BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2016 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–14

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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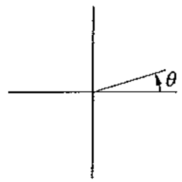
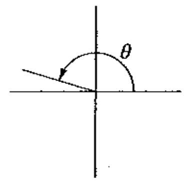
1 For the angle θ, $\sin\theta = \frac{7}{25}$ and $\cos\theta = -\frac{24}{25}$.

Which diagram best shows the angle θ?

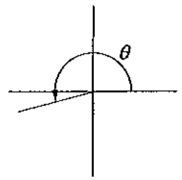
(A)



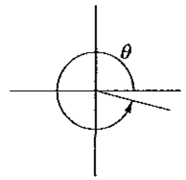
(B)



(C)



(D)



2 In a raffle, 30 tickets are sold and there is one prize to be won.

What is the probability that someone buying 6 tickets wins the prize?

(A) $\frac{1}{30}$

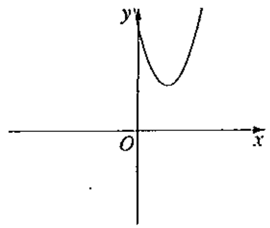
(B) $\frac{1}{6}$

(C) $\frac{1}{5}$

(D) $\frac{1}{4}$

3 Which diagram best shows the graph of the parabola $y = 3 - (x - 2)^2$?

(A)



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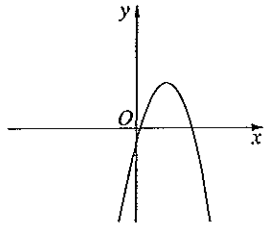
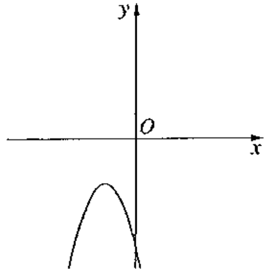
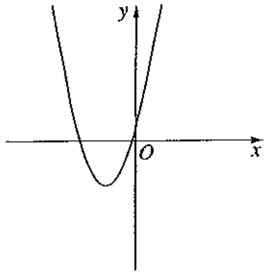


(C)

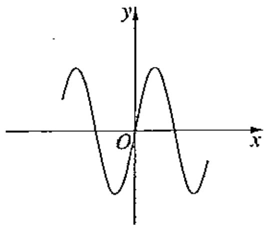


(D)

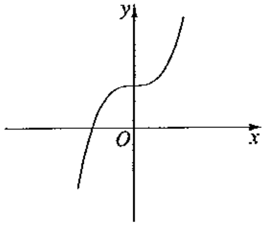


4 Which diagram shows the graph of an odd function?

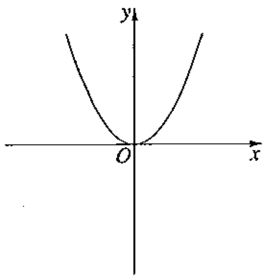
(A)



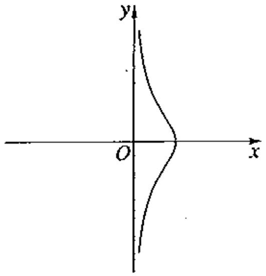
(B)



(C)



(D)



5 What is the derivative of $\ln(\cos x)$?

(A) $-\sec x$

(B) $-\tan x$

(C) $\sec x$

(D) $\tan x$

6 What is the period of the function $f(x) = \tan(3x)$?

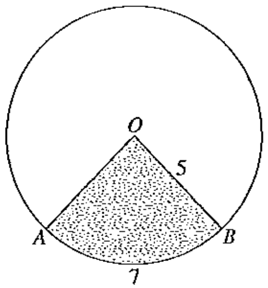
(A) $\dfrac{\pi}{3}$

(B) $\dfrac{2\pi}{3}$

(C) 3π

(D) 6π

7 The circle centred at O has radius 5. Arc AB has length 7 as shown in the diagram.



What is the area of the shaded sector OAB?

(A) $\dfrac{35}{2}$

(B) $\dfrac{35}{2}\pi$

(C) $\dfrac{125}{14}$

(D) $\dfrac{125}{14}\pi$

8 How many solutions does the equation $|\cos(2x)| = 1$ have for $0 \le x \le 2\pi$?

(A) 1

(B) 3

(C) 4

(D) 5

9 What is the value of $\int_{-3}^{2} |x+1|\,dx$?

(A) $\frac{5}{2}$

(B) $\frac{11}{2}$

(C) $\frac{13}{2}$

(D) $\frac{17}{2}$

10 Which expression is equivalent to $4 + \log_2 x$?

(A) $\log_2(2x)$

(B) $\log_2(16+x)$

(C) $4\log_2(2x)$

(D) $\log_2(16x)$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Sketch the graph of $(x-3)^2 + (y+2)^2 = 4$. **2**

(b) Differentiate $\dfrac{x+2}{3x-4}$. **2**

(c) Solve $|x-2| \le 3$. **2**

(d) Evaluate $\int_0^1 (2x+1)^3\,dx$. **2**

(e) Find the points of intersection of $y = -5 - 4x$ and $y = 3 - 2x - x^2$. **3**

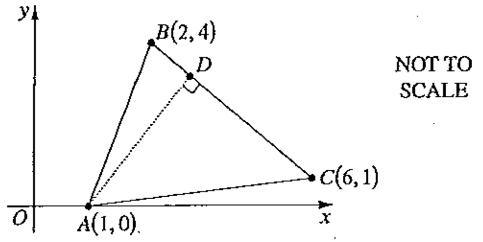
(f) Find the gradient of the tangent to the curve $y = \tan x$ at the point where $x = \frac{\pi}{8}$. **2**

Give your answer correct to 3 significant figures.

(g) Solve $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$ for $0 \le x \le 2\pi$. **2**

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) The diagram shows points $A(1, 0)$, $B(2, 4)$ and $C(6, 1)$. The point D lies on BC such that $AD \perp BC$.

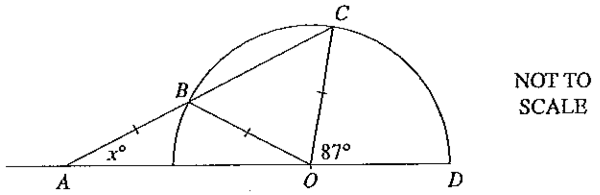


(i) Show that the equation of BC is $3x + 4y - 22 = 0$. **2**

(ii) Find the length of AD. **2**

(iii) Hence, or otherwise, find the area of $\triangle ABC$. **2**

(b) The diagram shows a semicircle with centre O. It is given that $AB = OB$, $\angle COD = 87°$ and $\angle BAO = x°$.



(i) Show that $\angle CBO = 2x°$, giving reasons. **1**

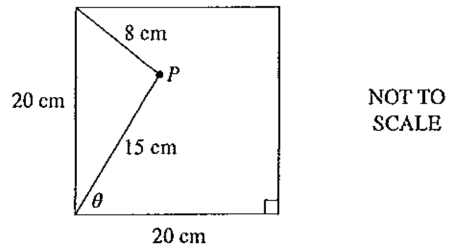
(ii) Find the value of x, giving reasons. **2**

Question 12 continues on page 8

(c) Square tiles of side length 20 cm are being used to tile a bathroom. **3**

The tiler needs to drill a hole in one of the tiles at a point P which is 8 cm from one corner and 15 cm from an adjacent corner.

To locate the point P the tiler needs to know the size of the angle θ shown in the diagram.



Find the size of the angle θ to the nearest degree.

(d) (i) Differentiate $y = xe^{3x}$. **1**

(ii) Hence find the exact value of $\int_0^2 e^{3x}(3+9x)\,dx$. **2**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) Consider the function $y = 4x^3 - x^4$.

(i) Find the two stationary points and determine their nature. **4**

(ii) Sketch the graph of the function, clearly showing the stationary points and the x and y intercepts. **2**

(b) Consider the parabola $x^2 - 4x = 12y + 8$.

(i) By completing the square, or otherwise, find the focal length of the parabola. **2**

(ii) Find the coordinates of the focus. **1**

(c) A radioactive isotope of Curium has a half-life of 163 days. Initially there are 10 mg of Curium in a container.

The mass $M(t)$ in milligrams of Curium, after t days, is given by

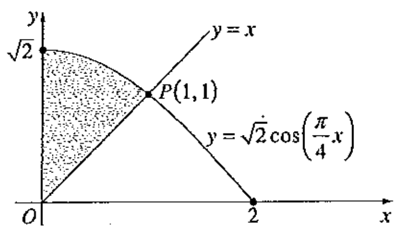
$$M(t) = Ae^{-kt},$$

where A and k are constants.

(i) State the value of A. **1**

(ii) Given that after 163 days only 5 mg of Curium remain, find the value of k. **2**

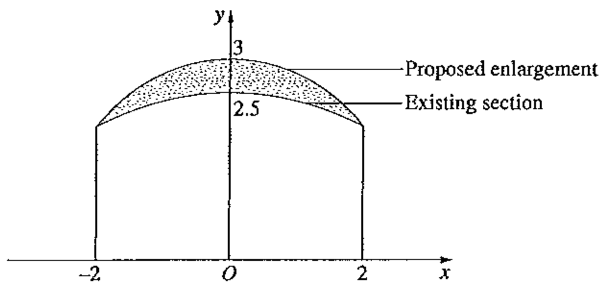
(d) The curve $y = \sqrt{2}\cos\left(\frac{\pi}{4}x\right)$ meets the line $y = x$ at $P(1, 1)$, as shown in the diagram. **3**



Find the exact value of the shaded area.

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) The diagram shows the cross-section of a tunnel and a proposed enlargement. **3**



The heights, in metres, of the existing section at 1 metre intervals are shown in Table A.

Table A: Existing heights

x	–2	–1	0	1	2
y	2	2.38	2.5	2.38	2

The heights, in metres, of the proposed enlargement are shown in Table B.

Table B: Proposed heights

x	–2	–1	0	1	2
y	2	2.78	3	2.78	2

Use Simpson's rule with the measurements given to calculate the approximate increase in area.

Question 14 continues on page 11

(b) A gardener develops an eco-friendly spray that will kill harmful insects on fruit trees without contaminating the fruit. A trial is to be conducted with 100 000 insects. The gardener expects the spray to kill 35% of the insects each day and that exactly 5000 new insects will be produced each day.

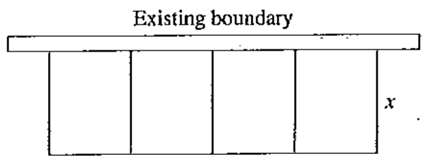
The number of insects expected at the end of the nth day of the trial is A_n.

(i) Show that $A_2 = 0.65(0.65 \times 100\,000 + 5000) + 5000$. **2**

(ii) Show that $A_n = 0.65^n \times 100\,000 + 5000\dfrac{\left(1 - 0.65^n\right)}{0.35}$. **1**

(iii) Find the expected insect population at the end of the fourteenth day, correct to the nearest 100. **1**

(c) A farmer wishes to make a rectangular enclosure of area 720 m^2. She uses an existing straight boundary as one side of the enclosure. She uses wire fencing for the remaining three sides and also to divide the enclosure into four equal rectangular areas of width x m as shown.



(i) Show that the total length, ℓ m, of the wire fencing is given by **1**

$$\ell = 5x + \frac{720}{x}.$$

(ii) Find the minimum length of wire fencing required, showing why this is the minimum length. **3**

(d) By summing the geometric series $1 + x + x^2 + x^3 + x^4$, or otherwise, **2**

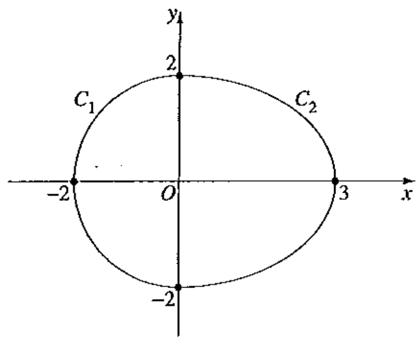
find $\lim\limits_{x \to 1} \dfrac{x^5 - 1}{x - 1}$.

(e) Write $\log 2 + \log 4 + \log 8 + \cdots + \log 512$ in the form $a \log b$ where a and b are integers greater than 1. **2**

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) The diagram shows two curves C_1 and C_2. The curve C_1 is the semicircle $x^2 + y^2 = 4$, $-2 \le x \le 0$. The curve C_2 has equation $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$, $0 \le x \le 3$. **4**



An egg is modelled by rotating the curves about the x-axis to form a solid of revolution.

Find the exact value of the volume of the solid of revolution.

(b) An eight-sided die is marked with numbers 1, 2, …, 8. A game is played by rolling the die until an 8 appears on the uppermost face. At this point the game ends.

(i) Using a tree diagram, or otherwise, explain why the probability of the game ending before the fourth roll is **2**

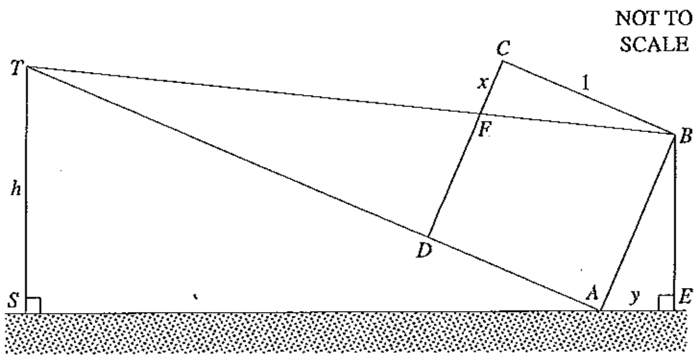
$$\frac{1}{8} + \frac{7}{8} \times \frac{1}{8} + \left(\frac{7}{8}\right)^2 \times \frac{1}{8}.$$

(ii) What is the smallest value of n for which the probability of the game ending before the nth roll is more than $\dfrac{3}{4}$? **3**

Question 15 continues on page 13

(c) Maryam wishes to estimate the height, h metres, of a tower, ST, using a square, $ABCD$, with side length 1 metre.

She places the point A on the horizontal ground and ensures that the point D lies on the line joining A to the top of the tower T. The point F is the intersection of the line joining B and T and the side CD. The point E is the foot of the perpendicular from B to the ground. Let CF have length x metres and AE have length y metres.



Copy or trace the diagram into your writing booklet.

(i) Show that $\triangle FCB$ and $\triangle BAT$ are similar. **2**

(ii) Show that $\triangle TSA$ and $\triangle AEB$ are similar. **2**

(iii) Find h in terms of x and y. **2**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet.

(a) A particle moves in a straight line. Its velocity v m s^{-1} at time t seconds is given by

$$v = 2 - \frac{4}{t+1}.$$

(i) Find the initial velocity. **1**

(ii) Find the acceleration of the particle when the particle is stationary. **2**

(iii) By considering the behaviour of v for large t, sketch a graph of v against t for $t \geq 0$, showing any intercepts. **2**

(iv) Find the exact distance travelled by the particle in the first 7 seconds. **3**

(b) Some yabbies are introduced into a small dam. The size of the population, y, of yabbies can be modelled by the function

$$y = \frac{200}{1+19e^{-0.5t}},$$

where t is the time in months after the yabbies are introduced into the dam.

(i) Show that the rate of growth of the size of the population is **2**

$$\frac{1900e^{-0.5t}}{\left(1+19e^{-0.5t}\right)^2}.$$

(ii) Find the range of the function y, justifying your answer. **2**

(iii) Show that the rate of growth of the size of the population can be rewritten as **1**

$$\frac{y}{400}(200-y).$$

(iv) Hence, find the size of the population when it is growing at its fastest rate. **2**

End of paper

2016 Higher School Certificate Solutions Mathematics

Section I

Multiple Choice Summary

1 B	2 C	3 B	4 A	5 B
6 A	7 A	8 D	9 C	10 D

Note:
* indicates that the information can be found on the Reference Sheet

Multiple Choice Solutions

1. B

If $\sin\theta > 0$ and $\cos\theta < 0$ then θ lies in the second quadrant.

2. C

$P(\text{win}) = \frac{6}{30} = \frac{1}{5}$

3. B

Rearranging $y = 3 - (x-2)^2$

$\therefore\ y - 3 = -(x-2)^2$

The parabola has a vertex at $(2,3)$ and is concave down.

4. A

An odd function has point symmetry about the origin.
i.e. the graph remains unchanged upon a rotation of 180° about the origin.

5. B

$$\frac{d}{dx}\left[\ln(\cos x)\right] = \frac{\frac{d}{dx}(\cos x)}{\cos x} = \frac{-\sin x}{\cos x} = -\tan x$$

6. A

The period of $y = \tan nx$ is $\frac{\pi}{n}$.
For $y = \tan 3x$, $n = 3$

$\therefore$ period $= \frac{\pi}{3}$

7. A

$l = r\theta$ and $r = 5$, $l = 7$

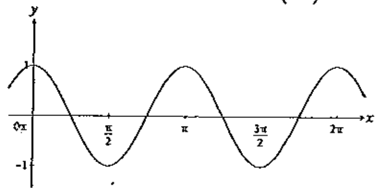
$7 = 5\theta \Rightarrow \theta = \frac{7}{5}$

$$A_{\text{sector}} = \tfrac{1}{2}r^2\theta = \tfrac{1}{2}(5)^2\left(\tfrac{7}{5}\right) = \tfrac{35}{2}$$

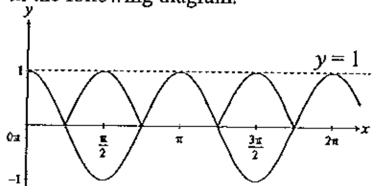
8. D

Method 1

Consider the graph of $y = \cos(2x)$



The graph of $y = |\cos(2x)|$ differs only in that all parts below the x-axis are reflected in the x-axis and as illustrated in the following diagram.



In the domain $0 \le x \le 2\pi$, this graph at intersects $y = 1$ on 5 occasions namely $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

$\therefore$ there are 5 solutions

Method 2

$|\cos(2x)| = 1$

$\therefore\ \cos(2x) = \pm 1$

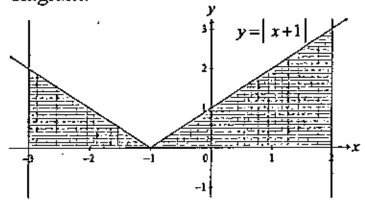
$2x = 0, \pi, 2\pi, 3\pi, 4\pi$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

$\therefore$ there are 5 solutions

9. C

$y = |x+1|$ is illustrated in the diagram.



$\therefore \int_{-3}^{2} |x+1|\,dx$ = area of the two shaded triangles

$$\therefore \int_{-3}^{2} |x+1|\,dx = \frac{1}{2}\times 2\times 2 + \frac{1}{2}\times 3\times 3 = \frac{13}{2}$$

10. D

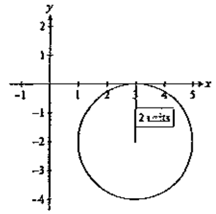
Since $\log_2 2 = 1$

$$4 + \log_2 x = 4\log_2 2 + \log_2 x = \log_2 2^4 + \log_2 x = \log_2 16 + \log_2 x = \log_2(16x)$$

Section II

Question 11

(a) $(x-3)^2 + (y+2)^2 = 4$ represents a circle with centre $(3,-2)$ and radius 2 units.
This is illustrated in the diagram on the right.



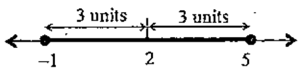
(b) Using the quotient rule*: Let $u = x+2$, $v = 3x-4$

then $\frac{du}{dx} = 1$, $\frac{dv}{dx} = 3$

$$\frac{d}{dx}\left(\frac{x+2}{3x-4}\right) = \frac{(3x-4)(1)-(x+2)(3)}{(3x-4)^2} = \frac{3x-4-3x-6}{(3x-4)^2} = -\frac{10}{(3x-4)^2}$$

(c) **Method 1**

$|x-2| \le 3$ means that the distance between x and 2 is 3 units or less.



$\therefore\ -1 \le x \le 5$

Method 2

$|x-2| \le 3$

$\therefore -3 \le x-2 \le 3$

$-3+2 \le x \le 2+2$

$-1 \le x \le 5$

(d) $\int_0^1 (2x+1)^3\,dx = \left[\frac{(2x+1)^4}{2\times 4}\right]_0^1$

$= \frac{3^4}{8} - \frac{1^4}{8}$

$= 10$

(e) Equating the two functions:

$-5-4x = 3-2x-x^2$

$\therefore\ x^2-2x-8=0$

$(x-4)(x+2)=0$

$x=4,-2$

Substituting into $y=-5-4x$:

If $x=4: y=-5-4(4)=-21 \Rightarrow (4,-21)$

If $x=-2: y=-5-4(-2)=3 \Rightarrow (-2,3)$

$\therefore$ $(4,-21)$ and $(-2,3)$ are the points of intersection

(f) $y=\tan x$ $\quad \therefore \frac{dy}{dx} = \sec^2 x$

At $x=\frac{\pi}{8}$: $\quad \frac{dy}{dx} = \sec^2\frac{\pi}{8} \doteqdot 1.1715...$

Note: $\sec^2\frac{\pi}{8} = \frac{1}{\cos^2\frac{\pi}{8}} = \frac{1}{\left(\cos\frac{\pi}{8}\right)^2}$

$\therefore$ the gradient is 1.17 (3 sig. fig.)

(g) For $\sin\frac{x}{2} = \frac{1}{2}$ if $0 \le x \le 2\pi \Rightarrow 0 \le \frac{x}{2} \le \pi$

$\therefore\ \frac{x}{2} = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$\frac{x}{2} = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore\ x = \frac{\pi}{3}, \frac{5\pi}{3}$

Question 12

(a) (i) $B(2,4)$ and $C(6,1)$

$m_{BC} = \frac{1-4}{6-2} = -\frac{3}{4}$

Using the point-gradient formula*:

$\therefore\ y-4 = -\frac{3}{4}(x-2)$

$4y-16 = -3x+6$

$\therefore\ 3x+4y-22=0$

(ii) $A(1,0)=(x_1,y_1)$ and $ax_1+by_1+c = 3x+4y-22$ and using $d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$*

$AD = \frac{|3\times 1+4\times 0-22|}{\sqrt{3^2+4^2}}$

$= \frac{19}{5}$

$\therefore\ AD = 3.8$ units

(iii) Using the distance formula*: $BC = \sqrt{(6-2)^2+(1-4)^2} = 5$

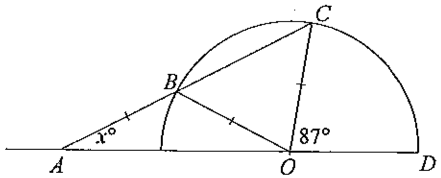
Area $= \frac{1}{2}\times AD\times BC$

$= \frac{1}{2}\times\frac{19}{5}\times 5$

$= 9.5$

$\therefore$ the area is 9.5 unit^2

(b)



NOT TO SCALE

(i) $\angle BOA = x°$ [base angles of isosceles $\triangle ABO$]

$\angle CBO = 2x°$ [exterior angle of $\triangle ABO$]

(ii) $\angle ACO = 2x°$ [base angles of isosceles $\triangle COB$]

$\therefore\ 87° = 2x° + x°$ [exterior angle of $\triangle AOC$]

$\therefore\ x = 29$

(c) Using the cosine rule* to find α

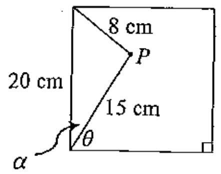
$\cos\alpha = \frac{20^2+15^2-8^2}{2\times 20\times 15}$ $\quad$ but $\alpha = 90° - \theta$

$\therefore\ \cos(90°-\theta) = \frac{20^2+15^2-8^2}{2\times 20\times 15}$

$90° - \theta = 20.7718...°$

$\theta = 69.28...°$

$\therefore\ x = 69°$ (nearest degree)



NOT TO SCALE

(d) (i) Using the product rule*:

$\frac{d}{dx}(xe^{3x}) = e^{3x}\times 1 + x\times 3e^{3x}$

$= e^{3x} + 3xe^{3x}$

$= e^{3x}(1+3x)$

(ii) $$\int_0^2 e^{3x}(3+9x)\,dx = 3\int_0^2 e^{3x}(1+3x)\,dx$$
$$= 3\left[xe^{3x}\right]_0^2$$
$$= 3\left(2e^6 - 0\right)$$
$$= 6e^6$$

Question 13

(a) (i) $y = 4x^3 - x^4 = x^3(4-x)$

$$\frac{dy}{dx} = 12x^2 - 4x^3 = 4x^2(3-x) \qquad \frac{d^2y}{dx^2} = 24x - 12x^2 = 12x(2-x)$$

Stationary points occur when $\frac{dy}{dx} = 0$: $4x^2(3-x) = 0$
$$x = 0, 3$$

At $x = 3$: $y = 4(3)^3 - 3^4 = 27 \Rightarrow (3,27)$

$$\frac{d^2y}{dx^2} = 12\times 3(2-3) < 0 \Rightarrow \text{ a local maximum}$$

At $x = 0$: $y = 0 \Rightarrow (0,0)$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow \text{a possible point of inflexion}$$

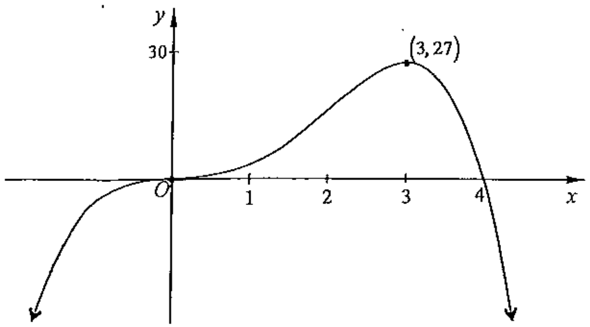
Checking concavity:

x	-1	0	1
$\frac{d^2y}{dx^2} = 12x(2-x)$	-36	0	12

$\therefore$ there is a change in concavity from concave down to concave up

$\therefore$ $(0,0)$ is a stationary point of inflexion and $(3,27)$ is a local maximum.

(ii) The x-intercepts are 0 and 4. The y-intercept is 0.



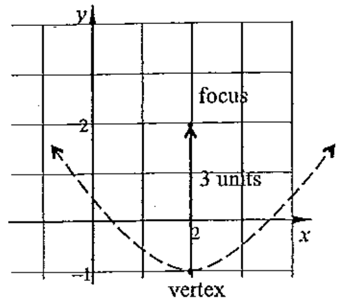
(b) (i) $x^2 - 4x = 12y + 8$
$$\therefore x^2 - 4x + 4 = 12y + 8 + 4$$
$$(x-2)^2 = 12y + 12$$
$$(x-2)^2 = 12(y+1)$$

Comparing to the equation of a parabola*: $4a = 12 \Rightarrow a = 3$
$\therefore$ the focal length is 3 units

(ii) From part (i), the vertex is at $(2,-1)$.

The parabola is concave up so the focus is above the vertex at $(2,-1+3)$ as illustrated in the diagram.

i.e. the focus is at $(2,2)$.



(c) (i) A is the initial amount
i.e. $A = 10$

(ii) $5 = 10e^{-163k}$
$$e^{-163k} = \tfrac{1}{2}$$
$$-163k = \ln\left(\tfrac{1}{2}\right)$$
$$163k = \ln 2$$
$$\therefore \quad k = \tfrac{1}{163}\ln 2$$
$$\doteqdot 4.252\times 10^{-3} \quad \text{(to 3 sig. fig.)}$$

(d) $$A = \int_0^1 \left[\sqrt{2}\cos\left(\frac{\pi}{4}x\right) - x\right] dx$$
$$= \left[\frac{4\sqrt{2}}{\pi}\sin\left(\frac{\pi}{4}x\right) - \frac{1}{2}x^2\right]_0^1$$
$$= \frac{4\sqrt{2}}{\pi}\sin\left(\frac{\pi}{4}\right) - \frac{1}{2} - \left(\frac{4\sqrt{2}}{\pi}\sin(0) - \frac{1}{2}(0)^2\right)$$
$$= \frac{4\sqrt{2}}{\pi}\times\frac{1}{\sqrt{2}} - \frac{1}{2}$$
$$= \left(\frac{4}{\pi} - \frac{1}{2}\right)$$

$\therefore$ the shaded area is $\left(\frac{4}{\pi} - \frac{1}{2}\right)$ unit^2

Question 14

(a) Subtracting the y-values of the two tables produces the following table of height differences:

x	−2	−1	0	1	2
y	0	0.4	0.5	0.4	0
weight	1	4	2	4	1

Using two applications of Simpson's rule with $\frac{b-a}{6}=\frac{0-(-2)}{6}=\frac{1}{3}$ or $\frac{h}{3}=\frac{1}{3}$

Method 1

$$\Delta A \doteqdot \frac{1}{3}\left[1(0)+4(0.4)+2(0.5)+4(0.4)+1(0)\right]$$
$$=1.4$$

Method 2

$$\Delta A \doteqdot \frac{1}{3}\left[1(0)+4(0.4)+0.5\right]+\frac{1}{3}\left[0.5+4(0.4)+0\right]$$
$$=1.4$$

$\therefore$ the increase in area is approximately 1.4 m^2

(b) (i)
$$A_0=100\,000$$
$$A_1=100\,000(1-0.35)+5000$$
$$=100\,000(0.65)+5000$$
$$\therefore A_2=A_1(0.65)+5000$$
$$=\left[100\,000(0.65)+5000\right](0.65)+5000$$
$$=0.65(0.65\times100\,000+5000)+5000 \quad \text{as required}$$

(ii) $A_2=100\,000(0.65)^2+5000(1+0.65)$

So after n days, $A_n=100\,000(0.65)^n+5000(1+0.65+\ldots+0.65^{n-1})$

Now $1+0.65+\ldots+0.65^{n-1}$ is geometric with $n=1$, $r=0.65$ and n terms.

Using the formula for the sum to n terms of a geometric series*:

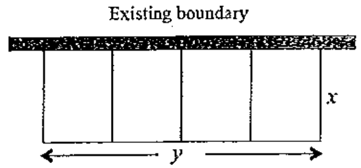
$$S_n=\frac{a(1-r^n)}{1-r}$$
$$=\frac{1-0.65^n}{0.35}$$

$$\therefore A_n=0.65^n\times100\,000-5000\times\frac{(1-0.65^n)}{0.35} \quad \text{as required}$$

(iii)
$$A_{14}=0.65^{14}\times100\,000+5000\times\frac{(1-0.65^{14})}{0.35}$$
$$\doteqdot 14491.70147$$
$$\doteqdot 14500 \quad \text{(nearest 100)}$$

$\therefore$ the expected insect population will be 14500 (to the nearest 100)

(c) (i) Let y be the breadth of the enclosure.



$$A=xy$$
$$\therefore 720=xy$$
$$y=\frac{720}{x}$$

$\therefore$ the side parallel to the existing boundary is $\frac{720}{x}$ m

The 5 parallel sides have total length of $5x$ m

$\therefore \ell=5x+\frac{720}{x}$ Note: $x>0$

(ii)
$$l=5x+\frac{720}{x}$$
$$=5x+720x^{-1}$$
$$\frac{dl}{dx}=5-720x^{-2} \qquad \frac{d^2l}{dx^2}=1440x^{-3}$$
$$=5-\frac{720}{x^2} \qquad =\frac{1440}{x^3}$$

The minimum length will occur when $\frac{dl}{dx}=0$.

$$\therefore 5-\frac{720}{x^2}=0$$
$$\frac{720}{x^2}=5$$
$$x^2=\frac{720}{5}$$
$$x^2=144$$
$$\therefore x=12 \quad \text{as } x>0$$

When $x=12$, $\frac{d^2l}{dx^2}=\frac{1440}{12^3}>0$ i.e. the minimum value of l is when $x=12$

$$\therefore l_{min}=5\times12+\frac{720}{12}=120$$

$\therefore$ the minimum length of wire fencing is 120 m

(d) $1+x+x^2+x^3+x^4$ is geometric with $a=1$, $r=x$ and $n=5$.

Using the formula for the sum to n terms of a geometric series*:

$$S_n=\frac{a(r^n-1)}{r-1}$$
$$\therefore S_5=\frac{x^5-1}{x-1}$$

$$\therefore \lim_{x\to1}\frac{x^5-1}{x-1}=\lim_{x\to1}\left(1+x+x^2+x^3+x^4\right)$$
$$=1+1+1+1+1$$
$$=5$$

(e) **Method 1**

$\log 2+\log 4+\log 8+\ldots+\log 512$ is an arithmetic series with $a=\log 2$, $d=\log 2$ and $n=9$ and using the formula for the sum of an arithmetic series*:

$$\log 2+\log 4+\log 8+\ldots+\log 512$$
$$=\frac{9}{2}\left(\log 2+\log 512\right)$$
$$=\frac{9}{2}\log 1024$$
$$=\frac{9}{2}\log 2^{10}$$
$$=\frac{9}{2}\times 10\log 2$$
$$=45\log 2$$

Method 2

Using log laws:

$$\log 2+\log 4+\log 8+\ldots+\log 512$$
$$=\log 2+\log 2^2+\log 2^3+\ldots+\log 2^9$$
$$=\log 2+2\log 2+3\log 2+\ldots+9\log 2$$
$$=\log 2\times(1+\ldots+9)$$
$$=\log 2\times\frac{9(1+9)}{2}$$
$$=45\log 2$$

Question 15

(a) **Method 1**

Rotating C_1 about the x-axis produces a hemisphere of radius 2 units as C_1 is a semicircle.

The formula for the volume of a sphere is $V=\frac{4}{3}\pi r^3$.

$$\therefore V_1=\frac{1}{2}\times\frac{4}{3}\pi\times 2^3$$
$$=\frac{16}{3}\pi$$

For C_2: $\quad \frac{x^2}{9}+\frac{y^2}{4}=1 \Rightarrow \frac{y^2}{4}=1-\frac{x^2}{9}$

$$\therefore\ y^2=4-\frac{4x^2}{9}$$

Rotating C_2 about the x-axis:

$$\therefore V_2=\pi\int_0^3 y^2\,dx$$
$$=\pi\int_0^3\left(4-\frac{4x^2}{9}\right)dx$$
$$=\pi\left[4x-\frac{4x^3}{27}\right]_0^3$$
$$=\pi(12-4)$$
$$=8\pi$$

Total volume $=V_1+V_2$

$$=\frac{16}{3}\pi+8\pi$$
$$=\frac{40}{3}\pi$$

$\therefore$ the volume is $\frac{40}{3}\pi$ unit^3

Method 2

For C_1: $y^2=4-x^2$ and for C_2: $y^2=4-\frac{4x^2}{9}$.

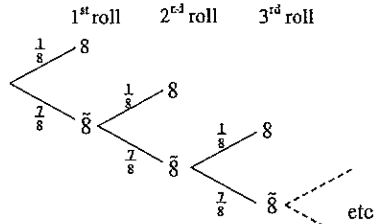
Using $V=\pi\int y^2\,dx$ for each curve: $V=\pi\int_{-2}^{0}\left(4-x^2\right)dx+\pi\int_0^3\left(4-\frac{4x^2}{9}\right)dx$

$$=\pi\left[4x-\frac{x^3}{3}\right]_{-2}^{0}+\pi\left[4x-\frac{4x^3}{(3)(9)}\right]_0^3$$
$$=\pi\left[0-\frac{0}{3}-\left(4(-2)-\frac{(-2)^3}{3}\right)\right]+\pi\left[4(3)-\frac{4(3)^3}{27}-\left(0-\frac{0}{27}\right)\right]$$
$$=\pi\left[8-\frac{8}{3}+12-4\right]$$
$$=\frac{40\pi}{3}$$

$\therefore$ the volume is $\frac{40}{3}\pi$ unit^3

(b) (i) **Method 1**

Note: In the tree diagram the notation $\tilde{8}$ is used for 'not 8'.



$$P(\text{ends before the } 4^{\text{th}} \text{ roll})=P(\text{ends roll 1})+P(\text{ends roll 2})+P(\text{ends roll 3})$$
$$=P(8)+P(\tilde{8}8)+P(\tilde{8}\tilde{8}8)$$
$$=\frac{1}{8}+\frac{7}{8}\times\frac{1}{8}+\frac{7}{8}\times\frac{7}{8}\times\frac{1}{8}$$
$$=\frac{1}{8}+\frac{7}{8}\times\frac{1}{8}+\left(\frac{7}{8}\right)^2\times\frac{1}{8} \qquad \text{as required}$$

Method 2

Probability of game ending on the 1st roll: $\frac{1}{8}$

Probability of game ending on 2nd roll, means not winning 1st roll:

i.e. $\underbrace{\frac{7}{8}}_{\text{not winning 1st}}\times\underbrace{\frac{1}{8}}_{\text{winning 2nd}}$

Probability of game ending on 3rd roll, means not winning previous rolls:

i.e. $\underbrace{\frac{7}{8}}_{\text{not winning 1st}}\times\underbrace{\frac{7}{8}}_{\text{not winning 2nd}}\times\underbrace{\frac{1}{8}}_{\text{winning 3rd}}$

$$\therefore\ P(\text{ends before the } 4^{\text{th}} \text{ roll})=\frac{1}{8}+\frac{7}{8}\times\frac{1}{8}+\left(\frac{7}{8}\right)^2\times\frac{1}{8}$$

(ii) Note that there are 2 factors of $\frac{7}{8}$ in last term of the series that gives the probability of ending before the 4th roll. Hence there will be $n-2$ factors of $\frac{7}{8}$ in the last term of the series that gives the probability of ending before the n^{th} roll.

$$P(\text{ends before the } n^{th} \text{ roll}) = \frac{1}{8} + \frac{7}{8}\times\frac{1}{8} + \left(\frac{7}{8}\right)^2 \times \frac{1}{8} + \ldots + \left(\frac{7}{8}\right)^{n-2} \times \frac{1}{8}$$

This is a geometric series with $a = \frac{1}{8}; r = \frac{7}{8}$; and $n-1$ terms and using the formula for the sum a geometric series*:

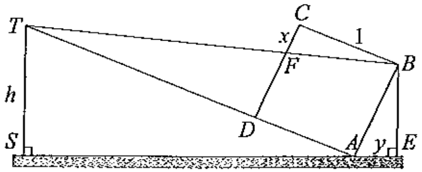
$$\therefore\ P(\text{ends before the } n^{th} \text{ roll}) = \frac{\frac{1}{8}\left[1-\left(\frac{7}{8}\right)^{n-1}\right]}{1-\frac{7}{8}}$$
$$= 1-\left(\tfrac{7}{8}\right)^{n-1}$$

Now $P(\text{win}) > \frac{3}{4}$

$$\therefore\ 1-\left(\tfrac{7}{8}\right)^{n-1} > \tfrac{3}{4}$$
$$\left(\tfrac{7}{8}\right)^{n-1} < \tfrac{1}{4}$$
$$n-1 > \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{7}{8}\right)}$$
$$n > \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{7}{8}\right)} + 1$$
$$n > 11.38\ldots$$

$\therefore$ 12 games are needed.

(c)



NOT TO SCALE

(i) In $\triangle FCB$ and $\triangle BAT$

1.	$\angle FCB = \angle BAT = 90°$	[angles of square $ABCD$]
2.	$\angle CBF = \angle BTA$	[alternate angles, $CB \parallel DA$]
	$\therefore \triangle FCB \parallel\!\mid \triangle BAT$	[equiangular]

(ii) In $\triangle TSA$ and $\triangle AEB$

1.	$\angle TSA = \angle BEA = 90°$	[given]
2.	$\angle STA + \angle TAS + 90° = 180°$	[angle sum $\triangle STA$]
	$\therefore \angle STA + \angle TAS = 90°$	
	$\angle BAE + \angle TAS + 90° = 180°$	[SAE a straight line]
	$\therefore \angle BAE + \angle TAS = 90°$	
	$\therefore \angle STA = \angle BAE$	[both complements of $\angle TAS$]
	$\therefore \triangle TSA \parallel\!\mid \triangle AEB$	[equiangular]

(iii) From part (i): $\frac{CB}{TA} = \frac{CF}{AB}$ [matching sides of similar triangles]

$$\therefore\ \frac{1}{TA} = \frac{x}{1}$$
$$TA = \frac{1}{x}$$

From part (ii): $\frac{ST}{AE} = \frac{TA}{AB}$ [matching sides of similar triangles]

$$\therefore\ \frac{h}{y} = \frac{\frac{1}{x}}{1}$$
$$\therefore\ h = \frac{y}{x}$$

Question 16

(a) (i) $v = 2 - \dfrac{4}{t+1}$

When $t = 0$: $v = 2 - \dfrac{4}{0+1} = -2$

$\therefore$ the initial velocity is -2 m/s

(ii) $v = 2 - 4(t+1)^{-1}$ $\qquad a = \dfrac{dv}{dt} = 4(t+1)^{-2}$
$$= \frac{4}{(t+1)^2}$$

The particle is stationary when $v = 0$: $\quad 2 - \dfrac{4}{t+1} = 0$
$$\frac{4}{t+1} = 2$$
$$4 = 2(t+1)$$
$$t+1 = 2$$
$$\therefore\ t = 1$$

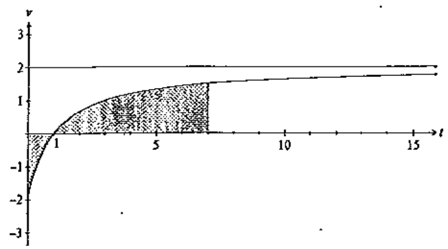
When $t = 1$, $a = \dfrac{4}{(1+1)^2} = 1$

$\therefore$ the acceleration is 1 m/s^2

(iii) As $t \to \infty$, $v \to 2$

Intercepts:
When $v = 0$: $t = 1$
When $t = 0$: $v = -2$

Note: The shading in the diagram is for part (iv).



(iv) The distance travelled is the area under the v-t graph.

$$D=\left|\int_0^1\left(2-\frac{4}{t+1}\right)dt\right|+\int_1^7\left(2-\frac{4}{t+1}\right)dt$$

$$=\left|\left[2t-4\ln(t+1)\right]_0^1\right|+\left[2t-4\ln(t+1)\right]_1^7$$

$$=\left|2-4\ln 2-(0-4\ln 1)\right|+(14-4\ln 8)-(2-4\ln 2)$$

$$=4\ln 2-2+14-4\ln 8-2+4\ln 2$$

$$=4\ln\left(\frac{2\times 2}{8}\right)+10$$

$$=10-4\ln 2$$

$\therefore$ the distance travelled is $(10-4\ln 2)$ m

(b) (i) $y=\dfrac{200}{1+19e^{-0.5t}}=200\left(1+19e^{-0.5t}\right)^{-1}$

Using the chain or function of a function rule*:

$$\therefore\ \frac{dy}{dt}=-200\left(1+19e^{-0.5t}\right)^{-2}\times\left(-0.5\times 19e^{-0.5t}\right)$$

$$=\frac{-200\times\left(-0.5\times 19e^{-0.5t}\right)}{\left(1+19e^{-0.5t}\right)^2}$$

$$=\frac{1900e^{-0.5t}}{\left(1+19e^{-0.5t}\right)^2}$$

(ii) Note that $\dfrac{dy}{dt}>0$ i.e. y is monotonic increasing

At $t=0$, $y=\dfrac{200}{1+19}=10$

As $t\to\infty$, $y\to\dfrac{200}{1+0}=200$

$\therefore\ 10\le y<200$

(iii) **Method 1** Substitute $y=\dfrac{200}{1+19e^{-0.5t}}$ into $\dfrac{y}{400}(200-y)$:

$$\frac{y}{400}(200-y)=\frac{\frac{200}{1+19e^{-0.5t}}}{400}\left(200-\frac{200}{1+19e^{-0.5t}}\right)$$

$$=\frac{1}{2\left(1+19e^{-0.5t}\right)}\left(\frac{200\left(1+19e^{-0.5t}\right)-200}{1+19e^{-0.5t}}\right)$$

$$=\frac{1}{2\left(1+19e^{-0.5t}\right)}\left(\frac{200+200\left(19e^{-0.5t}\right)-200}{1+19e^{-0.5t}}\right)$$

$$=\frac{1}{2\left(1+19e^{-0.5t}\right)}\left(\frac{2\times 1900e^{-0.5t}}{1+19e^{-0.5t}}\right)$$

$$\therefore\ \frac{y}{400}(200-y)=\frac{1900e^{-0.5t}}{\left(1+19e^{-0.5t}\right)^2}$$

$$=\frac{dy}{dt}$$

Method 2

$y=\dfrac{200}{1+19e^{-0.5t}}\ \Rightarrow\ 19e^{-0.5t}+1=\dfrac{200}{y}\qquad\therefore\ 19e^{-0.5t}=\dfrac{200}{y}-1$

Substituting this into the derivative: $\dfrac{dy}{dt}=\dfrac{1900e^{-0.5t}}{\left(1+19e^{-0.5t}\right)^2}$

$$=\frac{100\left(19e^{-0.5t}\right)}{\left(1+19e^{-0.5t}\right)^2}$$

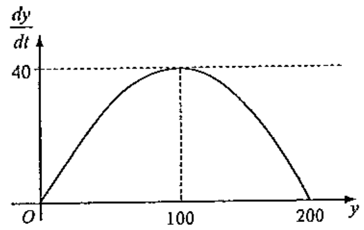
$$=\frac{100\left(\frac{200}{y}-1\right)}{\left(1+\frac{200}{y}-1\right)^2}$$

$$=\frac{100y^2}{200^2}\left(\frac{200}{y}-1\right)$$

$$=\frac{y^2}{400}\left(\frac{200}{y}-1\right)$$

$$=\frac{y}{400}(200-y)\qquad\text{as required}$$

(iv) The graph of $\dfrac{dy}{dt}=\dfrac{y}{400}(200-y)$ is a concave down parabola as illustrated.



$\therefore$ the maximum of $\dfrac{dy}{dt}=\dfrac{y}{400}(200-y)$ occurs at the vertex

Now $\dfrac{y}{400}(200-y)=0$ when $y=0,200$

$\therefore$ by symmetry the vertex will occur when $y=100$

$\therefore$ the population is growing fastest when $y=100$

End of solutions