BOARD OF STUDIES
NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

1995

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time allowed—Two hours
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Each question attempted is to be returned in a *separate* Writing Booklet clearly marked Question 1, Question 2, etc. on the cover. Each booklet must show your Student Number and the Centre Number.
- You may ask for extra Writing Booklets if you need them.

QUESTION 1. Use a *separate* Writing Booklet. **Marks**

(a) On a number plane, indicate the region specified by $y \le |x-1|$ and $y \le 1$. **3**

(b) Evaluate $\int_1^4 y\,dx$ if $xy = 1$. **2**

(c) Find $\lim_{x \to 0} \frac{\sin x}{5x}$. **1**

(d) Factorize $2^{n+1} + 2^n$, and hence write $\frac{2^{1001} + 2^{1000}}{3}$ as a power of 2. **2**

(e) Use the substitution $u = 9 - x^2$ to find $\int_0^1 6x\sqrt{9-x^2}\,dx$. **4**

QUESTION 2. Use a *separate* Writing Booklet. **Marks**

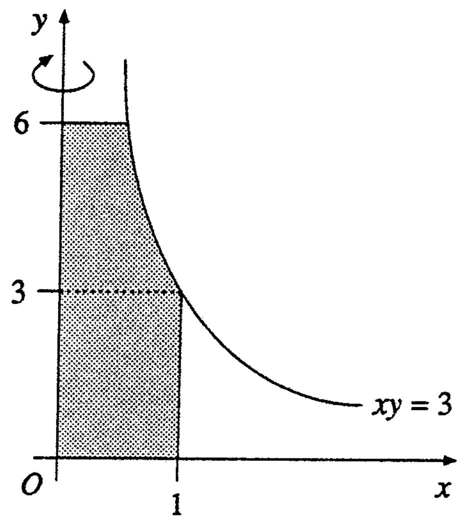
(a) Let $f(x) = x^3 + 5x^2 + 17x - 10$. The equation $f(x) = 0$ has only one real root. **4**

(i) Show that the root lies between 0 and 2.

(ii) Use one application of the 'halving the interval' method to find a smaller interval containing the root.

(iii) Which end of the smaller interval found in part (ii) is closer to the root? Briefly justify your answer.

(b) **4**



The shaded area is bounded by the curve $xy = 3$, the lines $x = 1$ and $y = 6$, and the two axes. A solid is formed by rotating the shaded area about the y axis.

Find the volume of this solid by considering separately the regions above and below $y = 3$.

(c) Consider the equation **4**

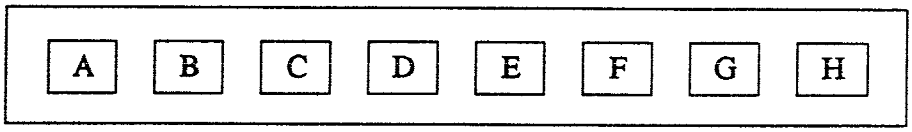
$$x^3 + 6x^2 - x - 30 = 0.$$

One of the roots of this equation is equal to the sum of the other two roots.

Find the values of the three roots.

QUESTION 3. Use a *separate* Writing Booklet. **Marks**

(a) **3**



A security lock has 8 buttons labelled as shown. Each person using the lock is given a 3-letter code.

(i) How many different codes are possible if letters can be repeated and their order is important?

(ii) How many different codes are possible if letters cannot be repeated and their order is important?

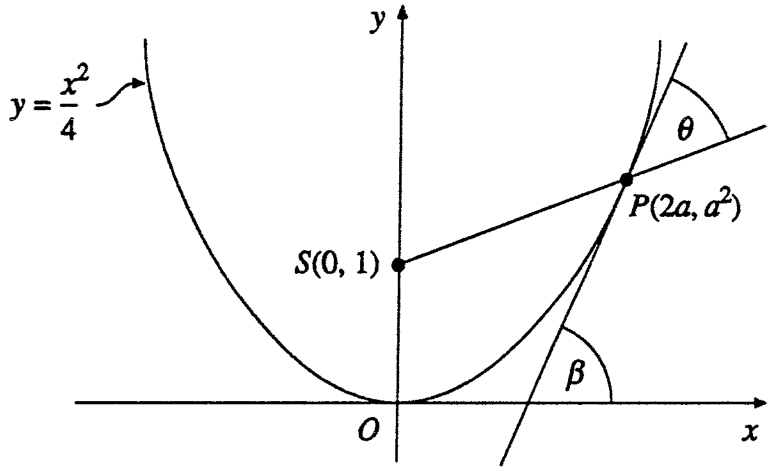
(iii) Now suppose that the lock operates by holding 3 buttons down together, so that order is NOT important. How many different codes are possible?

(b) Find the value of the term that does not depend on x in the expansion of **3**

$$\left(x^2 + \frac{3}{x}\right)^6.$$

QUESTION 3. (Continued) **Marks**

(c) **6**



Let $P(2a, a^2)$ be a point on the parabola

$$y = \frac{x^2}{4},$$

and let S be the point $(0, 1)$. The tangent to the parabola at P makes an angle of β with the x axis. The angle between SP and the tangent is θ. Assume that $a > 0$, as indicated.

(i) Show that $\tan\beta = a$.

(ii) Show that the gradient of SP is $\frac{1}{2}\left(a - \frac{1}{a}\right)$.

(iii) Show that $\tan\theta = \frac{1}{a}$.

(iv) Hence find the value of $\theta + \beta$.

(v) Find the coordinates of P if $\theta = \beta$.

QUESTION 4. Use a *separate* Writing Booklet. **Marks**

Consider the function $f(x)=\dfrac{e^x}{3+e^x}$.

Note that e^x is always positive, and that $f(x)$ is defined for all real x.

(a) Show that $f(x)$ has no stationary points. **2**

(b) Find the coordinates of the point of inflexion, given that $f''(x)=\dfrac{3e^x\left(3-e^x\right)}{\left(3+e^x\right)^3}$. **1**

(c) Show that $0<f(x)<1$ for all x. **2**

(d) Describe the behaviour of $f(x)$ for very large positive and very large negative values of x, i.e. as $x\to\infty$ and $x\to-\infty$. **2**

(e) Sketch the curve $y=f(x)$. **2**

(f) Explain why $f(x)$ has an inverse function. **1**

(g) Find the inverse function $y=f^{-1}(x)$. **2**

QUESTION 5. Use a *separate* Writing Booklet. **Marks**

(a) (i) Solve the equation $\sin 2x = 2\sin^2 x$ for $0 < x < \pi$. **6**

(ii) Show that if $0 < x < \frac{\pi}{4}$, then $\sin 2x > 2\sin^2 x$.

(iii) Find the area enclosed between the curves $y = \sin 2x$ and $y = 2\sin^2 x$ for $0 \le x \le \frac{\pi}{4}$.

(b) In a Jackpot Lottery, 1500 numbers are drawn from a barrel containing the 100 000 ticket numbers available. **6**

After all the 1500 prize-winning numbers are drawn, they are returned to the barrel and a jackpot number is drawn. If the jackpot number is the same as one of the 1500 numbers that have already been selected, then the additional jackpot prize is won.

The probability that the jackpot prize is won in a given game is thus

$$p = \frac{1500}{100\,000} = 0{\cdot}015.$$

(i) Calculate the probability that the jackpot prize will be won *exactly* once in 10 independent lottery games.

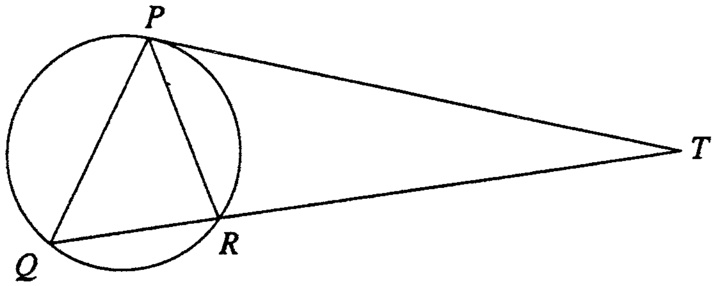
(ii) Calculate the probability that the jackpot prize will be won *at least* once in 10 independent lottery games.

(iii) The jackpot prize is initially \$8000, and it increases by \$8000 each time the prize is NOT won.

Calculate the probability that the jackpot prize will exceed \$200 000 when it is finally won.

QUESTION 6. Use a *separate* Writing Booklet. **Marks**

(a) **3**



PT is a tangent to the circle PRQ, and QR is a secant intersecting the circle in Q and R. The line QR intersects PT at T.

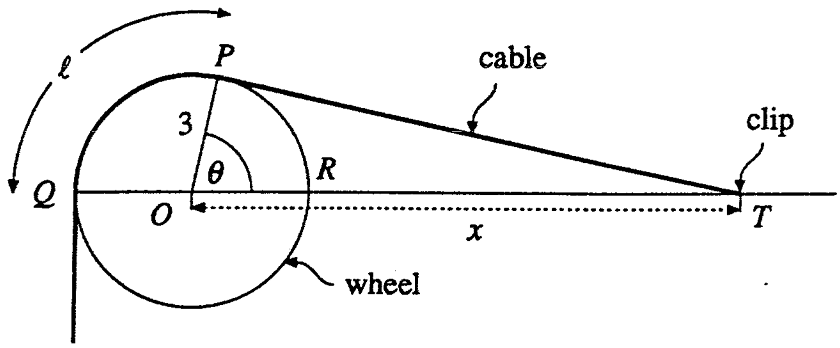
Copy or trace the diagram into your Writing Booklet.

(i) Prove that the triangles PRT and QPT are similar.

(ii) Hence prove that $PT^2 = QT \times RT$.

QUESTION 6. (Continued) **Marks**

(b) **9**



A long cable is wrapped over a wheel of radius 3 metres and one end is attached to a clip at T. The centre of the wheel is at O, and QR is a diameter. The point T lies on the line OR at a distance x metres from O.

The cable is tangential to the wheel at P and Q as shown. Let $\angle POR = \theta$ (in radians).

The length of cable in contact with the wheel is ℓ metres; that is, the length of the arc between P and Q is ℓ metres.

(i) Explain why $\cos\theta = \dfrac{3}{x}$.

(ii) Show that $\ell = 3\left[\pi - \cos^{-1}\left(\dfrac{3}{x}\right)\right]$.

(iii) Show that $\dfrac{d\ell}{dx} = \dfrac{-9}{x\sqrt{x^2-9}}$.

What is the significance of the fact that $\dfrac{d\ell}{dx}$ is negative?

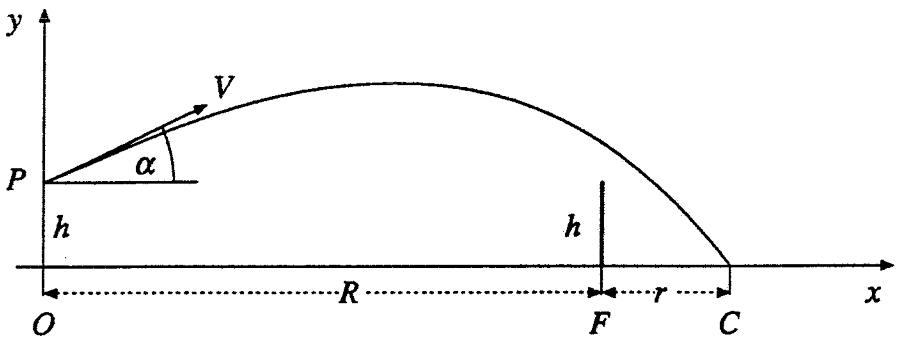
(iv) Let $s = \ell + PT$.

Using part (a), or otherwise, express s in terms of x.

(v) The clip at T is moved away from O along the line OR at a constant speed of 2 metres per second.

Find the rate at which s changes when $x = 10$.

QUESTION 7. Use a *separate* Writing Booklet. **Marks**



A cap C is lying outside a softball field, r metres from the fence F, which is h metres high. The fence is R metres from the point O, and the point P is h metres above O. Axes are based at O, as shown.

At time $t = 0$, a ball is hit from P at a speed V metres per second and at an angle α to the horizontal, towards the cap.

(a) The equations of motion of the ball are **4**

$$\ddot{x} = 0, \quad \ddot{y} = -g.$$

Using calculus, show that the position of the ball at time t is given by

$$x = Vt\cos\alpha$$

$$y = Vt\sin\alpha - \tfrac{1}{2}gt^2 + h.$$

(b) Hence show that the trajectory of the ball is given by **1**

$$y = h + x\tan\alpha - x^2 \frac{g}{2V^2\cos^2\alpha}.$$

(c) The ball clears the fence. Show that **2**

$$V^2 \geq \frac{gR}{2\sin\alpha\cos\alpha}.$$

(d) After clearing the fence, the ball hits the cap C. Show that **3**

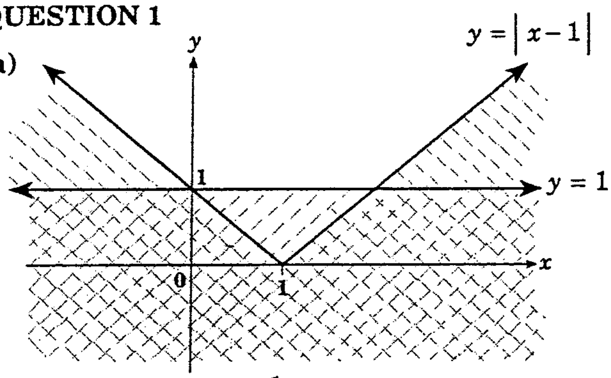
$$\tan\alpha \geq \frac{Rh}{(R+r)r}.$$

(e) Suppose that the ball clears the fence, and that $V \leq 50$, $g = 10$, $R = 80$, and $h = 1$. What is the closest point to the fence where the ball can land? **2**

1995 Higher School Certificate Solutions
3 Unit (Additional) and 3/4 Unit (Common) Mathematics

QUESTION 1

(a)



(b) $xy = 1 \quad \therefore \quad y = \dfrac{1}{x}$

$$\therefore \quad \int_1^4 y\,dx = \int_1^4 \frac{1}{x}\,dx$$
$$= \left[\ln x\right]_1^4$$
$$= \ln 4 - \ln 1$$
$$= \ln 4 \approx 1{\cdot}386.$$

(c) $$\lim_{x\to 0}\frac{\sin x}{5x} = \lim_{x\to 0}\frac{1}{5}.\frac{\sin x}{x}$$
$$= \frac{1}{5}\lim_{x\to 0}\frac{\sin x}{x}$$
$$= \frac{1}{5}\times 1$$
$$= \frac{1}{5}.$$

(d) $$2^{n+1} + 2^n = 2^n(2+1)$$
$$= 3\times 2^n \qquad \text{—①}$$

From ①: $$\frac{2^{1001} + 2^{1000}}{3} = \frac{3\times 2^{1000}}{3}$$
$$= 2^{1000}.$$

(e) $$I = \int_0^1 6x\sqrt{9-x^2}\,dx$$

Put $u = 9 - x^2$

$\therefore \quad \dfrac{du}{dx} = -2x$

$\therefore \quad dx = -\dfrac{du}{2x}$ When $x = 0,\ u = 9$ and $x = 1,\ u = 8$.

$$\therefore \quad I = \int_9^8 6x.u^{\frac{1}{2}}.\left(-\frac{du}{2x}\right)$$
$$\therefore \quad = -3\int_9^8 u^{\frac{1}{2}}\,du$$
$$\therefore \quad = 3\int_8^9 u^{\frac{1}{2}}.du$$
$$= 3\left[\frac{2}{3}u^{\frac{3}{2}}\right]_8^9$$
$$= 3\left[\frac{2}{3}\left(9^{\frac{3}{2}} - 8^{\frac{3}{2}}\right)\right]$$
$$= 2\left(27 - 16\sqrt{2}\right).$$

QUESTION 2

(a) (i) $f(x) = x^3 + 5x^2 + 17x - 10$

$f(0) = -10$

$f(2) = 8 + 20 + 34 - 10$

$= 52.$

Since the function is continuous, and changes sign between 0 and 2, the root lies between 0 and 2.

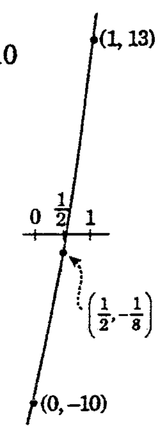
(ii) $f(1) = 1 + 5 + 17 - 10$

$= 13.$

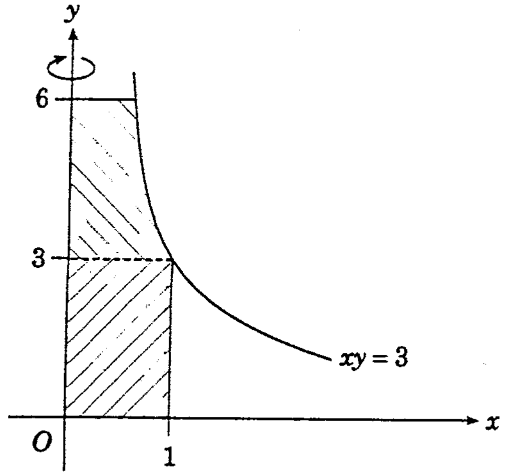
$\therefore$ The root lies between 0 and 1.

(iii) $$f\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{5}{4} + \frac{17}{2} - 10$$
$$= -\frac{1}{8}.$$

Thus, the root is between $\frac{1}{2}$ and 1, so the end closer to the root is 1 (not 0).



(b)



The lower region is a circular cylinder, radius 1 unit, height 3 units.

$\therefore$ Volume $= \pi r^2 h$
$= \pi \times 1^2 \times 3$
$= 3\pi$ units^3.

Volume of the upper region is

$$\int_3^6 \pi x^2 \, dy = \pi \int_3^6 \left(\frac{3}{y}\right)^2 dy$$
$$= 9\pi \int_3^6 y^{-2} \, dy$$
$$= 9\pi \left[\frac{y^{-1}}{-1}\right]_3^6$$
$$= 9\pi \left[-\frac{1}{6} + \frac{1}{3}\right]$$
$$= \frac{9\pi}{6}$$
$$= \frac{3\pi}{2} \text{ units}^3.$$

$\therefore$ Volume $= 3\pi + \frac{3\pi}{2}$
$= \frac{9\pi}{2}$ units^3.

(c) $x^3 + 6x^2 - x - 30 = 0$

Let the roots be α, β, γ, where $\alpha = \beta + \gamma$, then

$\alpha + \beta + \gamma = -\frac{b}{a} = -6$

$\therefore$ $2\alpha = -6$, $\alpha = -3$,

and $\alpha\beta\gamma = -\frac{d}{a} = 30$

$\therefore$ $\beta\gamma = -10$,

and $\beta + \gamma = -3$.

By inspection, $\beta = -5$, $\gamma = 2$ (or $\beta = 2$, $\gamma = -5$).

$\therefore$ The three roots are $-5, -3, 2$.

QUESTION 3

(a) (i) The first code letter can be chosen eight ways. The second and third can also be chosen in each of eight ways.

$\therefore$ The number of different codes is

$$8 \times 8 \times 8 = 512$$

(ii) The first code letter can be chosen eight ways, the second only seven ways, and the third only 6 ways.

$\therefore$ The number of codes is

$$8 \times 7 \times 6 = 336$$

(iii) The number of ways of choosing 3 from 8 is:

$${}^8C_3 = \frac{8 \times 7 \times 6}{1 \times 2 \times 3}$$
$$= 56.$$

(b) The general term is ${}^6C_n \, (x^2)^{6-n} \left(\frac{3}{x}\right)^n$.

If this is independent of x,

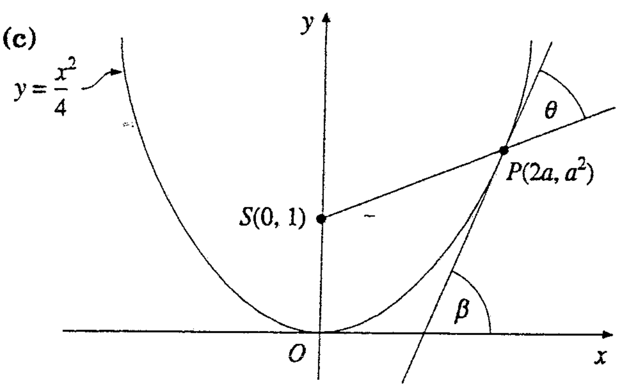
then $x^{2(6-n)} \times x^{-n} = x^0$

$\therefore$ $12 - 3n = 0$

$\therefore$ $n = 4$.

$\therefore$ The term is ${}^6C_4 . x^4 . \frac{3^4}{x^4} = 1215$.

(c)



(i) $y = \frac{x^2}{4}$

$$\frac{dy}{dx} = \frac{2x}{4}$$
$$= \frac{x}{2}$$
$$= \frac{2a}{2} \text{ at } P(2a, a^2)$$
$$= a.$$

But the gradient of the tangent at P is given by $\tan\beta$, since the tangent cuts the x-axis with angle β.

$\therefore$ $\tan\beta = a$.

(ii) Gradient of $SP = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{a^2 - 1}{2a - 0}$

$= \dfrac{a^2 - 1}{2a}$

$= \dfrac{1}{2}\left(a - \dfrac{1}{a}\right).$

(iii) θ is the angle between two lines with gradients $m_2 = \tan\beta = a$ for the tangent and $m_1 = \dfrac{1}{2}\left(a - \dfrac{1}{a}\right)$ for SP.

$$\therefore \quad \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$= \frac{a - \frac{1}{2}\left(a - \frac{1}{a}\right)}{1 + \frac{1}{2}\left(a - \frac{1}{a}\right).a}$$

$$= \frac{2a - a + \frac{1}{a}}{2 + a^2 - 1}$$

$$= \frac{a + \frac{1}{a}}{1 + a^2}$$

$$= \frac{(a^2 + 1)}{a(1 + a^2)}$$

$$= \frac{1}{a}.$$

(iv) $$\tan(\theta + \beta) = \frac{\tan\theta + \tan\beta}{1 - \tan\theta\tan\beta}$$

$$= \frac{\frac{1}{a} + a}{1 - \frac{1}{a} \times a}$$

$$= \frac{\frac{1}{a} + a}{0}.$$

That is, $\tan(\theta + \beta)$ is undefined, hence $\theta + \beta = 90°$.

(v) ***Method 1*:**

If $\theta = \beta = 45°$,

then $a = \tan\beta = \tan 45° = 1$,

$\therefore \quad P(2a, a^2) = P(2, 1).$

***Method 2*:**

If $\theta = \beta$, then from the diagram, $SP \| x$-axis.

$\therefore \quad P(2a, a^2) = (x, 1)$

$\therefore \quad a = 1$

so $\quad x = 2$

$\therefore \quad P(2, 1).$

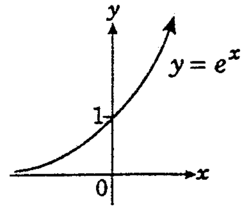
QUESTION 4

(a) $$f(x) = \frac{e^x}{3 + e^x}$$

$$f'(x) = \frac{(3 + e^x).e^x - e^x(e^x)}{(3 + e^x)^2}$$

$$= \frac{3e^x + e^{2x} - e^{2x}}{(3 + e^x)^2}$$

$$= \frac{3e^x}{(3 + e^x)^2}.$$



Since $e^x \neq 0$

then $f'(x) \neq 0$

$\therefore$ there are no stationary points.

(b) $$f''(x) = \frac{3e^x(3 - e^x)}{(3 + e^x)^3}$$

$= 0$ when $3 - e^x = 0$

that is, $e^x = 3$

i.e., $x = \ln 3.$

When $x = \ln 3$, $y = \dfrac{e^x}{3 + e^x}$

$= \dfrac{3}{3 + 3}$

$= \dfrac{1}{2}.$

i.e., possible point of inflexion at $\left(\ln 3, \frac{1}{2}\right)$.

Now, to test sign change at $x = \ln 3$.

When $x < \ln 3$,

say $x = 1$, then $3 - e^1 > 0$.

When $x > \ln 3$,

say $x = 2$, then $3 - e^2 < 0$.

That is, $f''(x)$ will change sign.

Hence the inflexion is $\left(\ln 3, \frac{1}{2}\right)$.

OR

Since the question states there is a point of inflexion, it must be where $f''(x) = 0$.

$\therefore$ The inflexion is at $\left(\ln 3, \frac{1}{2}\right)$.

(c) Since $e^x > 0$, $3 + e^x > e^x$

$\therefore \quad f(x) = \dfrac{e^x}{3 + e^x} < 1$ for all x.

But since $e^x > 0$ and $3 + e^x > 0$

then $\dfrac{e^x}{3 + e^x} > 0$ for all x.

i.e., $0 < f(x) < 1$ for all x.

(d) $f(x) = \frac{e^x}{3+e^x} = \frac{1}{3e^{-x}+1}$

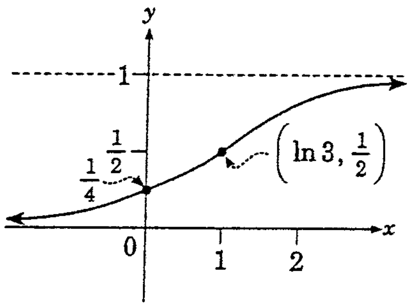
As $x \to \infty$, $e^{-x} \to 0$ $\quad \therefore \; f(x) \to \frac{1}{0+1}$

$\therefore \; f(x) \to 1$ but is always less than 1.

As $x \to -\infty$, $e^x \to 0$ $\quad \therefore \; f(x) \to \frac{0}{3+0}$

$\therefore \; f(x) \to 0$ but is always above zero.

(e)



Asymptotes at $y=0$ and $y=1$.

(f) $f(x)$ has an inverse function because it is an increasing function. Graphically, any horizontal line cuts it in one point only, so the inverse function (reflection in $y=x$) will satisfy the vertical line test.

(g) $f(x) = \frac{e^x}{3+e^x}$ i.e. $y = \frac{e^x}{3+e^x}$

$\therefore$ The inverse will be $x = \frac{e^y}{3+e^y}$

$\therefore \quad 3x + xe^y = e^y$

$e^y(1-x) = 3x$

$e^y = \frac{3x}{1-x}$

$\therefore \quad y = \ln\left(\frac{3x}{1-x}\right)$

QUESTION 5

(a) (i) $\sin 2x = 2\sin^2 x$

$\therefore \; 2\sin x \cos x - 2\sin^2 x = 0$

$2\sin x(\cos x - \sin x) = 0$ —①

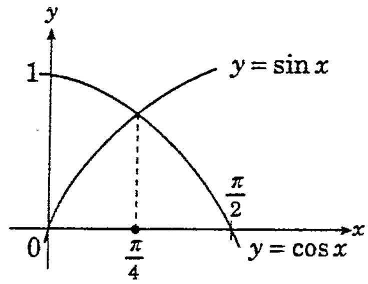
$\therefore \; \sin x = 0$ or $\sin x = \cos x$

i.e. $\sin x = 0$ or $\tan x = 1$

$\therefore \; x = 0, \pi,$ or $x = \frac{\pi}{4}$.

So $x = \frac{\pi}{4}$ is the only solution for $0 < x < \pi$.

(ii) *Method 1*:



From ①, when $0 < x < \frac{\pi}{4}$,

$\sin x > 0$, and $\cos x > \sin x$

i.e., $2\sin x(\cos x - \sin x) > 0$

$\therefore \; 2\sin x\cos x - 2\sin^2 x > 0$

$\sin 2x > 2\sin^2 x$.

Method 2:

For $0 \le x \le \frac{\pi}{4}$, $\sin 2x = 2\sin^2 x$ only when $x = 0, \frac{\pi}{4}$.

$\therefore \; \sin 2x > 2\sin^2 x$ or $\sin 2x < 2\sin^2 x$ for $0 < x < \frac{\pi}{4}$.

Let $x = \frac{\pi}{6}$, say:

$\sin\frac{\pi}{3} \approx 0{\cdot}87$ and $2\sin^2\frac{\pi}{6} = 0{\cdot}5$

i.e., $\sin 2\left(\frac{\pi}{6}\right) > 2\sin^2\left(\frac{\pi}{6}\right)$

$\therefore \; \sin 2x > 2\sin^2 x$ for $0 < x < \frac{\pi}{4}$.

(iii) Since $\sin 2x > 2\sin^2 x$

$\sin 2x - 2\sin^2 x > 0$ for $0 < x < \frac{\pi}{4}$.

$\therefore$ Area between the curves

$= \int_0^{\frac{\pi}{4}} \sin 2x - 2\sin^2 x \, dx$

$= \int_0^{\frac{\pi}{4}} \sin 2x \, dx + \int_0^{\frac{\pi}{4}} (\cos 2x - 1)\, dx$

$\left(\begin{array}{l}\text{Since } \cos 2x = 1 - 2\sin^2 x \\ \text{i.e., } -2\sin^2 x = \cos 2x - 1.\end{array}\right)$

$= \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}} + \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} - \left[x\right]_0^{\frac{\pi}{4}}$

$= -\frac{1}{2}\cos\frac{\pi}{2} + \frac{1}{2}\cos 0 + \frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\sin 0 - \frac{\pi}{4} + 0$

$= 0 + \frac{1}{2} + \frac{1}{2} - 0 - \frac{\pi}{4}$

$= 1 - \frac{\pi}{4}$ units^2.

(b) We are considering ten trials, so the terms of $(q+p)^{10}$ will be considered, where $p = 0{\cdot}015,\ q = 0{\cdot}985$.

(i) For exactly one win:

$$\text{Probability} = {}^{10}C_1\, q^9 p^1 = 10 \times 0{\cdot}985^9 \times 0{\cdot}015 = 0{\cdot}130\,923\cdots \approx 0{\cdot}1309.$$

(ii) For at least one win, we want

$$1 - P(\text{no wins}) = 1 - {}^{10}C_0\, q^{10} p^0 = 1 - 0{\cdot}8597 = 0{\cdot}140\,269\cdots \approx 0{\cdot}1403.$$

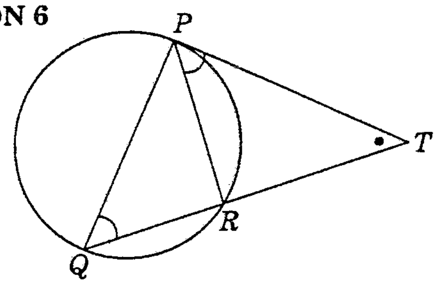
(iii) \$200 000 would need the initial \$8000 plus 24 no-wins, so for *more than* \$200 000 we need 25 no-wins.

Now, probability that it is not won is 0·985. So if not won for 25 times in sequence, it will have a probability of

$$(0{\cdot}985)^{25} = 0{\cdot}685\,339\cdots \approx 0{\cdot}6853.$$

QUESTION 6

(a)



In ΔPRT, ΔQPT

$\angle PTR = \angle PTQ$ (same angle)

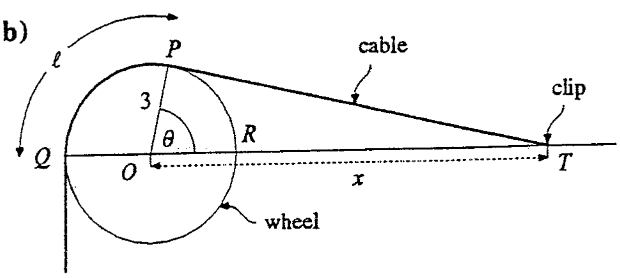
$\angle TPR = \angle PQR$ (angle between tangent and chord equals angle in the alternate segment)

$\therefore$ The triangles are equiangular, hence similar.

$\therefore \dfrac{PT}{QT} = \dfrac{RT}{PT}$ (corresponding sides in similar triangles)

$\therefore PT^2 = QT \times RT$.

b)



(i) Radius $OP \perp$ tangent PT

$\therefore \Delta OPT$ is right angled

$$\therefore \cos\theta = \frac{OP}{OT} = \frac{3}{x}.$$

(ii) $\text{arc } \ell = r(\angle POQ)$

$= 3(\pi - \theta)$ since QOR is a straight line

$$= 3\left[\pi - \cos^{-1}\left(\frac{3}{x}\right)\right].$$

(iii) $$\frac{d\ell}{dx} = 3 \times \frac{d}{dx}\left(-\cos^{-1}\frac{3}{x}\right)$$

Put $u = \dfrac{3}{x} \Rightarrow \dfrac{du}{dx} = -3x^{-2}$

$$\therefore \frac{d}{dx}\left(-\cos^{-1}\frac{3}{x}\right) = \frac{d}{du}\left(-\cos^{-1}u\right) \times \frac{du}{dx} = \frac{+1}{\sqrt{1-u^2}}.(-3x^{-2}) = \frac{-3}{x^2\sqrt{1-\frac{9}{x^2}}} = \frac{-3x}{x^2\sqrt{x^2-9}}$$

$$\therefore \frac{d\ell}{dx} = \frac{-9}{x\sqrt{x^2-9}}.$$

The significance of the negative value is that ℓ is decreasing as x increases. That is, the length of cable in contact with the wheel decreases as x increases.

(iv) $s = \ell + PT$

Now, from (a),

$$PT = \sqrt{QT \times RT} = \sqrt{(x+3)(x-3)} = \sqrt{x^2-9}$$

OR

Since $\angle OPT = 90°$ (from (i)),

$$PT^2 + 3^2 = x^2$$

$$\therefore PT = \sqrt{x^2-9}$$

$$\therefore s = 3\left(\pi - \cos^{-1}\frac{3}{x}\right) + \sqrt{x^2-9}.$$

(v) We want $\frac{ds}{dt}$ when $x=10$, given that $\frac{dx}{dt}=2$. Now, $\frac{ds}{dt}=\frac{ds}{dx}\cdot\frac{dx}{dt}$

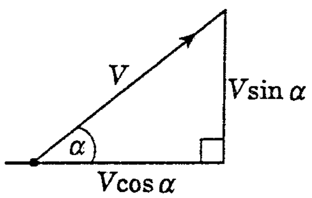
$$\therefore\ \frac{ds}{dx}=\frac{-9}{x\sqrt{x^2-9}}+\frac{1}{2}(x^2-9)^{-\frac{1}{2}}\times 2x$$
$$=\frac{-9}{x\sqrt{x^2-9}}+\frac{x}{\sqrt{x^2-9}}$$
$$=\frac{-9+x^2}{x\sqrt{x^2-9}}$$
$$=\frac{\sqrt{x^2-9}}{x}.$$

$$\frac{ds}{dt}=\frac{ds}{dx}\times\frac{dx}{dt}$$
$$=\frac{\sqrt{x^2-9}}{x}\times 2$$

$\therefore$ When $x=10$,
$$\frac{ds}{dt}=\frac{\sqrt{91}}{10}\times 2$$
$$=\frac{\sqrt{91}}{5}$$
$$\approx 1{\cdot}908\ \text{m/sec.}$$

QUESTION 7

(a) At $t=0$:



$$\ddot{x}=0$$
$\therefore\ \dot{x}=c_1$

and when $t=0$, $\dot{x}=V\cos\alpha$

$\therefore\ \dot{x}=V\cos\alpha$

$x=Vt\cos\alpha+c_2$

when $t=0, x=0,\ \therefore\ c_2=0$

$\therefore\ x=Vt\cos\alpha.$

$$\ddot{y}=-g$$
$\therefore\ \dot{y}=-gt+k_1$

and when $t=0$, $\dot{y}=V\sin\alpha$

$\therefore\ V\sin\alpha=k_1$

$\therefore\ \dot{y}=-gt+V\sin\alpha$

$y=-\frac{1}{2}gt^2+Vt\sin\alpha+k_2$

when $t=0, y=h,\ \therefore\ k_2=h$

$\therefore\ y=-\frac{1}{2}gt^2+Vt\sin\alpha+h.$

(b) Since $x=Vt\cos\alpha$

$$\therefore\ t=\frac{x}{V\cos\alpha}$$
$$\therefore\ y=-\frac{1}{2}g\cdot\frac{x^2}{V^2\cos^2\alpha}+V\sin\alpha\cdot\frac{x}{V\cos\alpha}+h$$
$$\therefore\ y=h+x\tan\alpha-x^2\cdot\frac{g}{2V^2\cos^2\alpha}\qquad —①$$

(c) Since the ball just clears the fence, we have when $x=R$, $y\ge h$

$$\therefore\ h+R\tan\alpha-R^2\frac{g}{2V^2\cos^2\alpha}\ge h$$
$$\therefore\ V^2\tan\alpha-\frac{Rg}{2\cos^2\alpha}\ge 0$$
$$\therefore\ V^2\ge\frac{Rg}{2\tan\alpha\,.\,\cos^2\alpha}$$
i.e. $V^2\ge\dfrac{Rg}{2\sin\alpha\cos\alpha}.$

(d) When $x=r+R$, $y=0$. Substituting in ①:

$$\therefore\ 0=h+(R+r)\tan\alpha-(R+r)^2\frac{g}{2V^2\cos^2\alpha}$$
i.e. $\dfrac{(R+r)^2 g}{2V^2\cos^2\alpha}=h+(R+r)\tan\alpha$

$$\therefore\ \frac{g}{2V^2\cos^2\alpha}=\frac{1}{(R+r)^2}\left(h+(R+r)\tan\alpha\right).$$

But $V^2\tan\alpha\ge\dfrac{Rg}{2\cos^2\alpha}$ (from (c))

$$\therefore\ \frac{g}{2V^2\cos^2\alpha}\le\frac{\tan\alpha}{R}$$
$$\therefore\ \frac{1}{(R+r)^2}\left(h+(R+r)\tan\alpha\right)\le\frac{\tan\alpha}{R}$$
$$\therefore\ R\left(h+(R+r)\tan\alpha\right)\le(R+r)^2\tan\alpha$$
$$\therefore\ Rh+R(R+r)\tan\alpha\le(R+r)^2\tan\alpha$$
$$\therefore\ Rh\le\tan\alpha\left((R+r)^2-R(R+r)\right)$$
$$\therefore\ \tan\alpha\ge\frac{Rh}{(R+r)\left(R+r-R\right)}$$
i.e. $\tan\alpha\ge\dfrac{Rh}{r(R+r)}.$

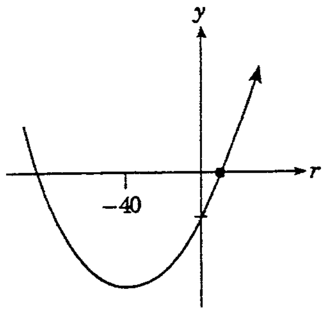
(e) When $V\le 50$, $g=10$, $R=80$, and $h=1$.

Then, since $V^2\ge\dfrac{gR}{2\sin\alpha\cos\alpha}$

$$\therefore\ \sin 2\alpha\ge\frac{gR}{V^2},$$

and when $V\le 50$, $\sin 2\alpha\ge\dfrac{10\times 80}{50^2}\ge\dfrac{8}{25}.$

Also, if it hits the ground outside the fence



$$\tan\alpha \ge \frac{Rh}{(R+r)r}$$

$$\therefore\ (R+r)r \ge \frac{Rh}{\tan\alpha}$$

$$\therefore\ r^2+80r \ge \frac{80}{\tan\alpha}$$

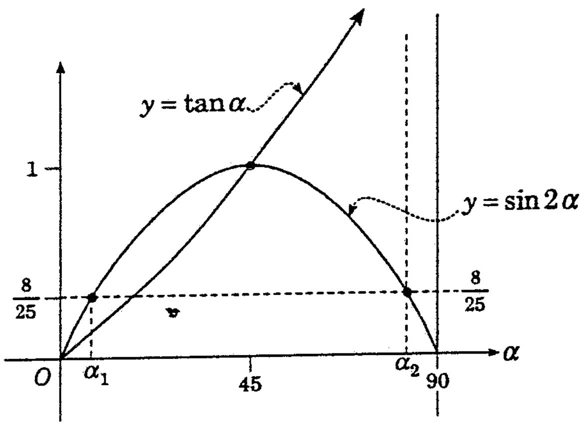
i.e. $r^2+80r-\frac{80}{\tan\alpha} \ge 0$

$$\therefore\ r = \frac{-80 \pm \sqrt{6400+\frac{320}{\tan\alpha}}}{2}$$

at extreme values.

For least r, we want $\tan\alpha$ to be a max.

Method 1:
We also know $\sin 2\alpha \ge \frac{8}{25}$.



Now, $\sin 2\alpha \ge \frac{8}{25}$ for $\alpha_1 \le \alpha \le \alpha_2$.

From the diagram, $\tan\alpha$ is a max for $\alpha = \alpha_2$, that is, for

$$\sin 2\alpha = \frac{8}{25}$$

where 2α is 'obtuse',

$$\therefore\quad 2\alpha = 160{\cdot}3375°$$
$$\alpha = 80{\cdot}6685°.$$

Hence, $$r = \frac{-80 \pm \sqrt{6400+\frac{320}{\tan 80{\cdot}67°}}}{2}$$

$$= \frac{-80 \pm 80{\cdot}3279}{2}$$

$$= \frac{0{\cdot}3279}{2}$$

$$= 0{\cdot}163\,98$$

Distance ≈ 0·16 m.

Method 2:

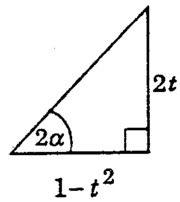
Put $t = \tan\alpha$, $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2t}{1-t^2}$

$$\therefore\ \sin 2\alpha = \frac{2t}{1+t^2}$$

$$\therefore\ \frac{2t}{1+t^2} \ge \frac{8}{25}$$

$$\therefore\quad 50t \ge 8+8t^2$$

$$\therefore\ 8t^2-50t+8 \le 0$$



Now $8t^2-50t+8=0$

when $$t = \frac{50 \pm \sqrt{2500-4\times 64}}{16}$$

$$= \frac{50 \pm 47{\cdot}37\cdots}{16}$$

$$= 6{\cdot}0856\cdots \text{ or } 0{\cdot}16\cdots$$

$\therefore\ 8t^2-50t+8 \le 0$ for $0{\cdot}16\cdots \le t \le 6{\cdot}08\cdots$

$\therefore$ max $\tan\alpha = 6{\cdot}0856\cdots$

$\therefore$ As before,

$$r = \frac{-80 \pm \sqrt{6400+\frac{320}{6{\cdot}0856\cdots}}}{2}$$

$$= 0{\cdot}163\,98 \text{ m}$$

Distance ≈ 0·16 m.