BOARD OF STUDIES
NEW SOUTH WALES

**2001**
HIGHER SCHOOL CERTIFICATE
EXAMINATION

# Mathematics Extension 1

**General Instructions**
- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

**Total marks – 84**
- Attempt Questions 1–7
- All questions are of equal value



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**Attempt Questions 1–7**
**All questions are of equal value**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

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**Marks**

**Question 1** (12 marks) Use a SEPARATE writing booklet.

(a) Use the table of standard integrals to find the exact value of **2**

$$\int_0^2 \frac{dx}{\sqrt{16-x^2}}\ .$$

(b) Find $\dfrac{d}{dx}\left(x\sin^2 x\right)$ . **2**

(c) Evaluate $\displaystyle\sum_{n=4}^{7}(2n+3)$ . **1**

(d) Let $A$ be the point $(-2, 7)$ and let $B$ be the point $(1, 5)$. Find the coordinates of the point $P$ which divides the interval $AB$ externally in the ratio $1:2$. **2**

(e) Is $x+3$ a factor of $x^3-5x+12$? Give reasons for your answer. **2**

(f) Use the substitution $u=1+x$ to evaluate **3**

$$15\int_{-1}^{0} x\sqrt{1+x}\, dx\ .$$

Marks

**Question 2** (12 marks) Use a SEPARATE writing booklet.

(a) Let $f(x)=3x^2+x$. Use the definition **2**

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

to find the derivative of $f(x)$ at the point $x=a$.

(b) Find

(i) $\int \frac{e^x}{1+e^x}\,dx$ **1**

(ii) $\int_0^{\pi} \cos^2 3x\,dx$. **3**

(c) The letters $A$, $E$, $I$, $O$, and $U$ are vowels.

(i) How many arrangements of the letters in the word ALGEBRAIC are possible? **1**

(ii) How many arrangements of the letters in the word ALGEBRAIC are possible if the vowels must occupy the 2nd, 3rd, 5th and 8th positions? **2**

(d) Find the term independent of $x$ in the binomial expansion of **3**

$$\left(x^2-\frac{1}{x}\right)^9.$$



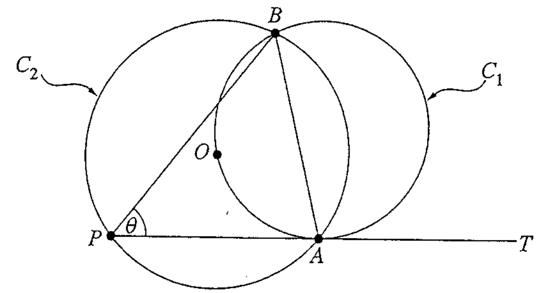
Marks

**Question 3** (12 marks) Use a SEPARATE writing booklet.

(a) The function $f(x)=\sin x+\cos x-x$ has a zero near $x=1.2$ **3**

Use one application of Newton's method to find a second approximation to the zero. Write your answer correct to three significant figures.

(b)




Two circles, $C_1$ and $C_2$, intersect at points $A$ and $B$. Circle $C_1$ passes through the centre $O$ of circle $C_2$. The point $P$ lies on circle $C_2$ so that the line $PAT$ is tangent to circle $C_1$ at point $A$. Let $\angle APB=\theta$.

Copy or trace the diagram into your writing booklet.

(i) Find $\angle AOB$ in terms of $\theta$. Give a reason for your answer. **1**

(ii) Explain why $\angle TAB = 2\theta$. **1**

(iii) Deduce that $PA=BA$. **2**

(c) (i) Starting from the identity $\sin(\theta+2\theta)=\sin\theta\cos 2\theta+\cos\theta\sin 2\theta$, and using the double angle formulae, prove the identity **2**

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

(ii) Hence solve the equation **3**

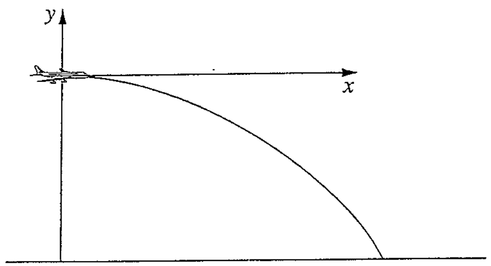
$$\sin 3\theta = 2\sin\theta \quad \text{for} \quad 0\le\theta\le 2\pi.$$

Marks

**Question 4** (12 marks) Use a SEPARATE writing booklet.

(a) Solve $\dfrac{3x}{x-2} \le 1$. **3**

(b) An aircraft flying horizontally at $V\,\text{m s}^{-1}$ releases a bomb that hits the ground 4000 m away, measured horizontally. The bomb hits the ground at an angle of 45° to the vertical. **4**




Assume that, $t$ seconds after release, the position of the bomb is given by

$$x = Vt, \quad y = -5t^2.$$

Find the speed $V$ of the aircraft.

(c) A particle, whose displacement is $x$, moves in simple harmonic motion. **5**

Find $x$ as a function of $t$ if

$$\ddot{x} = -4x$$

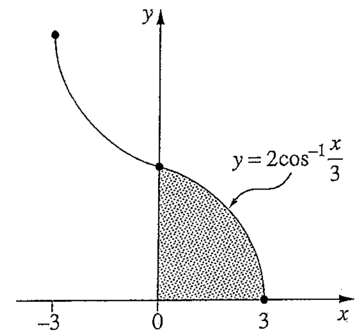
and if $x = 3$ and $\dot{x} = -6\sqrt{3}$ when $t = 0$.



Marks

**Question 5** (12 marks) Use a SEPARATE writing booklet.

(a)




The sketch shows the graph of the curve $y = f(x)$ where $f(x) = 2\cos^{-1}\frac{x}{3}$.
The area under the curve for $0 \le x \le 3$ is shaded.

(i) Find the $y$ intercept. **1**

(ii) Determine the inverse function $y = f^{-1}(x)$, and write down the domain $D$ of this inverse function. **2**

(iii) Calculate the area of the shaded region. **2**

(b) By using the binomial expansion, show that **3**

$$(q+p)^n - (q-p)^n = 2\binom{n}{1}q^{n-1}p + 2\binom{n}{3}q^{n-3}p^3 + \cdots$$

What is the last term in the expansion when $n$ is odd? What is the last term in the expansion when $n$ is even?

(c) A fair six-sided die is randomly tossed $n$ times.

(i) Suppose $0 \le r \le n$. What is the probability that exactly $r$ 'sixes' appear in the uppermost position? **2**

(ii) By using the result of part (b), or otherwise, show that the probability that an odd number of 'sixes' appears is **2**

$$\frac{1}{2}\left\{1 - \left(\frac{2}{3}\right)^n\right\}.$$

Marks

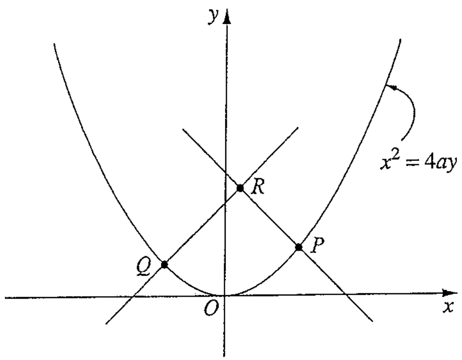
**Question 6** (12 marks) Use a SEPARATE writing booklet.

(a) Prove by induction that **3**

$$n^3+(n+1)^3+(n+2)^3$$

is divisible by 9 for $n=1, 2, 3, \ldots$

(b)




Consider the variable point $P(2at, at^2)$ on the parabola $x^2=4ay$.

(i) Prove that the equation of the normal at $P$ is $x+ty=at^3+2at$. **2**

(ii) Find the coordinates of the point $Q$ on the parabola such that the normal at $Q$ is perpendicular to the normal at $P$. **1**

(iii) Show that the two normals of part (ii) intersect at the point $R$, whose coordinates are **4**

$$x=a\left(t-\frac{1}{t}\right), \quad y=a\left(t^2+1+\frac{1}{t^2}\right).$$

(iv) Find the equation in Cartesian form of the locus of the point $R$ given in part (iii). **2**



Marks

**Question 7** (12 marks) Use a SEPARATE writing booklet.

(a) A particle moves in a straight line so that its acceleration is given by

$$\frac{dv}{dt} = x-1$$

where $v$ is its velocity and $x$ is its displacement from the origin.

Initially, the particle is at the origin and has velocity $v=1$.

(i) Show that $v^2=(x-1)^2$. **2**

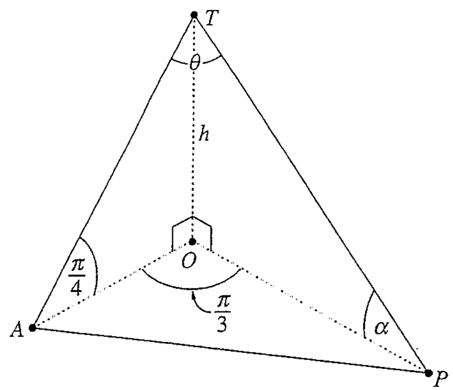
(ii) By finding an expression for $\frac{dt}{dx}$, or otherwise, find $x$ as a function of $t$. **2**

**Question 7 continues on page 9**

Marks

Question 7 (continued)

(b)




Consider the diagram, which shows a vertical tower $OT$ of height $h$ metres, a fixed point $A$, and a variable point $P$ that is constrained to move so that angle $AOP$ is $\frac{\pi}{3}$ radians. The angle of elevation of $T$ from $A$ is $\frac{\pi}{4}$ radians.

Let the angle of elevation of $T$ from $P$ be $\alpha$ radians and let angle $ATP$ be $\theta$ radians.

(i) By considering triangle $AOP$, show that **1**

$$AP^2 = h^2 + h^2\cot^2\alpha - h^2\cot\alpha .$$

(ii) By finding a second expression for $AP^2$, deduce that **3**

$$\cos\theta = \frac{1}{\sqrt{2}}\sin\alpha + \frac{1}{2\sqrt{2}}\cos\alpha .$$

(iii) Sketch a graph of $\theta$ for $0 < \alpha < \frac{\pi}{2}$, identifying and classifying any turning points. Discuss the behaviour of $\theta$ as $\alpha \to 0$ and as $\alpha \to \frac{\pi}{2}$. **4**

**End of paper**



## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

# 2001 Higher School Certificate Solutions

# MATHEMATICS EXTENSION 1

**QUESTION 1**

(a) $\displaystyle\int_0^2 \frac{dx}{\sqrt{16-x^2}} = \left[\sin^{-1}\frac{x}{4}\right]_0^2$

$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1} 0$

$= \frac{\pi}{6} - 0$

$= \frac{\pi}{6}.$

(b) $\frac{d}{dx}\left(x\sin^2 x\right) = 1\,.\sin^2 x + x\,.\,2\sin x\cos x$

$= \sin^2 x + 2x\sin x\cos x.$

(c) $\displaystyle\sum_{n=4}^{7}\left(2n+3\right) = \left(2\times4+3\right)+\left(2\times5+3\right)$

$+\left(2\times6+3\right)+\left(2\times7+3\right)$

$= 11+13+15+17$

$= 56.$

(d) $A(-2,\,7)\quad B(1,\,5)$

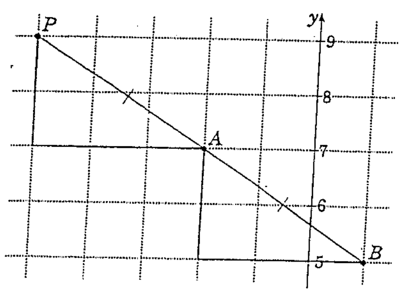
$-1:2$

$P$ has coordinates

$\left(\frac{2\times(-2)+(-1)\times1}{2-1},\ \frac{2\times7+(-1)\times5}{2-1}\right)$

$= (-5,\,9).$

Alternatively, an external ratio of 1 : 2 means that $PA = AB$, and the result can be found easily from a diagram.




(e) Let $P(x) = x^3 - 5x + 12$

$P(-3) = (-3)^3 - 5(-3) + 12$

$= 0.$

$\therefore$ $x+3$ is a factor of $P(x)$ by the Factor Theorem.

OR

$$
\begin{array}{r|l}
 & x^2 - 3x + 4 \\
x+3 & x^3 \qquad\quad - 5x + 12 \\
 & x^3 + 3x^2 \\
 & -3x^2 - 5x \\
 & -3x^2 - 9x \\
 & 4x + 12 \\
 & 4x + 12 \\
 & 0
\end{array}
$$

$\therefore\ x^3 - 5x + 12 = (x+3)\left(x^2 - 3x + 4\right)$

ie. $x+3$ is a factor.

(f) $u = 1 + x$

$du = dx.$

$x = -1 \quad u = 0$

$x = 0 \quad u = 1.$

$\therefore\ 15\displaystyle\int_{-1}^{0} x\sqrt{1+x}\ dx$

$= 15\displaystyle\int_0^1 (u-1)\sqrt{u}\ du$

$= 15\displaystyle\int_0^1\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du$

$= 15\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_0^1$

$= 15\left[\left(\frac{2}{5}-\frac{2}{3}\right)-0\right]$

$= -4.$

Note that the integral is negative since $x\sqrt{1+x} \le 0$ on the domain $-1 \le x \le 0$.



**QUESTION 2**

(a) $f(x) = 3x^2 + x$

$f'(a) = \displaystyle\lim_{h\to0}\frac{f(a+h)-f(a)}{h}$

$= \displaystyle\lim_{h\to0}\frac{\left[3(a+h)^2+(a+h)\right]-\left(3a^2+a\right)}{h}$

$= \displaystyle\lim_{h\to0}\frac{3a^2+6ah+3h^2+a+h-3a^2-a}{h}$

$= \displaystyle\lim_{h\to0}\frac{6ah+3h^2+h}{h}$

$= \displaystyle\lim_{h\to0}\left(6a+3h+1\right)$

$= 6a+1.$

(b) (i) $\displaystyle\int\frac{e^x}{1+e^x}\ dx = \ln\left(1+e^x\right)+c.$

(ii) $\displaystyle\int_0^{\pi}\cos^2 3x\ dx$

$= \frac{1}{2}\displaystyle\int_0^{\pi}(1+\cos 6x)\ dx$

$= \frac{1}{2}\left[x+\frac{\sin 6x}{6}\right]_0^{\pi}$

$= \frac{1}{2}\left[\left(\pi+\frac{\sin 6\pi}{6}\right)-\left(0+\frac{\sin 0}{6}\right)\right]$

$= \frac{\pi}{2}.$

*Note* $\frac{1}{2}\displaystyle\int_0^{\pi}(1+\cos 6x)\ dx = \frac{1}{2}\int_0^{\pi}1\ dx,$

since $\cos 6x$ has a period of $\frac{\pi}{3}$ and the integration is over 3 complete periods.

(c) There are 2*A*s.
The other 7 letters are unique.

(i) The number of arrangements $= \frac{9!}{2!}$

$= 181\,440.$

(ii) The number of arrangements of vowels $= \frac{4!}{2!}.$

The number of arrangements of consonants $= 5!.$

Total number of arrangements $= \frac{4!}{2!}\times5!$

$= 1440.$

(d) In the expansion of $\left(x^2-\frac{1}{x}\right)^9,$

$T_{k+1} = \binom{9}{k}\left(x^2\right)^{9-k}\left(-\frac{1}{x}\right)^k$

$= \binom{9}{k}(-1)^k x^{18-3k}.$

For term independent of $x$, $18-3k = 0$

$k = 6.$

$\therefore$ Required term $= \binom{9}{6}(-1)^6$

$= \binom{9}{6}$

$= 84.$

**QUESTION 3**

(a) $f(x) = \sin x + \cos x - x$

$f'(x) = \cos x - \sin x - 1$

$f(1{\cdot}2) = 0{\cdot}094\,39\ \ldots$

$f'(1{\cdot}2) = -1{\cdot}569\,68\ \ldots$

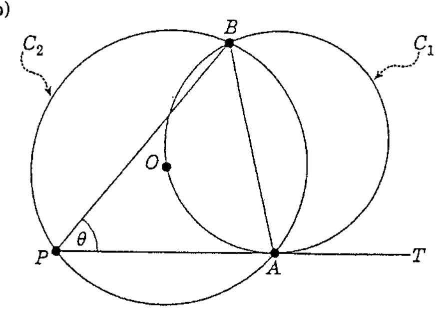
Let $x_2$ be a second approximation:

$x_2 = 1{\cdot}2 - \frac{f(1{\cdot}2)}{f'(1{\cdot}2)}$ by Newton's method

$= 1{\cdot}2 - \frac{0{\cdot}094\,39\ \ldots}{-1{\cdot}569\,68\ \ldots}$

$= 1{\cdot}26$ (3 sig. figs).

(b)




(i) $\angle AOB = 2\theta$ (Angle at the centre is twice angle at the circumference standing on the same arc.)

(ii) $\angle TAB$ is the angle between the tangent to $C_1$ and the chord $AB$.
$\angle AOB$ is the $\angle$ in the alternate segment.
$\therefore\ \angle TAB = \angle AOB = 2\theta.$

(iii) $\angle TAB = \angle APB + \angle ABP$ (Ext. $\angle$ of a $\Delta$ = sum of int. opp. $\angle$s)

$2\theta = \theta + \angle ABP.$

$\therefore \angle ABP = \theta$

$\therefore \angle ABP = \angle APB = \theta.$

$\therefore\ PA = BA$ (Sides opp. equal $\angle$s in a $\Delta$ are equal).

(c) (i) $\sin(\theta + 2\theta)$

$= \sin\theta\cos 2\theta + \cos\theta\sin 2\theta$

$= \sin\theta\left(1 - 2\sin^2\theta\right) + \cos\theta \cdot 2\sin\theta\cos\theta$

$= \sin\theta - 2\sin^3\theta + \cos^2\theta \cdot 2\sin\theta$

$= \sin\theta - 2\sin^3\theta + 2\sin\theta\left(1 - \sin^2\theta\right)$

$= 3\sin\theta - 4\sin^3\theta.$

(ii) $\sin 3\theta = 2\sin\theta, \quad 0 \le \theta \le 2\pi$

$3\sin\theta - 4\sin^3\theta = 2\sin\theta$

$\sin\theta - 4\sin^3\theta = 0$

$\sin\theta\left(1 - 4\sin^2\theta\right) = 0$

$\sin\theta\left(1 - 2\sin\theta\right)\left(1 + 2\sin\theta\right) = 0.$

$\therefore \sin\theta = 0, \frac{1}{2}, -\frac{1}{2}$

$\theta = 0, \pi, 2\pi, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}.$

In ascending order, the solutions are

$\theta = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi.$

## QUESTION 4

(a) $\frac{3x}{x-2} \le 1$

*Note* $x \ne 2$ since LHS is undefined.

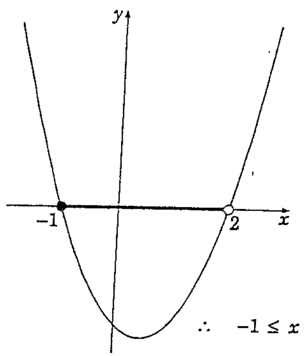
*METHOD 1*

$\frac{3x}{x-2} - 1 \le 0$

$\frac{2x+2}{x-2} \le 0$

$\frac{2(x+1)}{x-2} \times (x-2)^2 \le 0 \times (x-2)^2$

$2(x+1)(x-2) \le 0 \quad (x \ne 2).$



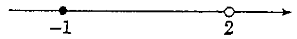

$\therefore \quad -1 \le x < 2.$

*METHOD 2 Use the critical point method*

Consider $\frac{3x}{x-2} = 1 \quad (x \ne 2)$

$3x = x - 2$

$x = -1.$




Now consider $\frac{3x}{x-2} \le 1.$

Test $x = -2$: $\quad \frac{-6}{-4} \le 1 \quad$ False

$x = 0$: $\quad 0 \le 1 \quad$ True

$x = 3$: $\quad \frac{9}{1} \le 1 \quad$ False

$\therefore \quad -1 \le x < 2.$

*METHOD 3*

$\frac{3x}{x-2} \le 1.$

If $x - 2 < 0$, ie. $x < 2$,

then $3x \ge x - 2$

$x \ge -1,$

ie. $-1 \le x < 2.$

If $x - 2 > 0$, ie. $x > 2$,

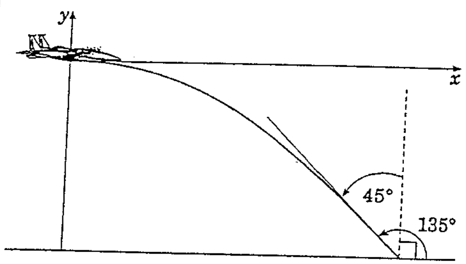
then $3x \le x - 2$

$x \le -1.$

$\therefore$ No further solutions.

$\therefore$ The solution is $-1 \le x < 2.$

(b)




*METHOD 1*

$x = Vt, \quad y = -5t^2.$

Eliminating the parameter $t$, $\quad t = \frac{x}{V}.$

$\therefore \quad y = -\frac{5x^2}{V^2}$

$\frac{dy}{dx} = -\frac{10x}{V^2}.$

*Note:* The angle between the direction of the bomb when it hits the ground and the positive $x$ axis is 135°.

When $x = 4000$, $\quad \frac{dy}{dx} = \tan 135° = -1.$

$\therefore \quad \frac{-10 \times 4000}{V^2} = -1$

$V^2 = 40\ 000$

$V = 200.$

$\left(V > 0,\right.$ since it is the speed of the aircraft.$\left.\right)$





*METHOD 2*

$x = Vt \qquad y = -5t^2$

$\dot{x} = V \qquad \dot{y} = -10t$

$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -\frac{10t}{V}.$

When $x = 4000$, $\quad t = \frac{4000}{V}$ and $\frac{dy}{dx} = -1.$

$\therefore \quad -1 = -\frac{10 \times 4000}{V^2}$

$V^2 = 40\ 000$

$\therefore \quad V = 200.$

*METHOD 3*

An angle of 45° means that the ground is a focal chord. Therefore the plane is flying at an altitude of $4000 \div 2 = 2000$ m.

Hence the bomb hits the ground at time $t$, given by:

$-2000 = -5t^2 \quad \left(\text{from } y = -5t^2\right)$

$t^2 = 400$

$\therefore \quad t = 20.$

Substituting $t = 20$, $x = 4000$

$4000 = 20V \quad \left(\text{from } x = Vt\right)$

$\therefore \quad V = 200.$

(c) $\quad \ddot{x} = -4x$

$= -n^2 x$

$\therefore \quad n = 2.$

$x = a\cos(nt + \alpha).$

$\dot{x} = -an\sin(nt + \alpha).$

When $t = 0$, $\quad x = 3$

$3 = a\cos\alpha. \quad$ —①

When $t = 0$, $\quad \dot{x} = -6\sqrt{3}$

$-6\sqrt{3} = -2a\sin\alpha$

$a\sin\alpha = 3\sqrt{3}. \quad$ —②

$\tan\alpha = \frac{3\sqrt{3}}{3} = \sqrt{3}$

$\therefore \quad \alpha = \frac{\pi}{3}.$

Substitute in ①: $\quad 3 = a\cos\frac{\pi}{3}$

$a = 6$

$\therefore \quad x = 6\cos\left(2t + \frac{\pi}{3}\right).$

Also correct are $x = 6\sin\left(2t + \frac{5\pi}{6}\right)$

and $x = 3\cos 2t - 3\sqrt{3}\sin 2t.$

## QUESTION 5

(a) (i) $y = 2\cos^{-1}\frac{x}{3}$

$x = 0, \quad y = 2\cos^{-1} 0$

$= 2 \times \frac{\pi}{2}$

$= \pi.$

$\therefore$ The $y$ intercept $= \pi.$

(ii) For the inverse, $\quad x = 2\cos^{-1}\frac{y}{3}$

$\frac{y}{3} = \cos\frac{x}{2}$

$y = 3\cos\frac{x}{2},$

$\therefore \quad f^{-1}(x) = 3\cos\frac{x}{2}.$

Since the range of $y = f(x)$ is $0 \le y \le 2\pi$, the domain of $y = f^{-1}(x)$ is $0 \le x \le 2\pi$.

(iii) $A = \int_0^{\pi} x \, dy$

$= \int_0^{\pi} 3\cos\frac{y}{2}\, dy$

$= 6\left[\sin\frac{y}{2}\right]_0^{\pi}$

$= 6\left(\sin\frac{\pi}{2} - \sin 0\right)$

$= 6.$

$\therefore$ The area is 6 square units.

(b) $(q + p)^n - (q - p)^n$

$= \left[q^n + \binom{n}{1}q^{n-1}p + \binom{n}{2}q^{n-2}p^2 + \binom{n}{3}q^{n-3}p^3 + \cdots + p^n\right] - \left[q^n - \binom{n}{1}q^{n-1}p + \binom{n}{2}q^{n-2}p^2 - \binom{n}{3}q^{n-3}p^3 + \cdots + (-1)^n p^n\right]$

$= 2\left[\binom{n}{1}q^{n-1}p + \binom{n}{3}q^{n-3}p^3 + \cdots\right]$

$\therefore (q + p)^n - (q - p)^n$

$= 2\binom{n}{1}q^{n-1}p + 2\binom{n}{3}q^{n-3}p + \cdots$

If $n$ is odd, $(-1)^n p^n$ is negative,

$\therefore$ last term is $2\binom{n}{n}p^n = 2p^n.$

If $n$ is even, $(-1)^n p^n$ is positive,

$\therefore$ last term is $2\binom{n}{n-1}qp^{n-1} = 2nqp^{n-1}.$

(c) (i) $\left(\frac{5}{6}+\frac{1}{6}\right)^n$ is the binomial probability function, since probability of a six $= \frac{1}{6}$.

$\therefore$ Probability of exactly $r$ sixes,

$$P(r) = \binom{n}{r}\left(\frac{5}{6}\right)^{n-r}\left(\frac{1}{6}\right)^r.$$

(ii) Probability of an odd number of sixes

$$= P(1)+P(3)+P(5)+\cdots$$
$$= \binom{n}{1}\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)+\binom{n}{3}\left(\frac{5}{6}\right)^{n-3}\left(\frac{1}{6}\right)^3+\cdots$$
$$= \frac{1}{2}\left[\left(\frac{5}{6}+\frac{1}{6}\right)^n-\left(\frac{5}{6}-\frac{1}{6}\right)^n\right], \quad \text{from (b),}$$
$$= \frac{1}{2}\left[1-\left(\frac{2}{3}\right)^n\right].$$

Note that the expression in (b) doubles up on probabilities for $r$ odd and omits probabilities for $r$ even.
Hence it was applicable to this question.

## QUESTION 6

(a) Let $S(n)$ be the statement that $n^3+(n+1)^3+(n+2)^3$ is divisible by 9.

For $n=1$, $n^3+(n+1)^3+(n+2)^3$
$= 1^3+2^3+3^3$
$= 36$, which is divisible by 9.

$\therefore$ $S(1)$ is true.

Assume $S(k)$ is true,

$\therefore$ $k^3+(k+1)^3+(k+2)^3$
$= 9M$, where $M$ is an integer.

Consider the case when $n=k+1$:

$(k+1)^3+(k+2)^3+(k+3)^3$
$= (k+1)^3+(k+2)^3+k^3+9k^2+27k+27$
$= \left[(k+1)^3+(k+2)^3+k^3\right]+9k^2+27k+27$
$= 9M+9k^2+27k+27$
$= 9\left(M+k^2+3k+3\right)$, which is divisible by 9, since $M+k^2+3k+3$ is an integer.

$\therefore$ $S(k+1)$ is true when $S(k)$ is true.

But the result is true when $n=1$, hence it is true when $n=2$, and so by mathematical induction the result is true for $n=1, 2, 3, \ldots$

(b) (i)
$$x^2 = 4ay$$
$$y = \frac{x^2}{4a}$$
$$\frac{dy}{dx} = \frac{x}{2a}.$$

At $P$, $\frac{dy}{dx} = \frac{2at}{2a} = t$.

Gradient of normal $= -\frac{1}{t}$.

Equation of normal is
$$y-at^2 = -\frac{1}{t}(x-2at)$$
$$ty-at^3 = -x+2at$$
$$x+ty = at^3+2at.$$

(ii) At $Q(2aq, aq^2)$, gradient of normal is $t$.

Also at $Q(2aq, aq^2)$, gradient of normal is $-\frac{1}{q}$.

$\therefore$ $t = -\frac{1}{q}$

$\therefore$ $q = -\frac{1}{t}$

$\therefore$ $Q$ is $\left(-\frac{2a}{t}, \frac{a}{t^2}\right)$.

(iii) Normal at $P$: $x+ty = at^3+2at$ —①

Normal at $Q$: $x-\frac{1}{t}y = -\frac{a}{t^3}-\frac{2a}{t}$ —②

① × $t$: $tx+t^2y = at^4+2at^2$ —③

② × $t^3$: $t^3x-t^2y = -a-2at^2$ —④

③ + ④:
$$\left(t+t^3\right)x = a\left(t^4-1\right)$$
$$t\left(t^2+1\right)x = a\left(t^2+1\right)\left(t^2-1\right)$$
$$x = \frac{a\left(t^2-1\right)}{t}$$
$$\therefore \quad x = a\left(t-\frac{1}{t}\right).$$

Substitute in ①:
$$a\left(t-\frac{1}{t}\right)+ty = at^3+2at$$
$$ty = at^3+at+\frac{a}{t}$$
$$y = a\left(t^2+1+\frac{1}{t^2}\right).$$

$$\therefore\ x = a\left(t-\frac{1}{t}\right),\quad y = a\left(t^2+1+\frac{1}{t^2}\right).$$

(iv) Parametric equations of the locus of $R$ are

$$x = a\left(t-\frac{1}{t}\right) \quad —⑤$$
$$y = a\left(t^2+1+\frac{1}{t^2}\right) \quad —⑥$$

From ⑤:
$$x^2 = a^2\left(t^2-2+\frac{1}{t^2}\right)$$
$$= a^2\left(t^2+1+\frac{1}{t^2}\right)-3a^2.$$

$\therefore$ $x^2 = ay-3a^2$ is the Cartesian equation of the locus.





## QUESTION 7

(a) (i)
$$\frac{dv}{dt} = x-1$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x-1.$$

Integrating, $\frac{1}{2}v^2 = \frac{x^2}{2}-x+c$.

When $x=0$, $v=1$:

$\therefore$ $\frac{1}{2} = c$

$\therefore$ $\frac{1}{2}v^2 = \frac{x^2}{2}-x+\frac{1}{2}$

$\therefore$ $v^2 = x^2-2x+1$

$\therefore$ $v^2 = (x-1)^2$.

(ii) $\therefore$ $v = \pm(x-1)$

But if $x=0$, $v=1$,

$\therefore$ $v = -(x-1)$
$$v = 1-x$$
$$\frac{dx}{dt} = 1-x$$
$$\frac{dt}{dx} = \frac{1}{1-x}$$
$$t = \int\frac{1}{1-x}\,dx$$
$$= -\log_e(1-x)+c_1.$$

When $t=0$, $x=0$,
$$0 = -\log_e 1+c_1$$
$$c_1 = 0$$

$\therefore$ $t = -\log_e(1-x)$
$$e^{-t} = 1-x$$

$\therefore$ $x = 1-e^{-t}$.

Checks: $\frac{dx}{dt} = e^{-t} = 1-x$.
$\frac{dv}{dt} = -e^{-t} = x-1$.

(b) (i) In $\Delta AOP$,
$$AP^2 = OA^2+OP^2-2OA\cdot OP\cos\frac{\pi}{3}.$$

From $\Delta AOT$, $AO = h$.

From $\Delta POT$, $\tan\alpha = \frac{h}{OP}$

$\therefore$ $OP = h\cot\alpha$.

$\therefore$ $AP^2 = h^2+h^2\cot^2\alpha-2h\cdot h\cdot\cot\alpha\cdot\frac{1}{2}$

$\therefore$ $AP^2 = h^2+h^2\cot^2\alpha-h^2\cot\alpha$.

(ii) In $\Delta ATP$,
$$AP^2 = AT^2+PT^2-2AT\cdot PT\cos\theta.$$

In $\Delta AOT$, $AT^2 = h^2+h^2$

$\therefore$ $AT^2 = 2h^2$.

In $\Delta POT$, $PT^2 = h^2+h^2\cot^2\alpha$

$\therefore$ $PT^2 = h^2\operatorname{cosec}^2\alpha$.

$\therefore$ $AP^2 = 2h^2+h^2+h^2\cot^2\alpha - 2\sqrt{2}\,h\cdot h\operatorname{cosec}\alpha\cdot\cos\theta$,

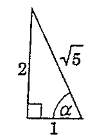
$\therefore$ $AP^2 = 3h^2+h^2\cot^2\alpha - 2\sqrt{2}h^2\operatorname{cosec}\alpha\cdot\cos\theta$.

Equating expressions for $AP^2$,
$$h^2+h^2\cot^2\alpha-h^2\cot\alpha$$
$$= 3h^2+h^2\cot^2\alpha-2\sqrt{2}\,h^2\operatorname{cosec}\alpha\cdot\cos\theta.$$
$$2\sqrt{2}h^2\operatorname{cosec}\alpha\cos\theta = 2h^2+h^2\cot\alpha \quad —①$$

① × $\frac{\sin\alpha}{h^2}$:
$$2\sqrt{2}\cos\theta = 2\sin\alpha+\cos\alpha$$
$$\therefore \quad \cos\theta = \frac{1}{\sqrt{2}}\sin\alpha+\frac{1}{2\sqrt{2}}\cos\alpha.$$

(iii)
$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\sin\alpha+\frac{1}{2\sqrt{2}}\cos\alpha\right)$$
$$\frac{d\theta}{d\alpha} = \frac{-\left(\frac{1}{\sqrt{2}}\cos\alpha-\frac{1}{2\sqrt{2}}\sin\alpha\right)}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\sin\alpha+\frac{1}{2\sqrt{2}}\cos\alpha\right)^2}}.$$

At stationary points, $\frac{d\theta}{d\alpha} = 0$.
$$\frac{1}{2\sqrt{2}}\sin\alpha-\frac{1}{\sqrt{2}}\cos\alpha = 0$$
$$\sin\alpha-2\cos\alpha = 0$$
$$\tan\alpha = 2$$
$$\alpha = \tan^{-1}2.$$



$\alpha \doteqdot 1{\cdot}107$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\cdot\frac{2}{\sqrt{5}}+\frac{1}{2\sqrt{2}}\cdot\frac{1}{\sqrt{5}}\right)$$
$$= \cos^{-1}\left(\frac{\sqrt{10}}{5}+\frac{\sqrt{10}}{20}\right)$$
$$= \cos^{-1}\frac{\sqrt{10}}{4}.$$

Stationary point at $\left(\tan^{-1}2,\ \cos^{-1}\frac{\sqrt{10}}{4}\right)$.

If $\alpha < \tan^{-1}2$, $\frac{d\theta}{d\alpha} < 0$
If $\alpha > \tan^{-1}2$, $\frac{d\theta}{d\alpha} > 0$
for $0 < \alpha < \frac{\pi}{2}$.

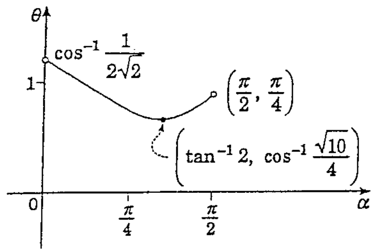
$\therefore$ Minimum turning point at $\left(\tan^{-1}2,\ \cos^{-1}\frac{\sqrt{10}}{4}\right)$.

If $\alpha = 0$, $\theta = \cos^{-1}\dfrac{1}{2\sqrt{2}}$ $(\doteqdot 1{\cdot}209)$.

If $\alpha = \dfrac{\pi}{2}$, $\theta = \cos^{-1}\dfrac{1}{\sqrt{2}} = \dfrac{\pi}{4}$.

As $\alpha \to 0$, $\theta \to \cos^{-1}\dfrac{1}{2\sqrt{2}}$.

As $\alpha \to \dfrac{\pi}{2}$, $\theta \to \dfrac{\pi}{4}$.




In an alternative approach to this question, we can write $\dfrac{1}{\sqrt{2}}\sin\alpha + \dfrac{1}{2\sqrt{2}}\cos\alpha$ in the form $a\cos(\alpha - \phi)$, leading to

$$\theta = \cos^{-1}\left[\frac{\sqrt{10}}{4}\cos\left(\alpha - \tan^{-1} 2\right)\right].$$

$\theta$ then has a minimum when

$\cos\left(\alpha - \tan^{-1} 2\right)$ has a maximum.

This occurs when

$$\cos\left(\alpha - \tan^{-1} 2\right) = 1$$

$$\alpha = \tan^{-1} 2$$

$$\theta = \cos^{-1}\frac{\sqrt{10}}{4}.$$

END OF MATHEMATICS EXTENSION 1 SOLUTIONS