B\right)=\frac{3}{5}\times\frac{3}{7}$

$=\frac{9}{35}.$

(ii) $P(\text{1 white or more})=1-P\left(r_Ar_B\right)$

$=1-\frac{9}{35}$

$=\frac{26}{35}.$

(iii) $P(\text{same colour})=P\left(r_Ar_B\right)+P\left(w_Aw_B\right)$

$=\frac{9}{35}+\frac{2}{5}\times\frac{4}{7}$

$=\frac{17}{35}.$

Question 14

(a) (i) $f(x)=3x^4+4x^3-12x^2$

$f'(x)=12x^3+12x^2-24x$

For $f'(x)=0$

$12x\left(x^2+x-2\right)=0$

$12x(x+2)(x-1)=0$

$x=-2,0,1.$

$f''(x)=36x^2+24x-24$

$=12\left(3x^2+2x-2\right)$

for $x=-2,\ f(-2)=-32,\ f''(-2)>($

for $x=0,\ f(0)=0,\quad f''(0)<0$

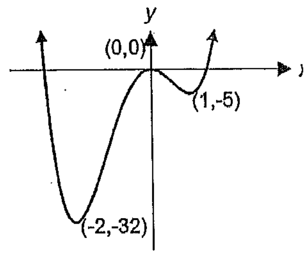
for $x=1,\ f(1)=-5,\quad f''(1)>0.$

$(-2,-32)$ is a minimum turning poi

$(0,0)$ is a maximum turning point.

$(1,-5)$ is a minimum turning point.

(ii)



(iii) From the graph, the function is increasing for $-2<x<0,\quad x>1$.

(iv) From the graph, the absolute minim is at $(-2,-32)$ and this would have be shifted vertically above the x-ax $\therefore\ k>32$.

(b)
$$V=\pi\int_a^b y^2\,dx$$
$$V=\pi\int_0^1\left(\frac{3}{(x+2)^2}\right)^2dx$$
$$=\pi\int_0^1\frac{9}{(x+2)^4}dx$$
$$=9\pi\int_0^1(x+2)^{-4}\,dx$$
$$=9\pi\left[-\frac{1}{3}(x+2)^{-3}\right]_0^1$$
$$=9\pi\times-\frac{1}{3}\left(\frac{1}{3^3}-\frac{1}{2^3}\right)$$
$$=-3\pi\times\left(\frac{8-27}{216}\right)$$
$$=\frac{19\pi}{72}\quad\text{units}^3.$$

(c) (i) When $t=20$, $N=2000$
$$N(t)=1000e^{kt}$$
$$2000=1000e^{20k}$$
$$2=e^{20k}$$
$$\ln 2=20k$$
$$k=\frac{\ln 2}{20}$$
$$=0\cdot 03465\ldots$$
$$=0\cdot 0347\quad(4\text{ dp}).$$

(ii) When $t=120$:
$$N=1000\,e^{120\times 0.0347}$$
$$\approx 64\,328\text{ bacteria.}$$

(iii)
$$N=1000e^{kt}$$
$$\frac{dN}{dt}=1000ke^{kt}$$
$$=0.0347\times 1000e^{120\times 0.0347}$$
$$=2232.192772\ldots$$
$$\approx 2232\text{ bacteria / min}.$$

(iv) When $N=100\,000$:
$$100\,000=1000e^{kt}$$
$$100=e^{kt}$$
$$\ln 100=kt$$
$$t=\frac{\ln 100}{0.0347}$$
$$=132\cdot 7138382\ldots$$
$$\approx 2\text{ hours }13\text{ min (nearest min)}.$$

Question 15

(a) (i) Geometric series, $a=10$, $r=0\cdot 96$
$$S_n=\frac{a(1-r^n)}{1-r}$$
$$S_{10}=\frac{10(1-0\cdot 96^{10})}{1-0\cdot 96}$$
$$=83\cdot 791841\ldots$$
$$=84\text{ cm}\quad(\text{nearest cm})$$

$\therefore$ the length of strip is 84 cm.

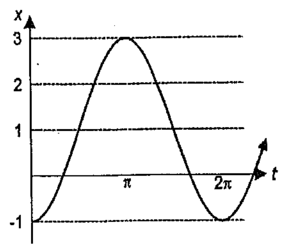
(ii)
$$S_\infty=\frac{a}{1-r}$$
$$=\frac{10}{1-0\cdot 96}$$
$$=250\text{ cm}$$

The length can never exceed 250 cm or 2.5m. $\therefore$ A strip of 3 m is sufficient.

(b) (i)
$$\dot{x}=1-2\cos t$$
when $t=0$,
$$\dot{x}=1-2\cos 0$$
$$=1-2\times 1$$
$$=-1.$$

$\therefore$ the initial velocity is -1 m/s.

(ii) *Method 1:*



From the graph the maximum velocity is 3 m/s.

OR

Method 2:
$$\ddot{x}=-2\times-\sin t$$
$$=2\sin t$$
For $\ddot{x}=0$
$$0=2\sin t$$
$$t=0,\pi,2\pi,\ldots$$
when $t=0$,
$$\dot{x}=1-2\cos 0$$
$$=-1$$
when $t=\pi$,
$$\dot{x}=1-2\cos\pi$$
$$=3$$

$\therefore$ maximum velocity of 3 m/s.

(iii)
$$x=\int\dot{x}\,dt$$
$$=\int 1-2\cos t\,dt$$
$$=t-2\sin t+c$$
when $t=0$, $x=3$
$$3=0-2\sin 0+c$$
$$c=3$$
$$\therefore x=t-2\sin t+3.$$

(iv) Particle is at rest when $\dot{x}=0$.
$$\dot{x}=1-2\cos t$$
$$0=1-2\cos t$$
$$2\cos t=1$$
$$\cos t=\frac{1}{2}$$
$$t=\frac{\pi}{3},\frac{5\pi}{3},\ldots$$

$\therefore$ the particle is at rest for the first time after $\frac{\pi}{3}$ seconds.
$$x=t-2\sin t+3$$
$$=\frac{\pi}{3}-2\sin\frac{\pi}{3}+3$$
$$=\frac{\pi}{3}-2\times\frac{\sqrt{3}}{2}+3$$
$$=\frac{\pi}{3}-\sqrt{3}+3\text{ m}\quad[\text{exact answer}]$$
$$\approx 2\cdot 315146744$$
$$\approx 2\cdot 3\text{ m }(1\text{ dp}).$$

(c) (i) 6% pa is $\frac{6}{12}\%=0\cdot 005$ per month
$$A_1=360\,000(1+0\cdot 005)-M$$
$$A_2=A_1(1\cdot 005)-M$$
$$=(360\,000(1\cdot 005)-M)1\cdot 005-M$$
$$=360\,000\times 1\cdot 005^2-1\cdot 005M-M$$
$$=360\,000\times 1\cdot 005^2-M(1\cdot 005+1).$$

(ii)
$$A_n=360\,000\times 1\cdot 005^n-M\begin{pmatrix}1+1\cdot 005+\ldots\\+1\cdot 005^{n-1}\end{pmatrix}$$
$$A_{300}=360\,000\times 1\cdot 005^{300}-M\begin{pmatrix}1+1\cdot 005+\ldots\\+1\cdot 005^{299}\end{pmatrix}$$
$$=360\,000\times 1\cdot 005^{300}-M\frac{(1\cdot 005^{300}-1)}{1\cdot 005-1}$$

But $A_{300}=0$ when the loan is paid in full.
$$M\frac{(1\cdot 005^{300}-1)}{0\cdot 005}=360\,000\times 1\cdot 005^{300}$$
$$M=\frac{360\,000\times 1\cdot 005^{300}\times 0\cdot 005}{(1\cdot 005^{300}-1)}$$
$$=2319\cdot 485045\ldots$$
$$=\$2319\cdot 50\ (\text{nearest }5c).$$

(iii) [Warning: The font size has been reduced to allow the calculation to fit into the available space.]
$$A_n<180\,000$$
$$360\,000\times 1\cdot 005^n-M\frac{(1\cdot 005^n-1)}{0\cdot 005}<180\,000$$
$$360\,000\times 1\cdot 005^n-\frac{2319.5(1\cdot 005^n-1)}{0\cdot 005}<180\,000$$
$$360\,000\times 1\cdot 005^n-463\,900(1\cdot 005^n-1)<180\,000$$
$$1\cdot 005^n(360\,000-463\,900)+463\,900<180\,000$$
$$1\cdot 005^n(360\,000-463\,900)<180\,000-463\,900$$

$$1{\cdot}005^n(-103\,900) < -283\,900$$

$$1{\cdot}005^n > \frac{283\,900}{103\,900}$$

$$n\ln 1{\cdot}005 > \ln\left(\frac{283\,900}{103\,900}\right)$$

$$n > \frac{\ln\left(\frac{283\,900}{103\,900}\right)}{\ln 1{\cdot}005}$$

$$n > 201{\cdot}5408119\ldots$$

$\therefore$ it takes 202 months for the amount owing to become less than $180 000.

Question 16

(a) (i) $\angle AED = \angle EBF$ (corresponding $\angle$s, $ED \parallel BC$)

$\angle EAD = \angle BEF$ (corresponding $\angle$s, $AC \parallel EF$)

$\angle ADE = \angle EFB$ ($\angle$ sum of Δ in both Δs)

$\therefore \Delta AED \parallel\!| \Delta EBF$ (equi-angular)

(ii) $\frac{ED}{BF} = \frac{DA}{FE}$ (matching sides of similiar Δs are in the same ratio)

$$\frac{x}{a-x} = \frac{b-x}{x}$$

$$x^2 = (a-x)(b-x)$$

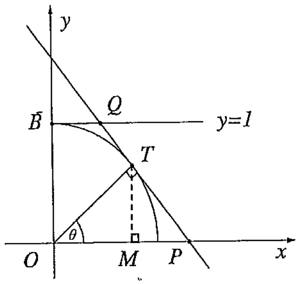
$$x^2 = ab - ax - bx + x^2$$

$$0 = ab - x(a+b)$$

$$x(a+b) = ab$$

$$x = \frac{ab}{a+b}.$$

(b) (i)



From T draw $TM \perp OM$

$OT = 1$, radius of unit circle

$$\sin\theta = \frac{TM}{OT} = TM$$

$$\cos\theta = \frac{OM}{OT} = OM$$

T is the point $(\cos\theta, \sin\theta)$

$\angle TPx = (90+\theta)°$ (exterior $\angle$ of ΔPTO)

Method 1:

$$m_{PT} = \tan(90+\theta)$$

$$= -\tan(180-(90+\theta))$$

$$= -\tan(90-\theta)$$

$$= -\cot\theta$$

$$= -\frac{\cos\theta}{\sin\theta}$$

OR

Method2:

gradient of OT $= \frac{\sin\theta}{\cos\theta}$

gradient of PT $= -\frac{\cos\theta}{\sin\theta}$

since $m_{PT} \times m_{OT} = -1$.

Equation of PT:

$$y - y_1 = m(x - x_1)$$

$$y - \sin\theta = -\frac{\cos\theta}{\sin\theta}(x - \cos\theta)$$

$$y\sin\theta - \sin^2\theta = -x\cos\theta + \cos^2\theta$$

$$x\cos\theta + y\sin\theta = \sin^2\theta + \cos^2\theta$$

$$x\cos\theta + y\sin\theta = 1.$$

$[\text{since } \sin^2\theta + \cos^2\theta = 1]$

(ii) Q is the intersection of PT with BQ

$x\cos\theta + y\sin\theta = 1$ and $y = 1$

$$x\cos\theta + 1.\sin\theta = 1$$

$$x\cos\theta = 1 - \sin\theta$$

$$x = \frac{1-\sin\theta}{\cos\theta}$$

$$BQ = \frac{1-\sin\theta}{\cos\theta}.$$

(iii) For the distance OP:

$x\cos\theta + y\sin\theta = 1$ and $y = 0$

$$x\cos\theta + 0.\sin\theta = 1$$

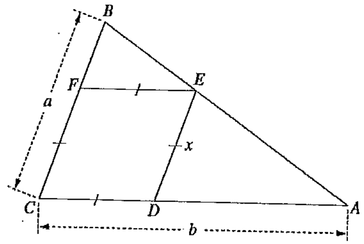
$$x\cos\theta = 1$$

(ii) Show that the monthly repayment is approximately $2319.50. **2**

(iii) After how many months will the amount owing, $\$A_n$, become less than $180 000 ? **3**

Question 16 (15 marks) **Marks**

(a) The diagram shows a triangle ABC with sides $BC = a$ and $AC = b$. The points D, E and F lie on the sides AC, AB and BC, respectively, so that $CDEF$ is a rhombus with sides of length x.



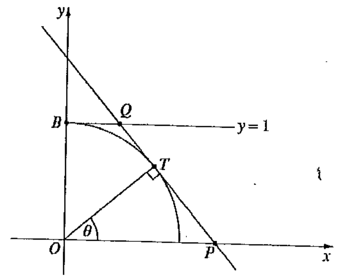
Copy or trace the diagram into your writing booklet.

(i) Prove that $\triangle EBF$ is similar to $\triangle AED$. **2**

(ii) Find an expression for x in terms of a and b. **2**

(b) The diagram shows a point T on the unit circle $x^2 + y^2 = 1$ at angle θ from positive x-axis, where $0 < \theta < \frac{\pi}{2}$.

The tangent to the circle at T is perpendicular to OT, and intersects the x-axis at P, and the line $y = 1$ at Q. The line $y = 1$ intersects the y-axis at B.



(i) Show that the equation of the line PT is $x\cos\theta + y\sin\theta = 1$

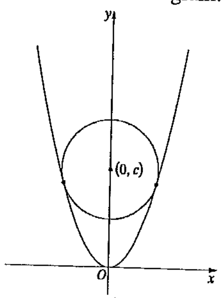
(ii) Find the length of BQ in terms of θ.

(iii) Show that the area, A, of the trapezium $OPQB$ is given by

$$A = \frac{2-\sin\theta}{2\cos\theta}$$

(iv) Find the angle θ that gives the minimum area of the trapezium.

(c) The circle $x^2 + (y-c)^2 = r^2$, where $c > 0$ and $r > 0$, lies inside the parabola $y = x^2$. The circle touches the parabola at exactly two points located symmetrically on opposite sides of the y-axis, as shown in the diagram.



(i) Show that $4c = 1 + 4r^2$

(ii) Deduce that $c > \frac{1}{2}$.

End of paper