2000 HIGHER SCHOOL CERTIFICATE EXAMINATION PAPER

3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS

QUESTION 1 — Marks

(a) Differentiate $x\sin^{-1}x$. — 2

(b) Find the acute angle between the lines $y = 2x - 1$ and $y = \frac{1}{3}x + 1$. — 2

(c) Find the value of k if $x - 3$ is a factor of
$$P(x) = x^3 - 3kx + 6.$$ — 2

(d) Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{4}{x^2+9}\,dx$. — 3

(e) Solve $\dfrac{5}{x+2} \le 1$. — 3

QUESTION 2

(a) How many arrangements of the letters of the word HOCKEYROO are possible? — 2

(b) Find the coefficient of x^6 in the expansion of $\left(5 + 2x^2\right)^7$. — 3

(c) Solve the equation
$$\cos 2\theta = \sin\theta,\ 0 \le \theta \le 2\pi.$$ — 4

(d) Use the substitution $u = 2 + x$ to find
$$\int \frac{x}{\sqrt{2+x}}\,dx.$$ — 3

QUESTION 3

(a) Use the definition
$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$
to find the derivative of x^3 where $x = a$. — 2

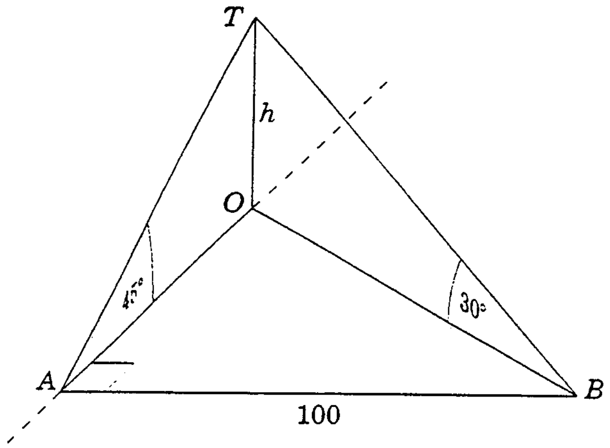
(b) Consider the function $f(x) = 3\tan^{-1}x$. — 5

(i) State the range of the function $y = f(x)$.

(ii) Sketch the graph of $y = f(x)$.

(iii) Find the gradient of the tangent to the curve $y = f(x)$ at $x = \dfrac{1}{\sqrt{3}}$.

(c) A surveyor stands at a point A, which is due south of a tower OT of height h m. The angle of elevation of the top of the tower from A is 45°. — 5



The surveyor then walks 100 m due east to point B, from where she measures the angle of elevation of the top of the tower to be 30°.

(i) Express the length of OB in terms of h.

(ii) Show that $h = 50\sqrt{2}$.

(iii) Calculate the bearing of B from the base of the tower.

QUESTION 4

(a) Use mathematical induction to prove that
$$1 + 3 + 6 + \cdots + \tfrac{1}{2}n(n+1) = \tfrac{1}{6}n(n+1)(n+2)$$
for all integers $n = 1, 2, 3, \ldots$. — 3

(b) We wish to find the interest rate r such that $(1+r)\left[(1+r)^{24} - 1\right] - 50r = 0$.
Use one step of Newton's method to estimate r. Take $r_1 = 0{\cdot}06$ as the first approximation. — 3

(c) The polynomial $P(x) = x^3 + px^2 + qx + r$ has real roots $\sqrt{k}, -\sqrt{k}$ and α. — 4

(i) Explain why $\alpha + p = 0$.

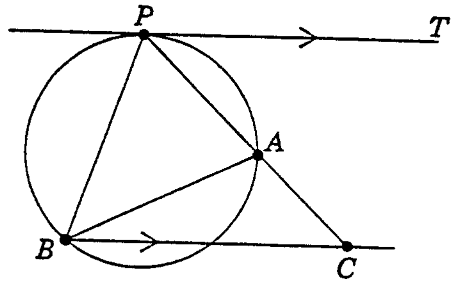
(ii) Show that $k\alpha = r$.

(iii) Show that $pq = r$.

(d) A particle is moving in simple harmonic motion about a fixed point O. Its amplitude is 3 cm and its period is 4π seconds.
Find its speed at the point O. — 2

QUESTION 5

(a) 4



In the diagram, A, P and B are points on the circle. The line PT is tangent to the circle at P, and PA is produced to C so that BC is parallel to PT. Copy the diagram.

(i) Show that $\angle PBA = \angle PCB$.

(ii) Deduce that $PB^2 = PA \times PC$.

(b) Consider the function $f(x) = \dfrac{x}{x+2}$. 8

(i) Show that $f'(x) > 0$ for all x in the domain.

(ii) State the equation of the horizontal asymptote of $y = f(x)$.

(iii) Without using any further calculus, sketch the graph of $y = f(x)$.

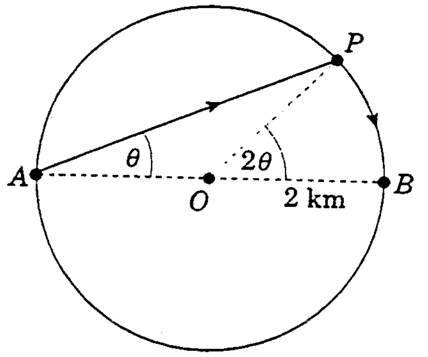
(iv) Explain why $f(x)$ has an inverse function $f^{-1}(x)$.

(v) Find an expression for $f^{-1}(x)$.

(vi) Write down the domain of $f^{-1}(x)$.

QUESTION 6

(a) 8



The diagram shows a circular lake, centre O, of radius 2 km with diameter AB. Pat can row at 3 km/h and can walk at 4 km/h and wishes to travel from A to B as quickly as possible. Pat considers the strategy of rowing direct from A to a point P and then walking around the edge of the lake to B.

Let $\angle PAB = \theta$ radians, and let the time taken for Pat to travel from A to B by this route be T hours.

(i) Show that $T = \frac{1}{3}(4\cos\theta + 3\theta)$.

(ii) Find the value of θ for which $\dfrac{dT}{d\theta} = 0$.

(iii) To what point P, if any, should Pat row to minimise T? Give reasons for your answer.

(b) A standard pack of 52 cards consists of 13 cards of each of the four suits: spades, hearts, clubs and diamonds. 4

(i) In how many ways can six cards be selected without replacement so that exactly two are spades and four are clubs? (Assume that the order of selection of the six cards is not important.)

(ii) In how many ways can six cards be selected without replacement if at least five must be of the same suit? (Assume that the order of selection of the six cards is not important.)

QUESTION 7

(a) The amount of fuel F in litres required *per hour* to propel a plane in level flight at constant speed u km/h is given by 4

$$F = Au^3 + \frac{B}{u},$$

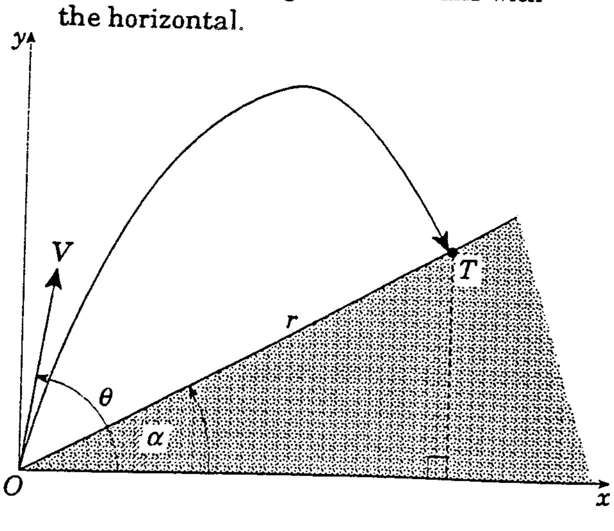
where A and B are positive constants.

(i) Show that a pilot wishing to remain in level flight for as long a period as possible should fly at

$$\left(\frac{B}{3A}\right)^{\frac{1}{4}} \text{ km/h}.$$

(ii) Show that a pilot wishing to fly as far as possible in level flight should fly approximately 32% faster than the speed given in part (i).

(b) The diagram shows an inclined plane that makes an angle of α radians with the horizontal. 8



A projectile is fired from O, at the bottom of the incline, with a speed of V m s^{-1} at an angle of elevation θ to the horizontal, as shown.

With the above axes, you may assume that the position of the projectile is given by

$$x = Vt\cos\theta$$

$$y = Vt\sin\theta - \frac{1}{2}gt^2,$$

where t is the time, in seconds, after firing, and g is the acceleration due to gravity.

For simplicity we assume that $\dfrac{2V^2}{g} = 1$.

(i) Show that the path of the trajectory of the projectile is $y = x\tan\theta - x^2\sec^2\theta$.

(ii) Show that the range of the projectile, $r = OT$ metres, up the inclined plane is given by

$$r = \frac{\sin(\theta-\alpha)\cos\theta}{\cos^2\alpha}.$$

(iii) Hence, or otherwise, deduce that the maximum range, R metres, up the incline is

$$R = \frac{1}{2(1+\sin\alpha)}.$$

[You may assume that
$2\sin A\cos B = \sin(A+B)+\sin(A-B)$.]

(iv) Consider the trajectory of the projectile for which the maximum range R is achieved.

Show that, for this trajectory, the initial direction is perpendicular to the direction at which the projectile hits the inclined plane.

2000 Higher School Certificate Solutions

3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS

QUESTION 1

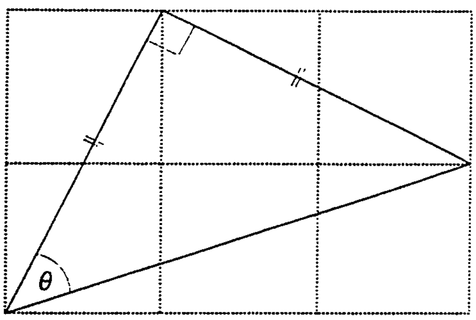
(a) $\frac{d}{dx}x\sin^{-1}x = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$.

(b) $m_1 = 2, m_2 = \frac{1}{3}$.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \frac{2 - \frac{1}{3}}{1 + 2\times\frac{1}{3}} = 1.$$

$\therefore$ The acute angle is $45°$ $\left(\frac{\pi}{4}\text{ radians}\right)$.

Alternative solution:



The sketch shows two lines with gradients 2 and $\frac{1}{3}$. By considering the triangle formed (right-angled isosceles), the angle between the two lines, θ, is $45°$.

(c) **Method 1:**

By the factor theorem, $P(3) = 0$ when $x - 3$ is a factor.

$$P(3) = 27 - 9k + 6 = 0$$
$$9k = 33$$
$$k = 3\tfrac{2}{3}.$$

Method 2:

$$P(x) = (x-3)(x^2 + 3x - 2) = x^3 - 11x + 6.$$

Equating coefficients, $k = \frac{11}{3} = 3\frac{2}{3}$.

Method 3:

$P(x) = x^3 - 3kx + 6$

Let roots be $3, \alpha, \beta$.

$$\alpha + \beta + 3 = 0 \quad ①$$
$$3\alpha + 3\beta + \alpha\beta = -3k \quad ②$$
$$3\alpha\beta = -6 \quad ③$$

From ①: $\alpha + \beta = -3$

From ③: $\alpha\beta = -2$

In ②: $3(\alpha + \beta) + \alpha\beta = -3k$

$$3(-3) - 2 = -3k$$
$$k = \frac{11}{3}.$$

(d) $$\int_0^{\sqrt{3}} \frac{4}{x^2 + 9}\,dx = \left[\frac{4}{3}\tan^{-1}\frac{x}{3}\right]_0^{\sqrt{3}} \text{ (standard integral)}$$
$$= \frac{4}{3}\left(\tan^{-1}\frac{\sqrt{3}}{3} - \tan^{-1}0\right) = \frac{4}{3}\left(\frac{\pi}{6} - 0\right) = \frac{2\pi}{9}.$$

(e) **Method 1:**

$\frac{5}{x+2} \le 1, \quad x \ne -2$

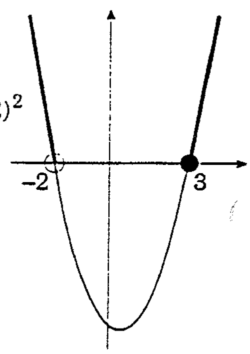
$\therefore \quad 5(x+2) \le (x+2)^2$

$(x+2)^2 - 5(x+2) \ge 0$

$\therefore \quad (x+2)(x+2-5) \ge 0$

$(x+2)(x-3) \ge 0.$

$\therefore \; x < -2 \text{ or } x \ge 3.$



Method 2:

$\frac{5}{x+2} \le 1, \quad x \ne -2$

Consider $\frac{5}{x+2} = 1$

$\therefore \quad 5 = x + 2$

$x = 3.$

$\therefore \; x < -2 \text{ or } x \ge 3.$

Method 3:

When $x + 2 > 0$, that is, $x > -2$,

$5 \le x + 2$,

that is, $x \ge 3$.

When $x + 2 < 0$, that is, $x < -2$,

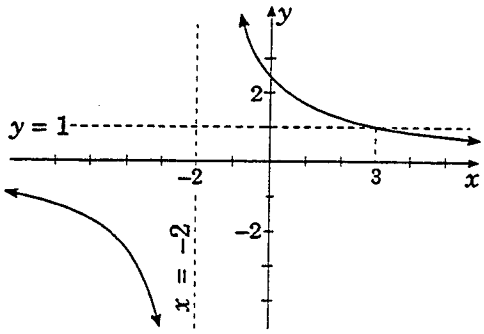
$5 \ge x + 2$,

that is, $x \le 3$. Hence $x < -2$.

$\therefore$ Solution is $x < -2$ or $x \ge 3$.

Method 4:

Graph $y = \dfrac{5}{x+2}$ and $y = 1$.



Point of intersection when $x = 3$ (see Method 2).

From the graph, $x < -2$ or $x \ge 3$.

QUESTION 2

(a) 'O' is repeated 3 times.

The other letters are unique.

$\therefore$ The number of arrangements $= \dfrac{9!}{3!}$

$= 60\,480.$

(b) $\left(5+2x^2\right)^7 = \sum_{r=0}^{7} {}^7C_r 5^{7-r}\left(2x^2\right)^r.$

For a term in x^6 we require $2r = 6$, $\therefore r = 3$.

$\therefore$ The coefficient of x^6 is ${}^7C_3\, 5^{7-3}\, 2^3 = 175\,000.$

(c) $\cos 2\theta = \sin\theta,$

$$\begin{aligned}\therefore\quad 1-2\sin^2\theta &= \sin\theta \\ 2\sin^2\theta+\sin\theta-1 &= 0 \\ (2\sin\theta-1)(\sin\theta+1) &= 0 \\ \sin\theta &= \frac{1}{2} \text{ or } -1.\end{aligned}$$

$\therefore\ \theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}.$

(d) $u = 2+x,\ x = u-2,$ $\qquad \therefore \dfrac{dx}{du} = 1.$

$$\begin{aligned}\int \frac{x}{\sqrt{2+x}}\,dx &= \int \frac{u-2}{u^{\frac{1}{2}}}\,du \\ &= \int\left(u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}\right)du \\ &= \frac{2u^{\frac{3}{2}}}{3} - 4u^{\frac{1}{2}} + C \\ &= \frac{2}{3}(2+x)^{\frac{3}{2}} - 4\sqrt{2+x} + C \\ &= \frac{2}{3}\sqrt{2+x}\left(2+x-\frac{3}{2}\times 4\right)+C \\ &= \frac{2}{3}\sqrt{2+x}\,(x-4)+C.\end{aligned}$$

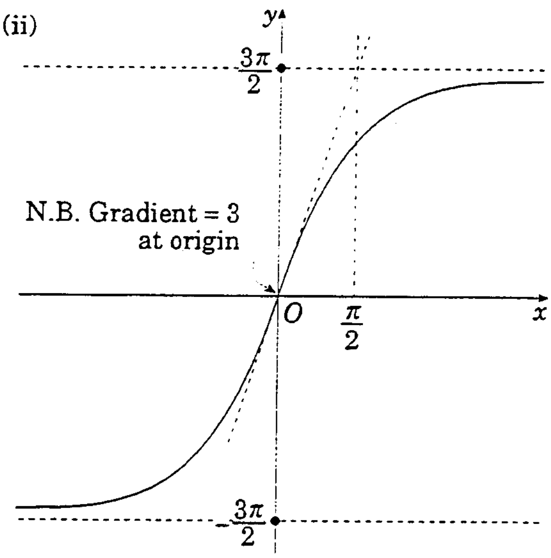
QUESTION 3

(a)
$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \\ &= \lim_{h\to 0}\frac{(a+h)^3-a^3}{h} \\ &= \lim_{h\to 0}\frac{a^3+3a^2h+3ah^2+h^3-a^3}{h} \\ &= \lim_{h\to 0}\frac{h\left(3a^2+3ah+h^2\right)}{h} \\ &= \lim_{h\to 0}\left(3a^2+3ah+h^2\right) \\ &= 3a^2. \end{aligned}$$

(b) $f(x) = 3\tan^{-1}x.$

(i) Range is $-\dfrac{3\pi}{2} \le y \le \dfrac{3\pi}{2}.$

(ii)



(iii) $f'(x) = \dfrac{3}{1+x^2}$

$f'\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{3}{1+\frac{1}{3}} = 2\frac{1}{4}.$

Therefore the gradient of the tangent at $x = \dfrac{1}{\sqrt{3}}$ is $2\frac{1}{4}$.

(c) (i) From ΔOTB, $\dfrac{h}{OB} = \tan 30° = \dfrac{1}{\sqrt{3}},$

$\therefore\ OB = \sqrt{3}h.$

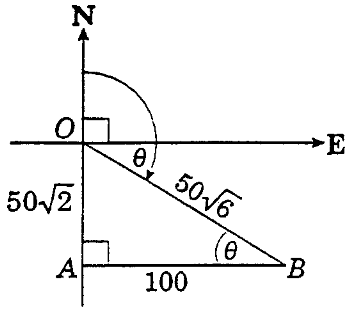
(Other answers are possible, such as $OB = \sqrt{100^2+h^2}$.)

(ii) From ΔOAT, $\dfrac{OA}{h} = \tan 45° = 1,$

$\therefore\ OA = h.$

From ΔOAB,
$$\begin{aligned} OB^2 &= AO^2 + AB^2 \\ 3h^2 &= h^2 + 100^2 \\ 2h^2 &= 100^2 \\ \therefore\quad h &= 50\sqrt{2}. \end{aligned}$$

(iii)



$$\sin\theta = \frac{OA}{OB} = \frac{50\sqrt{2}}{50\sqrt{6}} = \frac{1}{\sqrt{3}}.$$

$\therefore \quad \theta = 35°\,16'.$

Therefore the bearing is $125°\,16'$.

QUESTION 4

(a) Let $S(n)$ be the statement

$$1+3+6+\cdots+\tfrac{1}{2}n(n+1) = \tfrac{1}{6}n(n+1)(n+2).$$

When $n = 1$, LHS $= 1$

RHS $= \frac{1}{6}\times 1\times 2\times 3 = 1.$

$\therefore$ LHS $=$ RHS.

$\therefore$ $S(1)$ is true.

Assume $S(k)$ is true. That is,

$$1+3+6+\cdots+\tfrac{1}{2}k(k+1) = \tfrac{1}{6}k(k+1)(k+2).$$

Now

$$1+3+6+\cdots+\tfrac{1}{2}k(k+1)+\tfrac{1}{2}(k+1)(k+1+1)$$

$$= \tfrac{1}{6}k(k+1)(k+2)+\tfrac{1}{2}(k+1)(k+1+1) \quad \text{(by assumption)}$$

$$= \frac{(k+1)(k+2)}{6}(k+3)$$

$$= \tfrac{1}{6}(k+1)(k+1+1)(k+1+2).$$

This is $S(k+1)$.

$\therefore$ If $S(k)$ is true, then $S(k+1)$ is true.

But $S(1)$ is true, hence $S(2)$ is true, hence $S(3)$ is true and so on.

By the principle of mathematical induction, the result is true for all positive integral values of n.

(b) Let

$$f(r) = (1+r)\left[(1+r)^{24}-1\right]-50r$$
$$= (1+r)^{25}-(1+r)-50r$$
$$= (1+r)^{25}-1-51r.$$

$$f'(r) = 25(1+r)^{24}-51$$

$$r_1 = 0{\cdot}06,$$

then

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$

$$= 0{\cdot}06 - \frac{(1{\cdot}06)^{25}-1-51\times 0{\cdot}06}{25(1{\cdot}06)^{24}-51}$$

$$= 0{\cdot}055\,38\ldots$$

$$= 0{\cdot}055.$$

(c) $P(x) = x^3+px^2+qx+r.$

(i) Sum of roots $= -\frac{b}{a}.$

$\therefore \quad \sqrt{k}+\left(-\sqrt{k}\right)+\alpha = -p$

$\therefore \quad \alpha + p = 0.$

(ii) Product of roots $= -\frac{d}{a}.$

$\therefore \quad \sqrt{k}\times\left(-\sqrt{k}\right)\times\alpha = -r$

$\therefore \quad -k\alpha = -r$

$k\alpha = r.$

(iii) Product of roots in pairs $= \frac{c}{a}.$

$\therefore \quad \sqrt{k}\times\left(-\sqrt{k}\right)+\sqrt{k}\times\alpha+\left(-\sqrt{k}\right)\times\alpha = q$

$\therefore \quad -k = q.$

Since $\alpha = -p$, from (i), then $k\alpha = r$ from (ii) becomes $(-q)\times(-p) = r$, that is, $pq = r$.

(d) Period $T = \frac{2\pi}{n} = 4\pi,$

$\therefore \ n = \frac{1}{2}.$

Amplitude $a = 3$.

Since $v^2 = n^2\left(a^2-x^2\right)$ (putting origin at O)

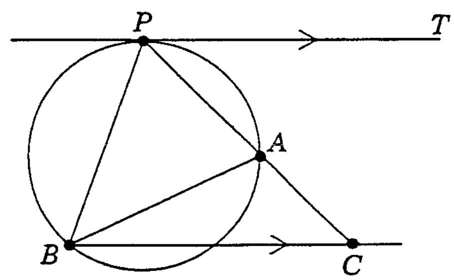
$$v^2 = \tfrac{1}{4}\left(9-x^2\right).$$

When $x=0$, at O, $v^2 = \frac{9}{4}.$

$\therefore$ Speed $= \frac{3}{2}$ cm s^{-1}.

QUESTION 5

(a)



(i) $\angle PBA = \angle TPA$ ($\angle$ between tangent and chord equals $\angle$ in alternate segment).

$\angle TPA = \angle PCB$ (alternate $\angle$s, $PT \parallel BC$),

$\therefore \ \angle PBA = \angle PCB.$

(ii) In ΔPBA and ΔPCB,

$\angle APB = \angle BPC$ (common)

$\angle PBA = \angle PCB$, from (i).

Therefore the triangles are equiangular, and hence similar.

$\therefore \quad \dfrac{PB}{PC} = \dfrac{PA}{PB}$ (corresponding sides are in the same ratio)

$\therefore \quad PB^2 = PA \times PC.$

(b) $f(x) = \dfrac{x}{x+2}$, defined for all $x \neq -2$.

(i) $f'(x) = \dfrac{(x+2)\times 1 - x\times 1}{(x+2)^2}$

$= \dfrac{2}{(x+2)^2} > 0$ for all $x \neq -2$,

since $(x+2)^2 > 0$ for all $x \neq -2$.

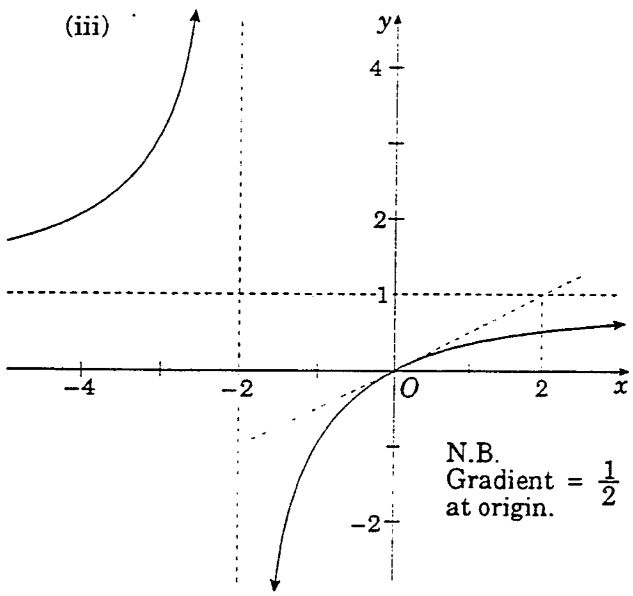
(ii) $f(x) = \dfrac{x+2-2}{x+2}$

$= 1 - \dfrac{2}{x+2}.$

As $x \to \pm\infty$, $\dfrac{2}{x+2} \to 0$.

Therefore the horizontal asymptote is $y = 1$.

(iii)



(iv) $f(x)$ is a one-to-one increasing function (it satisfies the horizontal-line test).

(v) $y = \dfrac{x}{x+2}$.

$\therefore$ The inverse is $x = \dfrac{y}{y+2}$.

$\therefore \quad xy + 2x = y$

$2x = y(1-x)$

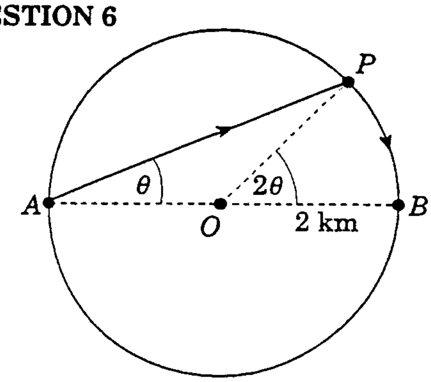
$y = \dfrac{2x}{1-x}$

$\therefore \quad f^{-1}(x) = \dfrac{2x}{1-x}.$

(vi) Domain of $f^{-1}(x)$ is the range of $f(x)$. That is, all real x, $x \neq 1$.

QUESTION 6

(a)



(i) $\angle APB = \dfrac{\pi}{2}$ ($\angle$ in a semicircle).

$\therefore \dfrac{AP}{AB} = \cos\theta$

$\therefore \quad AP = 4\cos\theta.$

$\text{Arc } PB = 2.2\theta = 4\theta.$

Time from A to B

= time for AP + time for PB.

That is, $\quad T = \dfrac{AP}{3} + \dfrac{PB}{4}$

$= \dfrac{4\cos\theta}{3} + \theta$

$= \dfrac{1}{3}(4\cos\theta + 3\theta).$

(ii) $\dfrac{dT}{d\theta} = \dfrac{1}{3}(-4\sin\theta + 3)$

$= 0$ when $\sin\theta = \dfrac{3}{4}$.

$\therefore \quad \theta = \sin^{-1}\left(\dfrac{3}{4}\right)$

$= 0{\cdot}848$ radians $(= 48°\,35')$.

(iii) $\dfrac{d^2T}{d\theta^2} = \dfrac{1}{3}(-4\cos\theta) < 0$ for θ acute.

Therefore this is a maximum stationary point, and so not the minimum.

Test the end points.

That is, row direct to B.

Time $= \dfrac{4}{3} = 1\dfrac{1}{3}$ hours.

Walk round the lake from A to B.

Time $= \dfrac{2\pi}{4} = 1{\cdot}57$ hours.

$\therefore$ Pat should row directly across the lake to B to minimise the time.

(b) (i) No. of ways for 2 spades and 4 clubs

$= {}^{13}C_2 \times {}^{13}C_4 = 55\,770.$

(ii) No. of ways with 5 of the same suit

$= 4 \times {}^{13}C_5 \times {}^{39}C_1 = 200\,772.$

No. of ways with 6 of the same suit

$= 4 \times {}^{13}C_6 = 6864.$

$\therefore$ Total number with at least 5 of the same suit $= 200\,772 + 6864 = 207\,636.$

QUESTION 7

(a) (i) $F = Au^3 + \frac{B}{u}$.

The maximum period of flight implies the minimum fuel used per hour.

$$\frac{dF}{du} = 3Au^2 - \frac{B}{u^2}.$$

$$\frac{d^2F}{du^2} = 6Au + \frac{2B}{u^3} > 0,$$

since $u > 0$ and A and B are positive.

Therefore we have a minimum when

$$\frac{dF}{du} = 0,$$

that is, when $3Au^2 = \frac{B}{u^2}$

$$u^4 = \frac{B}{3A}$$

$$u = \left(\frac{B}{3A}\right)^{\frac{1}{4}}.$$

(ii) Let the distance be s and the time t.

$$s = ut$$

$$= u\frac{k}{F} \quad \text{(where } k \text{ is a positive constant, the amount of fuel)}$$

$$= \frac{uk}{Au^3 + \frac{B}{u}}$$

$$= \frac{u^2k}{Au^4 + B}.$$

$$\frac{ds}{du} = \frac{2uk(Au^4 + B) - u^2k.4Au^3}{(Au^4 + B)^2}$$

$$= \frac{2Aku^5 + 2Bku - 4Aku^5}{(Au^4 + B)^2}$$

$$= \frac{2Bku - 2Aku^5}{(Au^4 + B)^2}$$

$$= \frac{2ku(B - Au^4)}{(Au^4 + B)^2}.$$

For a maximum s, $\frac{ds}{du} = 0$.

$\therefore\ u = 0$ or $u^4 = \frac{B}{A}$.

When $u = 0$, s is obviously a minimum value,

$\therefore$ maximum occurs when $u = \left(\frac{B}{A}\right)^{\frac{1}{4}}$.

$$\frac{\text{New speed}}{\text{Old speed}} = \frac{\left(\frac{B}{A}\right)^{\frac{1}{4}}}{\left(\frac{B}{3A}\right)^{\frac{1}{4}}}$$

$$= 3^{\frac{1}{4}}$$

$$= 1{\cdot}3162$$

$$= 132\% \text{ (nearest per cent).}$$

Therefore the speed for maximum distance is approximately 32% faster than the speed for maximum time.

(b) (i) Given $x = Vt\cos\theta$

$$\therefore\ t = \frac{x}{V\cos\theta}$$

$$y = Vt\sin\theta - \frac{1}{2}gt^2$$

$$= \frac{V.x\sin\theta}{V\cos\theta} - \frac{1}{2}g\frac{x^2}{V^2\cos^2\theta}$$

$$= x\tan\theta - x^2\sec^2\theta\left(\frac{g}{2V^2}\right)$$

$$= x\tan\theta - x^2\sec^2\theta, \text{ since } \frac{2V^2}{g} = 1.$$

(ii) On the inclined plane,

$$x = r\cos\alpha,\ y = r\sin\alpha.$$

$$\therefore\ r\sin\alpha = r\cos\alpha\tan\theta - \frac{r^2\cos^2\alpha}{\cos^2\theta}.$$

$\sin\alpha\cos^2\theta$

$= \cos\alpha\sin\theta\cos\theta - r\cos^2\alpha$ (since $r \neq 0$)

$\therefore\ r\cos^2\alpha = \cos\theta(\sin\alpha\cos\theta - \cos\alpha\sin\theta)$

$$\therefore\ r = \frac{\cos\theta.\sin(\theta - \alpha)}{\cos^2\alpha}.$$

(iii) Using the given identity with $A = \theta - \alpha$ and $B = \theta$, we have

$$r = \frac{\sin(\theta - \alpha)\cos\theta}{\cos^2\alpha}$$

$$= \frac{\sin(2\theta - \alpha) + \sin(-\alpha)}{2\cos^2\alpha}.$$

This will be a maximum when $\sin(2\theta - \alpha) = 1$.

(That is, when $2\theta - \alpha = \frac{\pi}{2}$.)

So the maximum range is

$$R = \frac{1 + \sin(-\alpha)}{2\cos^2\alpha}$$

$$= \frac{1 - \sin\alpha}{2(1 - \sin^2\alpha)}$$

$$= \frac{1 - \sin\alpha}{2(1 - \sin\alpha)(1 + \sin\alpha)}$$

$$= \frac{1}{2(1 + \sin\alpha)}.$$

(iv) **Method 1:**

Let m_O and m_T be the slopes of the tangents at O and T respectively.

$$y = x\tan\theta - x^2\sec^2\theta$$

$$y' = \tan\theta - 2x\sec^2\theta.$$

At O, $x = 0$, so $m_O = \tan\theta$.

Equation of OT is $y = x\tan\alpha$,

so T satisfies $x\tan\alpha = x\tan\theta - x^2\sec^2\theta$.

$\therefore\ x\sec^2\theta = \tan\theta - \tan\alpha$ ($x \neq 0$ at T)

$$x = \frac{\tan\theta - \tan\alpha}{\sec^2\theta}.$$

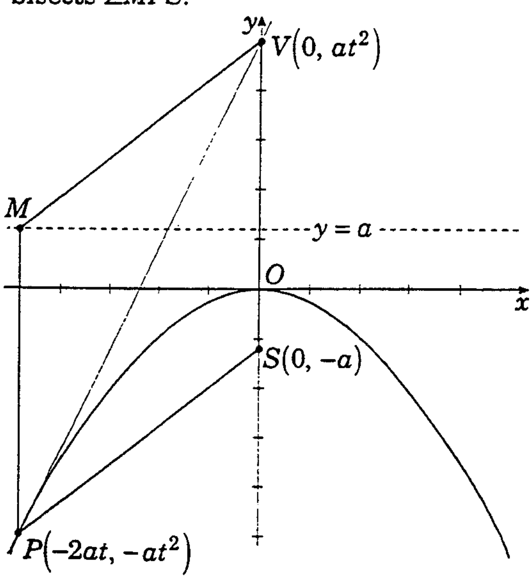
$$\begin{aligned}\therefore \quad m_T &= \tan\theta - 2\left(\frac{\tan\theta-\tan\alpha}{\sec^2\theta}\right).\sec^2\theta \\ &= 2\tan\alpha - \tan\theta \\ &= 2\tan\left(2\theta-\frac{\pi}{2}\right)-\tan\theta \quad \text{(from (iii))} \\ &= -2\cot 2\theta - \tan\theta \\ &= -2\left(\frac{1-\tan^2\theta}{2\tan\theta}\right)-\tan\theta \\ &= -\frac{1}{\tan\theta}+\tan\theta-\tan\theta \\ &= -\frac{1}{\tan\theta}.\end{aligned}$$

$$\begin{aligned} m_O m_T &= \tan\theta\times\left(-\frac{1}{\tan\theta}\right) \\ &= -1.\end{aligned}$$

Therefore the tangents are perpendicular.

Method 2:
The endpoints of a focal chord have tangents which are perpendicular, so it is enough to show that for maximum range OT is a focal chord.

Method 2a (focal chord from geometry):
The following diagram shows a general parabola $x^2 = -4ay$ with focus $S(0, -a)$.
The tangent at P $\left(y = tx + at^2\right)$ intersects the y axis at $V\left(0, at^2\right)$.
Therefore $PM = SV$ $\left(= at^2 + a\right)$.
Also, $PM = PS$ by definition of a parabola.
Therefore $PMVS$ is a rhombus, and PV bisects $\angle MPS$.



This only works because PS is a focal chord. For any other point S on the y axis, $\angle VPS$ would be different.
So if a tangent bisects the angle between the vertical and a chord, that chord must be a focal chord.

In part (iii) we showed that the maximum range occurs when

$$\begin{aligned} 2\theta-\alpha &= \frac{\pi}{2}. \\ \therefore \quad \theta-\alpha &= \frac{\pi}{2}-\theta.\end{aligned}$$

This means that the tangent at O (with angle θ) bisects the vertical (with angle $\frac{\pi}{2}$) and OT (with angle α).
Hence OT is a focal chord as required.

Method 2b (focal chord using algebra):

$$\begin{aligned} y &= x\tan\theta - x^2\sec^2\theta \\ \therefore \quad \cos^2\theta.y &= x\sin\theta\cos\theta - x^2.\end{aligned}$$

$$\begin{aligned}\left(x-\frac{\sin\theta\cos\theta}{2}\right)^2 &= \frac{\sin^2\theta\cos^2\theta}{4}-\cos^2\theta.y \\ &= -\cos^2\theta\left(y-\frac{\sin^2\theta}{4}\right).\end{aligned}$$

Focal length $= \dfrac{\cos^2\theta}{4}$.

Vertex $= \left(\dfrac{\sin\theta\cos\theta}{2}, \dfrac{\sin^2\theta}{4}\right)$.

Focus $F = \left(\dfrac{\sin\theta\cos\theta}{2}, \dfrac{\sin^2\theta-\cos^2\theta}{4}\right)$.

$$\begin{aligned}\text{Slope of } OF &= \frac{\sin^2\theta-\cos^2\theta}{2\sin\theta\cos\theta} \\ &= \frac{1}{2}\left(\tan\theta-\frac{1}{\tan\theta}\right).\end{aligned}$$

Now for maximum range, $2\theta-\alpha = \frac{\pi}{2}$.

$$\begin{aligned}\tan 2\theta &= \tan\left(\alpha+\frac{\pi}{2}\right) \\ &= -\frac{1}{\tan\alpha} \\ \therefore \quad \tan\alpha &= -\frac{1}{\tan 2\theta} \\ &= -\frac{1-\tan^2\theta}{2\tan\theta} \\ &= \frac{1}{2}\left(\tan\theta-\frac{1}{\tan\theta}\right).\end{aligned}$$

Therefore the slope of OF equals the slope of OT, and so OT is a focal chord, as required.

END OF 3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS SOLUTIONS