BOARD OF STUDIES
NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

1999
MATHEMATICS
3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON)

Time allowed—Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 8.
- Board-approved calculators may be used.
- Answer each question in a SEPARATE Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

QUESTION 1 Use a SEPARATE Writing Booklet. **Marks**

(a) Evaluate $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$. **2**

(b) Differentiate $\sin^3 x$. **2**

(c) The interval AB has end points $A(-2, 7)$ and $B(8, -8)$. **2**

Find the coordinates of the point P which divides the interval AB internally in the ratio 2 : 3.

(d) Write down the equation of the vertical asymptote of $y = \frac{4x}{(x-3)}$. **1**

(e) Find the remainder when the polynominal $P(x) = x^3 - 4x$ is divided by $x + 3$. **2**

(f) Use the substitution $u = \tan x$ to evaluate $\int_0^{\frac{\pi}{3}} \tan^2 x \sec^2 x \, dx$. **3**

QUESTION 2 Use a SEPARATE Writing Booklet.

(a) The staff in an office consists of 4 males and 7 females. **2**

How many committees of 5 staff can be chosen which contain exactly 3 females?

(b) Find all values of θ in the range $0 \leq \theta \leq 2\pi$ for which $\cos\theta + \sqrt{3}\sin\theta = 1$. **4**

(c) Let $f(x) = x + \log_e x$. **6**

(i) Write down the natural domain for $f(x)$.

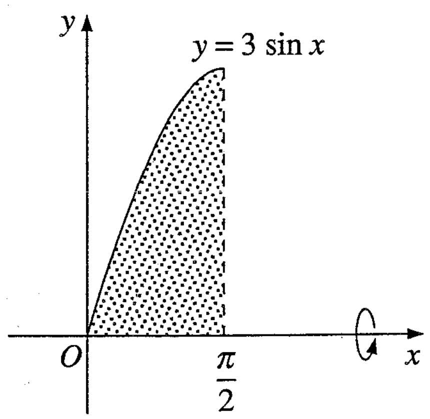
(ii) Show that, for all values of x in the natural domain, $y = f(x)$ is increasing.

(iii) Show that the curve $y = f(x)$ cuts the x axis between $x = 0{\cdot}5$ and $x = 1$.

(iv) Use Newton's method with a first approximation of $x = 0{\cdot}5$ to find a second approximation to the root of $x + \log_e x = 0$.

QUESTION 3 Use a SEPARATE Writing Booklet. **Marks**

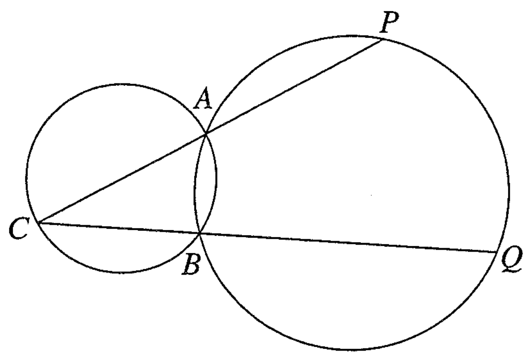
(a) 4



The shaded region bounded by $y = 3\sin x$, the x axis and the line $x = \frac{\pi}{2}$ is rotated about the x axis to form a solid. Calculate the volume of the solid.

(b) A fair, six-sided die is thrown seven times. What is the probability that a '6' occurs on exactly 2 of the 7 throws? 2

(c) 2



Two circles intersect at two points A and B as shown in the diagram. The diameter of one circle is CA and this line intersects the other circle at A and P. The line CB intersects the second circle at B and Q.

Copy or trace the diagram into your Writing Booklet.

Prove that $\angle CPQ$ is a right angle.

(d) (i) By equating the coefficients of $\sin x$ and $\cos x$, or otherwise, find constants A and B satisfying the identity 4

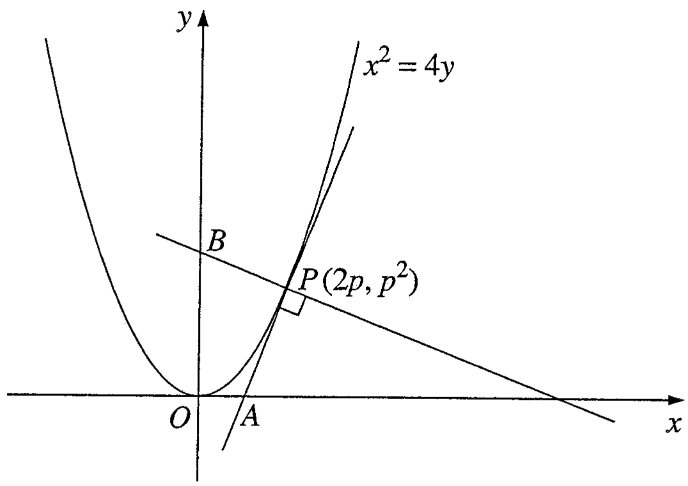
$$A(2\sin x + \cos x) + B(2\cos x - \sin x) \equiv \sin x + 8\cos x.$$

(ii) Hence evaluate $\displaystyle\int \frac{\sin x + 8\cos x}{2\sin x + \cos x}\,dx$.

QUESTION 4 Use a SEPARATE Writing Booklet. **Marks**

(a) Evaluate $\sum_{k=2}^{5}(-1)^k k$. **1**

(b) **6**



The diagram shows the graph of the parabola $x^2 = 4y$. The tangent to the parabola at $P(2p, p^2)$, $p > 0$, cuts the x axis at A. The normal to the parabola at P cuts the y axis at B.

(i) Derive the equation of the tangent AP.

(ii) Show that B has coordinates $(0, p^2 + 2)$.

(iii) Let C be the midpoint of AB. Find the cartesian equation of the locus of C.

(c) (i) Evaluate $\int_1^2 \frac{dx}{x}$. **5**

(ii) Use Simpson's rule with 3 function values to approximate $\int_1^2 \frac{dx}{x}$.

(iii) Use your results to parts (i) and (ii) to obtain an approximation for e. Give your answer correct to 3 decimal places.

QUESTION 5 Use a SEPARATE Writing Booklet. **Marks**

(a) Prove by induction that, for all integers $n \geq 1$, **3**

$$(n+1)(n+2)\cdots(2n-1)2n = 2^n\left[1 \times 3 \times \cdots \times (2n-1)\right].$$

(b) Consider the function $f(x) = e^x - 1 - x$. **9**

(i) Show that the minimum of $f(x)$ occurs at $x = 0$.

(ii) Deduce that $f(x) \geq 0$ for all x.

(iii) On the same set of axes, sketch $y = e^x - 1$ and $y = x$.

(iv) Find the inverse function of $g(x) = e^x - 1$.

(v) State the domain of $g^{-1}(x)$.

(vi) For what values of x is $\log_e(1+x) \leq x$? Justify your answer.

Please turn over

QUESTION 6 Use a SEPARATE Writing Booklet. **Marks**

(a) A particle moves in a straight line and its displacement x metres from the origin after t seconds is given by **6**

$$x = \cos^2 3t, \quad t > 0.$$

(i) When is the particle first at $x = \frac{3}{4}$?

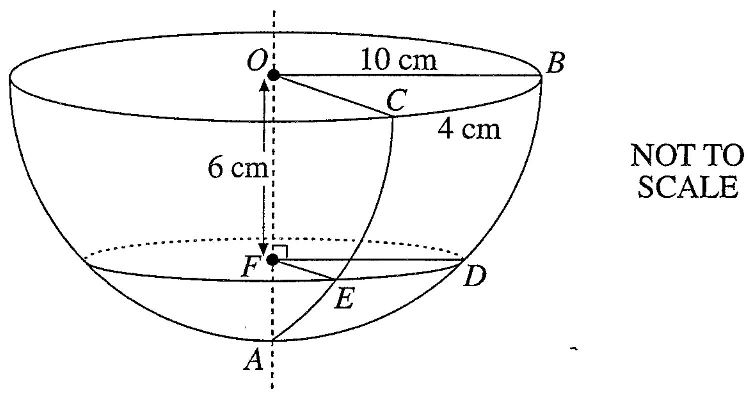
(ii) In what direction is the particle travelling when it is first at $x = \frac{3}{4}$?

(iii) Express the acceleration of the particle in terms of x.

(iv) Hence, or otherwise, show that the particle is undergoing simple harmonic motion.

(v) State the period of the motion.

(b) **6**



The diagram shows a hanging basket in the shape of a hemisphere with radius 10 cm. Let O be the centre of the sphere and let OA be the central axis. Two vertical wire supports, AB and AC, are shown on the diagram. The length of the arc BC is 4 cm.

A horizontal wire support is placed around the surface of the basket. This wire meets AB at D and AC at E. The plane through DE parallel to the plane OBC cuts OA at F. The length OF is 6 cm. Note that $\angle BOC = \angle DFE$.

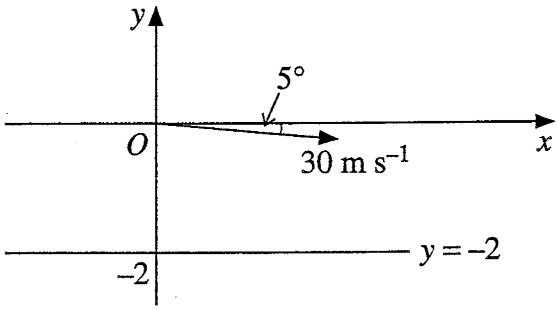
(i) Show that the length of FD is 8 cm.

(ii) Find $\angle DFE$ in radians.

(iii) Find the size of the angle $\angle DOE$ in radians, correct to 3 decimal places.

QUESTION 7 Use a SEPARATE Writing Booklet. **Marks**

(a) **8**



A cricket ball leaves the bowler's hand 2 metres above the ground with a velocity of 30 m s^{-1} at an angle of $5°$ below the horizontal. The equations of motion for the ball are

$$\ddot{x} = 0 \qquad \text{and} \qquad \ddot{y} = -10.$$

Take the origin to be the point where the ball leaves the bowler's hand.

(i) Using calculus, prove that the coordinates of the ball at time t are given by

$$x = 30t\cos(5°), \text{ and}$$
$$y = -30t\sin(5°) - 5t^2.$$

(ii) Find the time at which the ball strikes the ground.

(iii) Calculate the angle at which the ball strikes the ground.

(b) By considering $(1-x)^n\left(1+\frac{1}{x}\right)^n$, or otherwise, express **4**

$$\binom{n}{2}\binom{n}{0} - \binom{n}{3}\binom{n}{1} + \cdots + (-1)^n\binom{n}{n}\binom{n}{n-2}$$

in simplest form.

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

1999 Higher School Certificate Solutions

3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS

QUESTION 1

(a) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}}$ (standard integral)

$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}0$

$= \frac{\pi}{3} - 0$

$= \frac{\pi}{3}.$

(b) $\frac{d}{dx}\sin^3 x = 3\sin^2 x.\frac{d}{dx}\sin x$

$= 3\sin^2 x\cos x.$

(c) $A(-2, 7)$, $B(8, -8)$

Ratio 2 : 3

For P, $x = \frac{2\times 8 + 3\times(-2)}{2+3}$

$= \frac{10}{5}$

$= 2,$

and $y = \frac{2\times(-8)+3\times 7}{2+3}$

$= \frac{5}{5}$

$= 1.$

$\therefore$ P is $(2, 1)$.

(d) Asymptote when denominator is zero, that is, $x-3=0$ or $x=3$.

(e) $P(x) = x^3 - 4x$

$P(-3) = -27+12$

$= -15$ is the remainder.

(f) If $u = \tan x$

$\frac{du}{dx} = \sec^2 x \Rightarrow du = \sec^2 x\ dx.$

When $x = 0$, $u = 0$.

When $x = \frac{\pi}{3}$, $u = \sqrt{3}$.

$\therefore \int_0^{\frac{\pi}{3}} \tan^2 x \sec^2 x\ dx = \int_0^{\sqrt{3}} u^2\ du$

$= \left[\frac{u^3}{3}\right]_0^{\sqrt{3}}$

$= \sqrt{3}.$

QUESTION 2

(a) Number of ways of choosing 3 females from 7 is ${}^7C_3$. The other two must be male. The number of ways of choosing 2 from 4 is ${}^4C_2$.

$\therefore$ Number of committees $= {}^7C_3 \times {}^4C_2$

$= 210.$

(b) **Method 1:**

$\cos\theta + \sqrt{3}\sin\theta = 1$

Now $R\cos(\theta-\alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$\therefore$ $R\cos\alpha = 1$

$R\sin\alpha = \sqrt{3}.$

$\therefore$ $R^2\left(\sin^2\alpha + \cos^2\alpha\right) = 3+1 = 4.$

$\therefore$ $R = 2$

and $\frac{R\sin\alpha}{R\cos\alpha} = \frac{\sqrt{3}}{1},$

$\therefore$ $\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}.$

$\therefore$ $2\cos\left(\theta - \frac{\pi}{3}\right) = 1,\ -\frac{\pi}{3} \le \left(\theta-\frac{\pi}{3}\right) \le \frac{5\pi}{3}$

$\cos\left(\theta-\frac{\pi}{3}\right) = \frac{1}{2}$

$\theta - \frac{\pi}{3} = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}.$

$\therefore$ $\theta = 0, \frac{2\pi}{3}, 2\pi.$

Method 2:

If $\theta = \pi$, $\cos\pi + \sqrt{3}\sin\pi = -1+0 \ne 1$,

$\therefore$ $\theta = \pi$ is not a solution.

If $\theta \ne \pi$, let $t = \tan\frac{\theta}{2}$.

$\therefore$ $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}.$

$\therefore$ $\frac{1-t^2}{1+t^2} + \sqrt{3}\times\frac{2t}{1+t^2} = 1$

$1 - t^2 + 2\sqrt{3}t = 1+t^2$

$2t^2 - 2\sqrt{3}t = 0$

$2t\left(t-\sqrt{3}\right) = 0.$

$\therefore$ $t = 0,\ \sqrt{3}.$

That is, $\tan\frac{\theta}{2} = 0, \sqrt{3}$.

$\therefore \frac{\theta}{2} = 0, \frac{\pi}{3}, \pi \quad (0 \le \theta \le 2\pi)$.

$\therefore \theta = 0, \frac{2\pi}{3}, 2\pi$.

(c) $f(x) = x + \log_e x$

(i) The natural domain is $x > 0$ since $\log_e x$ is defined only for $x > 0$.

(ii) $y = f(x)$ is increasing if $f'(x) > 0$.

$\therefore f'(x) = 1 + \frac{1}{x} > 0$, since $x > 0$.

(iii) $f(0{\cdot}5) = 0{\cdot}5 + \log_e 0{\cdot}5$
$\fallingdotseq -0{\cdot}193 < 0$.

$f(1) = 1 + \log_e 1$
$= 1 > 0$.

The curve cuts the x axis between $x = 0{\cdot}5$ and $x = 1$, since the sign of $f(x)$ changes and $f(x)$ is continuous.

(iv) Let $f(x) = x + \log_e x$

$f'(x) = 1 + \frac{1}{x}$.

Let x_2 be a second approximation to the root of $x + \log_e x = 0$.

$\therefore x_2 = 0{\cdot}5 - \frac{f(0{\cdot}5)}{f'(0{\cdot}5)}$, by Newton's method,

$= 0{\cdot}5 - \frac{0{\cdot}5 + \log_e 0{\cdot}5}{1 + \frac{1}{0{\cdot}5}}$

$= 0{\cdot}564 \ldots$

N.B. You need to use Newton's method again to see how many of these digits are significant, but this is not required by the question.

QUESTION 3

(a) $V = \pi\int_0^{\frac{\pi}{2}} (3\sin x)^2\,dx$

$= 9\pi\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

$= \frac{9\pi}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2x)\,dx$

$= \frac{9\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$

$= \frac{9\pi}{2}\left[\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right]$

$= \frac{9\pi^2}{4}$.

$\therefore$ Volume $= \frac{9\pi^2}{4}$ cubic units.

(b) $P(6) = \frac{1}{6}$, $P(\bar{6}) = \frac{5}{6}$.

Probability of '6' on exactly 2 of 7 throws

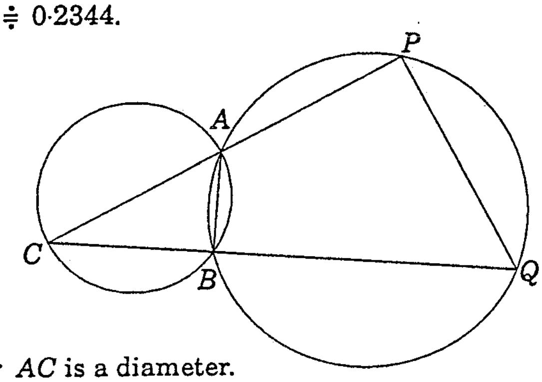
$= {}^7C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^5$

$= \frac{7\times 6}{1\times 2}\times\frac{1}{6}\times\frac{5^5}{6^5}$

$= \frac{21875}{93\,312}$

$\fallingdotseq 0{\cdot}2344$.

(c)



Data: AC is a diameter.

Construction: Join AB, PQ.

Proof: $\angle ABC = 90°$ (angle in semicircle, given AC is diameter)

$\angle CPQ = \angle ABC$ (exterior angle of cyclic quadrilateral equals interior opposite angle)

$\therefore \angle CPQ$ is a right angle.

(d) (i) $A(2\sin x + \cos x) + B(2\cos x - \sin x)$
$\equiv \sin x + 8\cos x$

$\therefore (2A - B)\sin x + (A + 2B)\cos x$
$\equiv \sin x + 8\cos x$.

Equating coefficients of $\sin x$ and $\cos x$,

$2A - B = 1$ —①

$A + 2B = 8$ —②

①×2 → $4A - 2B = 2$ —③

②+③ → $5A = 10$

$A = 2$.

Substitute $A = 2$ in ②:

$2B = 6$

$B = 3$.

$\therefore A = 2,\ B = 3$.

(ii) $\int \frac{\sin x + 8\cos x}{2\sin x + \cos x}\,dx$

$= \int \frac{2(2\sin x + \cos x) + 3(2\cos x - \sin x)}{2\sin x + \cos x}\,dx$ from (i)

$= \int 2\,dx + 3\int \frac{2\cos x - \sin x}{2\sin x + \cos x}\,dx$

$= 2x + 3\ln(2\sin x + \cos x) + C$.

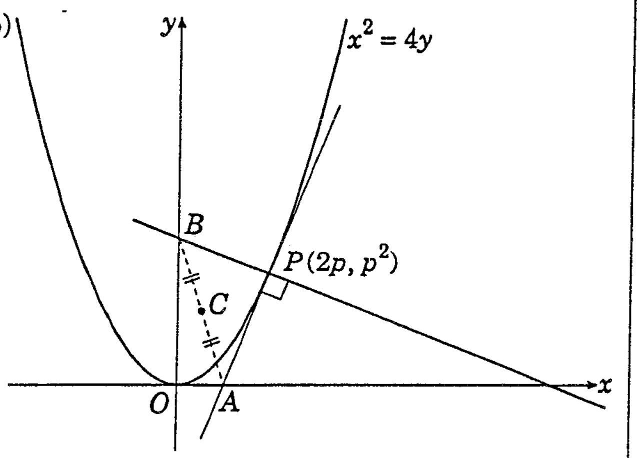
[*Note:* $\frac{d}{dx}(2\sin x + \cos x) = 2\cos x - \sin x$]

QUESTION 4

(a) $\sum_{k=2}^{5}(-1)^k k = (-1)^2\times 2+(-1)^3\times 3 + (-1)^4\times 4+(-1)^5\times 5$

$= -2.$

(b)



(i) $x^2 = 4y$

$y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{x}{2}.$

When $x = 2p$, $\frac{dy}{dx} = \frac{2p}{2} = p.$

Equation of tangent AP is

$y - y_1 = m(x - x_1)$

$y - p^2 = p(x - 2p)$

$y = px - p^2$ —①

(ii) Equation of normal BP is

$y - p^2 = -\frac{1}{p}(x-2p).$

B lies on BP at $x = 0$.

When $x = 0$, $y = p^2 - \frac{1}{p}(-2p)$

$= p^2 + 2.$

$\therefore$ B is $(0, p^2+2)$.

(iii) Substitute $y = 0$ in ①: $0 = px - p^2$

$x = p.$

$\therefore$ A is $(p, 0)$.

If $C(x, y)$ is the midpoint of $A(p, 0)$ and $B(0, p^2+2)$, $x = \frac{p+0}{2}$ and $y = \frac{0+(p^2+2)}{2}$.

$x = \frac{p}{2}$ —②

$y = \frac{p^2+2}{2}$ —③

From ②, $p = 2x.$

Substitute in ③: $y = \frac{4x^2+2}{2} = 2x^2+1.$

But $p > 0$, $\therefore$ $x > 0$.

$\therefore$ Cartesian equation of locus of C is $y = 2x^2+1$, $x > 0$.

(c) (i) $\int_1^2 \frac{dx}{x} = \left[\ln x\right]_1^2$

$= \ln 2 - \ln 1$

$= \ln 2.$

(ii) $\int_1^2 \frac{dx}{x} \doteqdot \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right],$

where $f(x) = \frac{1}{x}$, $a = 1$, $b = 2$.

$= \frac{2-1}{6}\left[\frac{1}{1}+4\times\frac{1}{1\cdot 5}+\frac{1}{2}\right]$

$= \frac{25}{36}$ $(= 0{\cdot}69\dot{4})$.

(iii) $\ln 2 \doteqdot \frac{25}{36}$

$2 \doteqdot e^{\frac{25}{36}}$

$2^{\frac{36}{25}} \doteqdot e$ (raising both sides to power $\frac{36}{25}$)

$\therefore$ $e \doteqdot 2{\cdot}7132\ldots$

$= 2{\cdot}713$ (3 dec. places).

QUESTION 5

(a) Prove $(n+1)(n+2)\cdots(2n-1)2n$

$= 2^n[1\times 3\times\cdots\times(2n-1)]$

If $n = 1$, LHS $= 1+1 = 2$

RHS $= 2^1\times 1 = 2.$

$\therefore$ The statement is true for $n = 1$.

Assume statement is true for $n = k$; that is,

assume $(k+1)(k+2)\cdots(2k-1)2k$

$= 2^k[1\times 3\times\cdots\times(2k-1)].$ —①

Hence prove statement is true for $n = k+1$, that is, prove

$(k+2)(k+3)\cdots(2k+1)(2k+2)$

$= 2^{k+1}[1\times 3\times\cdots\times(2k+1)].$ —②

Now LHS

$= (k+2)(k+3)\cdots(2k+1)(2k+2)$

$= \frac{(k+1)(k+2)(k+3)\cdots(2k-1)2k(2k+1)(2k+2)}{k+1}$

$= \frac{2^k}{k+1}[1\times 3\times\cdots\times(2k-1)](2k+1)(2k+2)$, from ①

$= \frac{2^k}{\cancel{k+1}}[1\times 3\times\cdots\times(2k-1)](2k+1)2\cancel{(k+1)}$

$= 2^{k+1}[1\times 3\times\cdots\times(2k-1)(2k+1)]$

$=$ RHS.

$\therefore$ If the statement is true for $n = k$, it is also true for $n = k+1$. But it is true for $n = 1$.

$\therefore$ It is true for $n = 1+1 = 2$ and so on, that is, it is true for all integers $n \geq 1$.

(b) $f(x) = e^x - 1 - x$

(i) $f'(x) = e^x - 1$

$= 0$ only when $x = 0$.

$\therefore$ There is only one stationary point (at $x = 0$).

$f''(x) = e^x > 0$ for all x.

$\therefore$ The graph of $f(x)$ is concave up for all x.

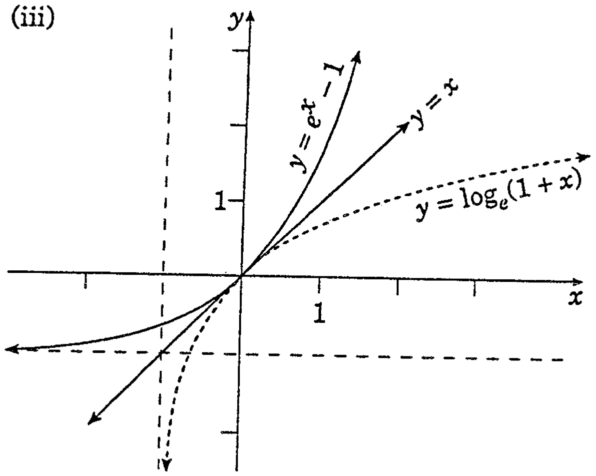
Since $f(x)$ is continuous for all x (being made up of the sum and difference of continuous functions), the stationary point at $x = 0$ is both a local and absolute minimum.

(ii) When $x = 0$, $f(x) = e^0 - 1 - 0 = 0$.

$\therefore$ The least value of $f(x) = 0$.

$\therefore$ $f(x) \geq 0$ for all x.

(iii)



N.B. The gradient of $y = e^x - 1$ at $x = 0$ is 1, so $y = x$ is a tangent at $(0, 0)$.

This is also implied by (ii).

(iv) Inverse relation of $y = e^x - 1$ is $x = e^y - 1$.

That is, $e^y = x + 1$

$y = \log_e(x + 1)$

$\therefore$ $g^{-1}(x) = \log_e(x + 1)$.

(v) Domain of $g^{-1}(x)$ is $x + 1 > 0$, that is, $x > -1$.

(vi) $g(x) = e^x - 1$

$g^{-1}(x) = \log_e(1 + x)$.

The graphs of a pair of inverse functions are symmetrical about the line $y = x$.
The graph of $y = g(x)$ is above the graph of $y = x$ except at $x = 0$ where they coincide.

$\therefore$ The graph of $y = g^{-1}(x)$ is below the graph of $y = x$ except at $x = 0$ where they coincide.

$\therefore$ $\log_e(1 + x) \leq x$ for all $x > -1$.

QUESTION 6

(a) $x = \cos^2 3t, \quad t > 0.$ —①

(i) Substitute $x = \frac{3}{4}$ in ①:

$$\frac{3}{4} = \cos^2 3t$$

$$\cos 3t = \pm\frac{\sqrt{3}}{2}$$

$$3t = \frac{\pi}{6}, \ldots$$

$$t = \frac{\pi}{18}, \ldots$$

Particle is first at $x = \frac{3}{4}$ after $\frac{\pi}{18}$ seconds.

(ii) $v = \frac{dx}{dt} = 2\cos 3t . -3\sin 3t$

$= -3\sin 6t$.

When $t = \frac{\pi}{18}$, $\quad v = -3\sin\left(6 \times \frac{\pi}{18}\right)$

$= -3\sin\frac{\pi}{3}$

$= \frac{-3\sqrt{3}}{2} < 0.$

Since $v < 0$, the particle is travelling in the negative direction.

(iii) $a = \frac{dv}{dt} = -3 \times 6\cos 6t$

$= -18\cos 6t$.

$\cos 6t = 2\cos^2 3t - 1$

(using $\cos 2x = 2\cos^2 x - 1$)

$= 2x - 1$, from ①.

$\therefore \quad a = -18(2x - 1)$.

(iv) $a = -18(2x - 1)$

$= -36\left(x - \frac{1}{2}\right)$

$= -6^2\left(x - \frac{1}{2}\right)$,

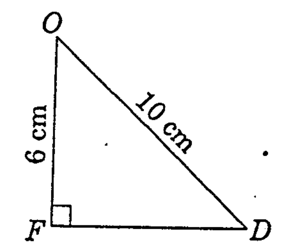
which is of the form $\ddot{x} = -n^2(x - b)$, indicating simple harmonic motion with centre of oscillation at $x = \frac{1}{2}$.

(v) Period $= \frac{2\pi}{n}$ seconds

$= \frac{2\pi}{6}$ seconds

$= \frac{\pi}{3}$ seconds.

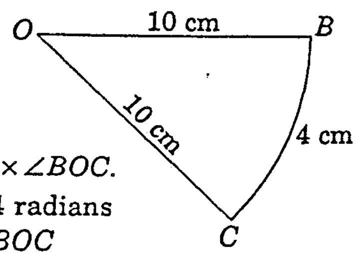
(b) (i)



$OD = 10$ cm (radius)

$\therefore$ $FD = 8$ cm (Pythagoras' theorem).

(ii) OBC is a sector of a circle, centre O, radius 10 cm.



$l = r\theta$

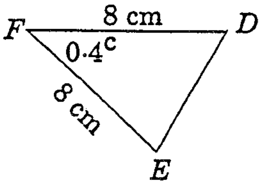
$\therefore \quad 4 = 10 \times \angle BOC.$

$\therefore \angle BOC = 0{\cdot}4$ radians

$\angle DFE = \angle BOC$

$\therefore \angle DFE = 0{\cdot}4$ radians.

(iii)

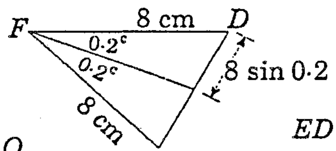


$ED^2 = FD^2 + FE^2 - 2 \times FD \times FE \cos 0{\cdot}4$ (by cosine rule)

$= 8^2 + 8^2 - 2 \times 8 \times 8 \cos 0{\cdot}4$

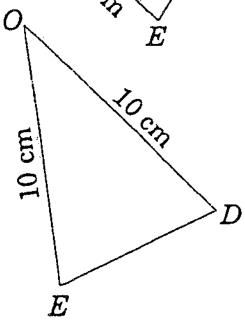
$= 128(1 - \cos 0{\cdot}4).$

{ *Alternatively:*



$ED = 2 \times 8 \sin 0{\cdot}2$

$= 16 \sin 0{\cdot}2.$ }



$$\cos \angle EOD = \frac{OE^2 + OD^2 - ED^2}{2 \times OE \times OD}$$

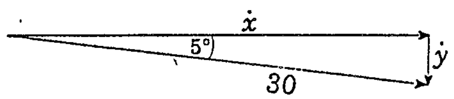
$$= \frac{10^2 + 10^2 - 128(1 - \cos 0{\cdot}4)}{2 \times 10 \times 10}$$

$\angle DOE = 0{\cdot}3192\ldots$

$\approx 0{\cdot}319$ radians (3 dec. places).

QUESTION 7

(a) Initial conditions for velocity:



When $t = 0$,

$\dot{x} = 30\cos(5°), \ \dot{y} = -30\sin(5°).$ —①

(i) $\ddot{x} = 0$

$\therefore \ \dot{x} = C_1$ (constant).

$\therefore \ \dot{x} = 30\cos(5°)$ from ①. —②

$$x = \int 30\cos(5°)\ dt$$

$= 30t\cos(5°) + C_2.$

When $t = 0, \ x = 0, \ \therefore \ C_2 = 0.$

$\therefore \quad x = 30t\cos(5°).$

$\ddot{y} = -10$

$$\therefore \quad \dot{y} = \int -10\ dt$$

$= -10t + D_1.$

When $t = 0, \ \dot{y} = -30\cos(5°)$ from ①.

$\therefore \quad D_1 = -30\sin(5°)$

$\therefore \quad \dot{y} = -10t - 30\sin(5°).$ —③

$$y = \int -10t - 30\sin(5°)\ dt$$

$= -5t^2 - 30t\sin(5°) + D_2.$

When $t = 0, \ y = 0, \ \therefore \ D_2 = 0.$

$\therefore \quad y = -30t\sin(5°) - 5t^2.$ —④

(ii) Ball strikes the ground when $y = -2$.
Substitute $y = -2$ in ④:

$-2 = -30t\sin 5° - 5t^2$

$5t^2 + 30t\sin 5° - 2 = 0$

$$t = \frac{-30\sin 5° \pm \sqrt{(-30\sin 5°)^2 - 4 \times 5 \times (-2)}}{2 \times 5}$$

$$= \frac{-30\sin 5° + \sqrt{900\sin^2 5° + 40}}{10}$$

(other answer negative and therefore irrelevant)

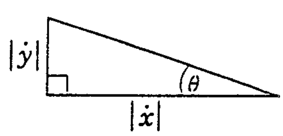
$= 0{\cdot}4229\ldots$

$\therefore$ The ball strikes the ground after $0{\cdot}42$ seconds (2 dec. places).

(iii) When $t = 0{\cdot}4229$,

$\dot{x} = 30\cos(5°)$ from ②,

$\dot{y} = -4{\cdot}229 - 30\sin(5°)$, from ③.



$$\tan\theta = \frac{4{\cdot}229 + 30\sin 5°}{30\cos 5°}$$

$= 0{\cdot}228\,99\ldots$

$\theta \doteqdot 12{\cdot}9°.$

Angle at which the ball strikes the ground is 13° (nearest degree).

(b) $(1-x)^n = \binom{n}{0} - \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n}(-1)^n x^n$

$\left(1+\frac{1}{x}\right)^n = \binom{n}{0}\left(\frac{1}{x}\right)^0 + \binom{n}{1}\left(\frac{1}{x}\right)^1 + \binom{n}{2}\left(\frac{1}{x}\right)^2 + \cdots + \binom{n}{n}(-1)^n\left(\frac{1}{x}\right)^n.$

The term in x^2 in $(1-x)^n\left(1+\frac{1}{x}\right)^n$ is

$\binom{n}{2}\binom{n}{0}x^2\left(\frac{1}{x}\right)^0 - \binom{n}{3}\binom{n}{1}x^3\left(\frac{1}{x}\right)^1 + \binom{n}{5}\binom{n}{3}x^5\left(\frac{1}{x}\right)^3 + \cdots + (-1)^n\binom{n}{n}\binom{n}{n-2}x^n\left(\frac{1}{x}\right)^n.$

$\therefore$ The coefficient of x^2 in $(1-x)^n\left(1+\frac{1}{x}\right)^n$ is

$\binom{n}{2}\binom{n}{0} - \binom{n}{3}\binom{n}{1} + \cdots + (-1)^n\binom{n}{n}\binom{n}{n-2},$

and this is the expression given in the question.

Now $(1-x)^n\left(1+\frac{1}{x}\right)^n = \left[(1-x)\left(1+\frac{1}{x}\right)\right]^n = \left(\frac{1}{x}-x\right)^n.$

The general term of $\left(\frac{1}{x}-x\right)^n$ is

$\binom{n}{r}\left(\frac{1}{x}\right)^{n-r}(-x)^r = \binom{n}{r}(-1)^r x^{2r-n}.$

The term in x^2 has $2r-n=2$

$r = \frac{n+2}{2}.$

$\therefore$ The coefficient of $x^2 = \binom{n}{\frac{n+2}{2}}(-1)^{\frac{n+2}{2}},$

and only exists if n is even, $\left(\frac{n+2}{2}\right.$ must be an integer$\left.\right)$.

$\therefore \binom{n}{2}\binom{n}{0} - \binom{n}{3}\binom{n}{1} + \cdots + (-1)^n\binom{n}{n}\binom{n}{n-2}$

$$= \begin{cases} \binom{n}{\frac{n+2}{2}}(-1)^{\frac{n+2}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

END OF 3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS SOLUTIONS