BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2016 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–13

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

1250

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which sum is equal to $\sum_{k=1}^{20}(2k+1)$?

(A) $1+2+3+4+\cdots+20$

(B) $1+3+5+7+\cdots+41$

(C) $3+4+5+6+\cdots+20$

(D) $3+5+7+9+\cdots+41$

2 What is the remainder when $2x^3-10x^2+6x+2$ is divided by $x-2$?

(A) -66

(B) -10

(C) $-x^3+5x^2-3x-1$

(D) x^3-5x^2+3x+1

3 Which expression is equivalent to $\dfrac{\tan 2x-\tan x}{1+\tan 2x\tan x}$?

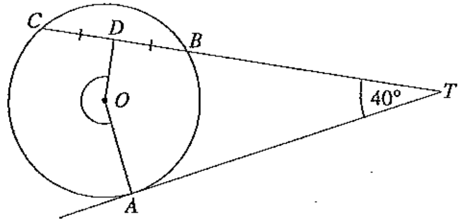
(A) $\tan x$

(B) $\tan 3x$

(C) $\dfrac{\tan 2x-1}{1+\tan 2x}$

(D) $\dfrac{\tan x}{1+\tan 2x\tan x}$

4 In the diagram, O is the centre of the circle ABC, D is the midpoint of BC, AT is the tangent at A and $\angle ATB = 40°$.



NOT TO SCALE

What is the size of the reflex angle DOA?

(A) 80°

(B) 140°

(C) 220°

(D) 280°

5 Which expression is equal to $\int \sin^2 2x\,dx$?

(A) $\frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + c$

(B) $\frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + c$

(C) $\frac{\sin^3 2x}{6} + c$

(D) $\frac{-\cos^3 2x}{6} + c$

6 What is the general solution of the equation $2\sin^2 x - 7\sin x + 3 = 0$?

(A) $n\pi - (-1)^n \frac{\pi}{3}$

(B) $n\pi + (-1)^n \frac{\pi}{3}$

(C) $n\pi - (-1)^n \frac{\pi}{6}$

(D) $n\pi + (-1)^n \frac{\pi}{6}$

7 The displacement x of a particle at time t is given by

$$x = 5\sin 4t + 12\cos 4t.$$

What is the maximum velocity of the particle?

(A) 13

(B) 28

(C) 52

(D) 68

8 A team of 11 students is to be formed from a group of 18 students. Among the 18 students are 3 students who are left-handed.

What is the number of possible teams containing at least 1 student who is left-handed?

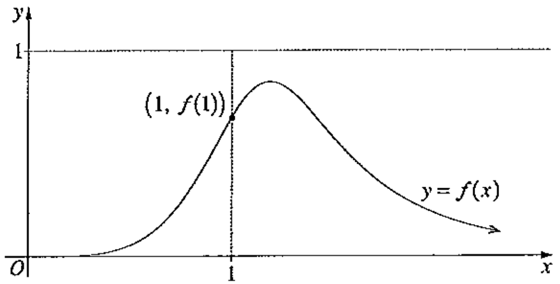
(A) 19 448

(B) 30 459

(C) 31 824

(D) 58 344

9 The diagram shows the graph of $y = f(x)$.



Which of the following is a correct statement?

(A) $f''(1) < f(1) < 1 < f'(1)$

(B) $f''(1) < f'(1) < f(1) < 1$

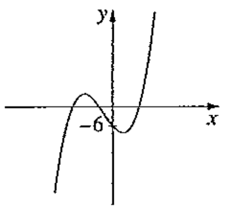
(C) $f(1) < 1 < f'(1) < f''(1)$

(D) $f'(1) < f(1) < 1 < f''(1)$

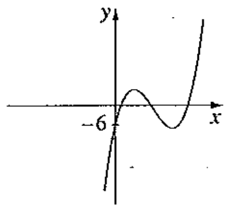
10 Consider the polynomial $p(x) = ax^3 + bx^2 + cx - 6$ with a and b positive.

Which graph could represent $p(x)$?

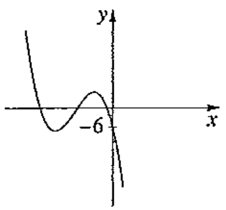
(A)



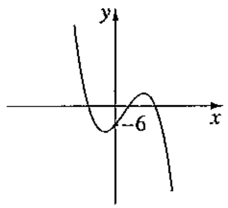
(B)



(C)



(D)



Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find the inverse of the function $y = x^3 - 2$. **2**

(b) Use the substitution $u = x - 4$ to find $\int x\sqrt{x-4}\,dx$. **3**

(c) Differentiate $3\tan^{-1}(2x)$. **2**

(d) Evaluate $\lim_{x \to 0}\left(\dfrac{2\sin x \cos x}{3x}\right)$. **2**

(e) Solve $\dfrac{3}{2x+5} - x > 0$. **3**

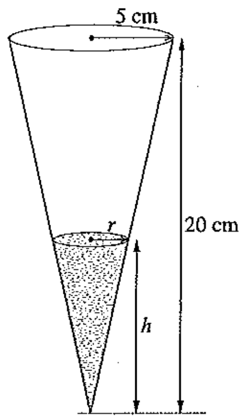
(f) A darts player calculates that when she aims for the bullseye the probability of her hitting the bullseye is $\frac{3}{5}$ with each throw.

(i) Find the probability that she hits the bullseye with exactly one of her first three throws. **1**

(ii) Find the probability that she hits the bullseye with at least two of her first six throws. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows a conical soap dispenser of radius 5 cm and height 20 cm.



At any time t seconds, the top surface of the soap in the container is a circle of radius r cm and its height is h cm.

The volume of the soap is given by $v = \frac{1}{3}\pi r^2 h$.

(i) Explain why $r = \frac{h}{4}$. **1**

(ii) Show that $\frac{dv}{dh} = \frac{\pi}{16}h^2$. **1**

The dispenser has a leak which causes soap to drip from the container. The area of the circle formed by the top surface of the soap is decreasing at a constant rate of $0.04\ \text{cm}^2\,\text{s}^{-1}$.

(iii) Show that $\frac{dh}{dt} = \frac{-0.32}{\pi h}$. **2**

(iv) What is the rate of change of the volume of the soap, with respect to time, when $h = 10$? **2**

Question 12 continues on page 8

(b) In a chemical reaction, a compound X is formed from a compound Y. The mass in grams of X and Y are $x(t)$ and $y(t)$ respectively, where t is the time in seconds after the start of the chemical reaction.

Throughout the reaction the sum of the two masses is 500 g.

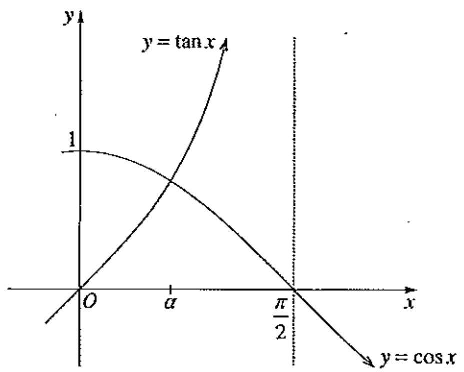
At any time t, the rate at which the mass of compound X is increasing is proportional to the mass of compound Y.

At the start of the chemical reaction, $x = 0$ and $\frac{dx}{dt} = 2$.

(i) Show that $\frac{dx}{dt} = 0.004(500 - x)$. **3**

(ii) Show that $x = 500 - Ae^{-0.004t}$ satisfies the equation in part (i), and find the value of A. **2**

(c) The graphs of $y = \tan x$ and $y = \cos x$ meet at the point where $x = \alpha$, as shown.



(i) Show that the tangents to the curves at $x = \alpha$ are perpendicular. **2**

(ii) Use one application of Newton's method with $x_1 = 1$ to find an approximate value for α. Give your answer correct to two decimal places. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The tide can be modelled using simple harmonic motion.

At a particular location, the high tide is 9 metres and the low tide is 1 metre.

At this location the tide completes 2 full periods every 25 hours.

Let t be the time in hours after the first high tide today.

(i) Explain why the tide can be modelled by the function $x = 5 + 4\cos\left(\frac{4\pi}{25}t\right)$. **2**

(ii) The first high tide tomorrow is at 2 am. **2**

What is the earliest time tomorrow at which the tide is increasing at the fastest rate?

Question 13 continues on page 10

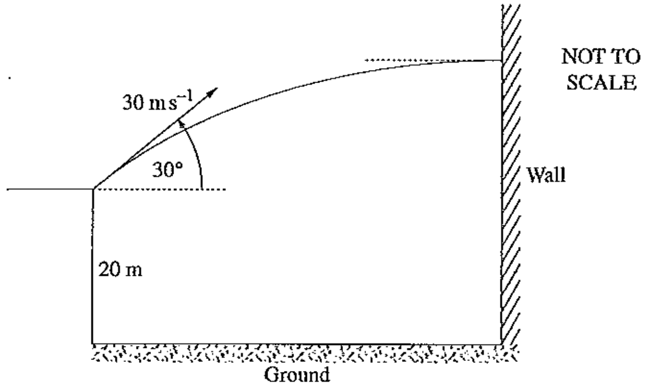
(b) The trajectory of a projectile fired with speed $u\text{ m s}^{-1}$ at an angle θ to the horizontal is represented by the parametric equations

$$x = ut\cos\theta \quad \text{and} \quad y = ut\sin\theta - 5t^2,$$

where t is the time in seconds.

(i) Prove that the greatest height reached by the projectile is $\dfrac{u^2\sin^2\theta}{20}$. **2**

A ball is thrown from a point 20 m above the horizontal ground. It is thrown with speed 30 m s^{-1} at an angle of 30° to the horizontal. At its highest point the ball hits a wall, as shown in the diagram.



(ii) Show that the ball hits the wall at a height of $\dfrac{125}{4}$ m above the ground. **2**

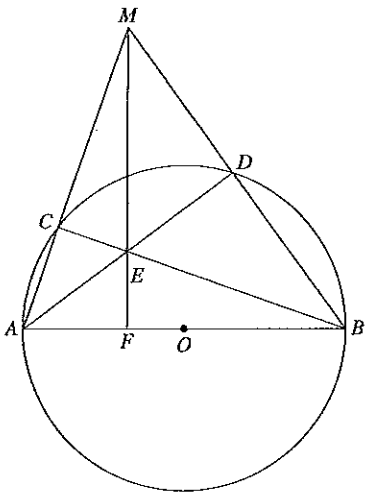
The ball then rebounds horizontally from the wall with speed 10 m s^{-1}. You may assume that the acceleration due to gravity is 10 m s^{-2}.

(iii) How long does it take the ball to reach the ground after it rebounds from the wall? **2**

(iv) How far from the wall is the ball when it hits the ground? **1**

Question 13 continues on page 11

(c) The circle centred at O has a diameter AB. From the point M outside the circle the line segments MA and MB are drawn meeting the circle at C and D respectively, as shown in the diagram. The chords AD and BC meet at E. The line segment ME produced meets the diameter AB at F.



Copy or trace the diagram into your writing booklet.

(i) Show that $CMDE$ is a cyclic quadrilateral. **2**

(ii) Hence, or otherwise, prove that MF is perpendicular to AB. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $4n^3 + 18n^2 + 23n + 9$ can be written as **1**

$$(n+1)(4n^2+14n+9).$$

(ii) Using the result in part (i), or otherwise, prove by mathematical induction that, for $n \geq 1$, **3**

$$1\times3+3\times5+5\times7+\cdots+(2n-1)(2n+1)=\frac{1}{3}n(4n^2+6n-1).$$

(b) Consider the expansion of $(1+x)^n$, where n is a positive integer.

(i) Show that $2^n = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}$. **1**

(ii) Show that $n2^{n-1} = \binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+n\binom{n}{n}$. **1**

(iii) Hence, or otherwise, show that $\sum_{r=1}^{n}\binom{n}{r}(2r-n) = n$. **2**

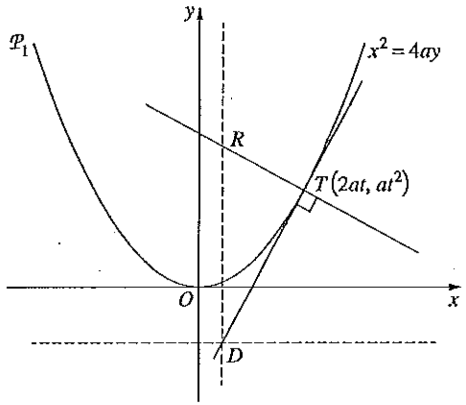
Question 14 continues on page 13

Question 14 (continued)

(c) The point $T(2at, at^2)$ lies on the parabola $\mathcal{P}_1$ with equation $x^2 = 4ay$.

The tangent to the parabola $\mathcal{P}_1$ at T meets the directrix at D.

The normal to the parabola $\mathcal{P}_1$ at T meets the vertical line through D at the point R, as shown in the diagram.



(i) Show that the point D has coordinates $\left(at - \frac{a}{t}, -a\right)$. **1**

(ii) Show that the locus of R lies on another parabola $\mathcal{P}_2$. **3**

(iii) State the focal length of the parabola $\mathcal{P}_2$. **1**

It can be shown that the minimum distance between R and T occurs when the normal to $\mathcal{P}_1$ at T is also the normal to $\mathcal{P}_2$ at R. (Do NOT prove this.)

(iv) Find the values of t so that the distance between R and T is a minimum. **2**

End of paper

2016 Higher School Certificate Solutions Mathematics Extension 1

Section I

Multiple Choice Summary

1 D	2 B	3 A	4 C	5 A
6 D	7 C	8 B	9 A	10 A

Multiple Choice Solutions

1. D

$$\sum_{k=1}^{20}(2k+1)=(2\times1+1)+(2\times2+1)+(2\times3+1)+\ldots+(2\times20+1)$$
$$=3+5+7+\ldots+41$$

2. B

Let $P(x)=2x^3-10x^2+6x+2$ then the remainder upon division by $x-2$ is

$$P(2)=2(2)^3-10(2)^2+6(2)+2$$
$$=-10$$

3. A

$\dfrac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\tan(\theta-\phi)$ and using $\theta=2x$ and $\phi=x$

$$\therefore \frac{\tan 2x-\tan x}{1+\tan 2x\tan x}=\tan(2x-x)$$
$$=\tan x$$

4. C

$\angle ODT=\angle OAT=90°$ [angle between tangent and radius]

$\therefore$ $ODTA$ is a cyclic quadrilateral [one pair of opposite angles supplementary]

$\therefore \angle DOA+40°=180°$ [opposite angles of a cyclic quad]

$\angle DOA=140°$

$\therefore$ reflex $\angle DOA=360°-140°=220°$

5. A

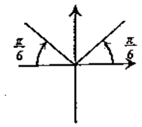
$\cos 2\theta=1-2\sin^2\theta \Rightarrow \sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$ and substituting $\theta=2x$

$$\int \sin^2 2x\,dx=\int\frac{1}{2}(1-\cos 4x)\,dx$$
$$=\frac{1}{2}\left(x-\frac{1}{4}\sin 4x\right)+c$$

6. D

$$2\sin^2 x-7\sin x+3=0$$
$$(2\sin x-1)(\sin x-3)=0$$

$\sin x=\dfrac{1}{2}$ or $\sin x=3$ but this has no solution, $-1\le\sin x\le1$



$$\therefore x=n\pi+(-1)^n\frac{\pi}{6},\ n\in\mathbb{Z}$$

7. C

$x=5\sin 4t+12\cos 4t$

Let $x=A\sin(4t+\alpha)$ where $A>0, 0<\alpha<\dfrac{\pi}{2}$

$$=A\sin 4t\cos\alpha+A\cos 4t\sin\alpha$$

Then $A\cos\alpha=5$ and $A\sin\alpha=12$

But $A^2\cos^2\alpha+A^2\sin^2\alpha=A^2$

$$\therefore A^2=5^2+12^2$$
$$A=13$$
$$\therefore x=13\sin(4t+\alpha)$$
$$\frac{dx}{dt}=52\cos(4t+\alpha)$$

But since $-1\le\cos\theta\le1$, the maximum value of $\dfrac{dx}{dt}=52$

8. B

Choose 11 from a group of 18 where 3 are left-handed

with at least 1 left-handed student = all possible teams – # teams with no one left-handed

$$={}^{18}C_{11}-{}^{15}C_{11}=30\,459$$

9. A

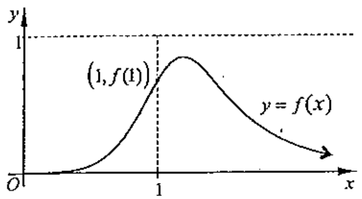
The graph is drawn to scale and so $0<f(1)<1$ as the point is below $y=1$.

The tangent at the point is steeper than $y=x$ and hence $f'(1)>1$.

$\therefore$ $0<f(1)<1<f'(1)$ eliminating B and D.

The point could be located either at a point of inflexion or on a part of the curve that is concave down and hence $f''(1)\le0$.

This leaves $f''(1)<f(1)<1<f'(1)$ as the only viable option.



10. A

Method 1

If $a>0$ then $p(x)\to\infty$ as $x\to\infty$ eliminating C and D.

$p(x)=ax^3+bx^2+cx-6$

$p'(x)=3ax^2+2bx+c$ $\therefore p'(0)=c$ of which we know nothing

$p''(x)=6ax+2b$ $\therefore p''(0)=2b>0$

$\Rightarrow$ at $x=0$ the curve must be concave up, leaving A as the only viable option.

Method 2

Since $a>0$ then the graph must be A or B.

Now $\alpha+\beta+\gamma=-\frac{b}{a}$ which is negative as $a>0$ and $b>0$. Then at least one zero must be negative. B has no negative roots leaving A as the only possible option.

Section II

Question 11

(a) $f: y=x^3-2$ Note that the function is 1-1 and so an inverse function exists.

$f^{-1}: x=y^3-2$

$y^3=x+2$

$y=\sqrt[3]{x+2}$

(b) If $u=x-4 \Rightarrow x=u+4$

$\frac{du}{dx}=1$

$\therefore du=dx$

$\therefore \int x\sqrt{x-4}\,dx=\int (u+4)\sqrt{u}\,du$

$=\int\left(u^{\frac{3}{2}}+4u^{\frac{1}{2}}\right)du$

$=\frac{2}{5}u^{\frac{5}{2}}+4\left(\frac{2}{3}\right)u^{\frac{3}{2}}+c$

$=\frac{2}{5}\sqrt{(x-4)^5}+\frac{8}{3}\sqrt{(x-4)^3}+c$

(c) $\frac{d}{dx}\left(3\tan^{-1}2x\right)=3\left(\frac{1}{1+(2x)^2}\right).2$

$=\frac{6}{1+4x^2}$

(d) **Method 1**

Since $\lim_{x\to0}\left(\frac{\sin\theta}{\theta}\right)=1$

$\lim_{x\to0}\left(\frac{2\sin x\cos x}{3x}\right)=\lim_{x\to0}\left(\frac{\sin 2x}{3x}\right)$

$=\frac{2}{3}\lim_{x\to0}\left(\frac{\sin 2x}{2x}\right)$

$=\frac{2}{3}\times1$

$=\frac{2}{3}$

Method 2

$\lim_{x\to0}\left(\frac{2\sin x\cos x}{3x}\right)=\frac{2}{3}\lim_{x\to0}\left(\frac{\sin x}{x}\times\cos x\right)$

$=\frac{2}{3}\lim_{x\to0}\left(\frac{\sin x}{x}\right)\times\lim_{x\to0}(\cos x)$

$=\frac{2}{3}\times1\times1$

$=\frac{2}{3}$

(e) **Method 1**

$\frac{3}{2x+5}-x>0 \quad \Rightarrow x\neq-\frac{5}{2}$

$\frac{3}{2x+5}\times(2x+5)^{2}-x(2x+5)^2>0$

$(2x+5)\left[3-x(2x+5)\right]>0$

$(2x+5)\left[3-2x^2-5x\right]>0$

$(2x+5)\left(3-5x-2x^2\right)>0$

$(2x+5)(3+x)(1-2x)>0$

Consider the cubic curve that passes through

$x=-3,-\frac{5}{2},\frac{1}{2}$

$\therefore\ x<-3$ or $-\frac{5}{2}<x<\frac{1}{2}$

Method 2

$\frac{3}{2x+5}-x>0 \quad \Rightarrow x\neq-\frac{5}{2}$

Find the other critical points:

$\frac{3}{2x+5}-x=0$

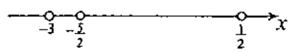
$3-x(2x+5)=0$

$3-2x^2-5x=0$

$2x^2+5x-3=0$

$(2x-1)(x+3)=0$

$\therefore\ x=\frac{1}{2},-3$



Test a value in each section of the number line:

If $x=-4$: $\frac{3}{2(-4)+5}-(-4)=3>0 \Rightarrow$ True

If $x=-2\frac{3}{4}$: $\frac{3}{2(-2\frac{3}{4})+5}-(-2\frac{3}{4})=-3\frac{1}{4}<0 \Rightarrow$ False

If $x=0$: $\frac{3}{2(0)+5}-0=0.6>0 \Rightarrow$ True

If $x=1$: $\frac{3}{2(1)+5}-1=-\frac{4}{7}<0 \Rightarrow$ False

$\therefore\ x<-3$ or $-\frac{5}{2}<x<\frac{1}{2}$

(f) $P(\text{bullseye})=\frac{3}{5}=0.6$ Probabilities are given by terms of $(0.6+0.4)^n$.

(i) When $n=3$:

$P(X=1)={}^3C_1(0.6)^1(0.4)^2$

$=0.288$

(ii) When $n=6$:

$P(X\geq2)=1-P(X<2)$

$=1-P(X=0)-P(X=1)$

$=1-{}^6C_0(0.6)^0(0.4)^6-{}^6C_1(0.6)^1(0.4)^5$

$=0.95904$

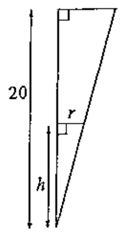
Question 12

(a) (i) Using similar triangles

$\frac{r}{h}=\frac{5}{20}$

$r=\frac{5h}{20}$

$r=\frac{h}{4}$ as required



(ii) $v=\frac{1}{3}\pi r^2 h$ and $r=\frac{h}{4}$

$\therefore v=\frac{1}{3}\pi\left(\frac{h}{4}\right)^2\times h=\frac{\pi h^3}{48}$

$\therefore \frac{dv}{dh}=\frac{3\pi h^2}{48}=\frac{\pi h^2}{16}$ as required

(iii) Given $\frac{dA}{dt}=-0.04$ and $A=\pi r^2$.

Method 1

Substitute $r=\frac{h}{4}$ into the area formula.

$\therefore A=\pi\left(\frac{h}{4}\right)^2=\frac{\pi h^2}{16}$

$\therefore \frac{dA}{dh}=\frac{2\pi h}{16}=\frac{\pi h}{8}$

Now $\frac{dh}{dt}=\frac{dh}{dA}\times\frac{dA}{dt}$ by the chain rule

$\therefore \frac{dh}{dt}=\frac{8}{\pi h}\times(-0.04)$

$=\frac{-0.32}{\pi h}$ as required

Method 2

$\frac{dA}{dr}=2\pi r=2\pi\left(\frac{h}{4}\right)=\frac{\pi h}{2}$

$\therefore \frac{dh}{dt}=\frac{dh}{dr}\times\frac{dr}{dA}\times\frac{dA}{dt}$

$=\frac{4}{1}\times\frac{2}{\pi h}\times -0.04$

$=\frac{-0.32}{\pi h}$ as required

(iv) We need $\frac{dv}{dt}$ when $h=10$

Now $\frac{dv}{dt}=\frac{dv}{dh}\times\frac{dh}{dt}$ by the chain rule.

$\therefore \frac{dv}{dt}=\frac{\pi h^2}{16}\times\frac{-0.32}{\pi h}$

$=-0.02h$

Substitute $h=10$: $\frac{dv}{dt}=-0.02\times 10=-0.2$

$\therefore$ the rate of change of volume is $-0.2\ \text{cm}^3\text{s}^{-1}$ i.e. decreasing at a rate of $0.2\ \text{cm}^3\text{s}^{-1}$

(b) (i) $\frac{dx}{dt}$ increasing in proportion to $y \Rightarrow \frac{dx}{dt}=ky$ for some constant k.

But $x+y=500 \Rightarrow y=500-x$

$\therefore \frac{dx}{dt}=k(500-x)$

Substituting $x=0$, $\frac{dx}{dt}=2$: $2=k(500-0) \Rightarrow k=\frac{2}{500}=0.004$

$\therefore \frac{dx}{dt}=0.004(500-x)$ as required

(ii) $x=500-Ae^{-0.004t} \Rightarrow Ae^{-0.004t}=500-x$ ①

$\frac{dx}{dt}=-(-0.004)Ae^{-0.004t}$

$=0.004Ae^{-0.004t}$

$=0.004(500-x)$ using ①

$\therefore x=500-Ae^{-0.004t}$ satisfies the equation in part (i)

Now when $t=0, x=0$

$\therefore 0=500-Ae^0$

$\therefore A=500$

(c) (i) $\frac{d}{dx}(\tan x)=\sec^2 x$ $\qquad$ $\frac{d}{dx}(\cos x)=-\sin x$

If $x=\alpha$: $m_1=\sec^2\alpha$ $\qquad$ $m_2=-\sin\alpha$

Now the curves meet at $x=\alpha$ $\therefore \tan\alpha=\cos\alpha$

$\frac{\sin\alpha}{\cos\alpha}=\cos\alpha$

$\cos^2\alpha=\sin\alpha$ ①

Now $m_1m_2=\sec^2\alpha(-\sin\alpha)$

$=-\frac{\sin\alpha}{\cos^2\alpha}$

$=-\frac{\sin\alpha}{\sin\alpha}$ using ①

$=-1$

$\Rightarrow$ the tangents are perpendicular at $x=\alpha$.

(ii) Consider the equation $\tan x=\cos x$. Then at $x=\alpha$, $\tan x-\cos x=0$.

Let $f(x)=\tan x-\cos x$ then $f'(x)=\sec^2 x+\sin x$

Taking the first approximation as $x_1=1$

$x_2=1-\frac{f(1)}{f'(1)}$

$=1-\frac{\tan 1-\cos 1}{\sec^2 1+\sin 1}$ $\qquad$ Note: We are using radian measure.

$=0.761633\ldots$

$\therefore \alpha \doteqdot 0.76$ (2 decimal places)

Question 13

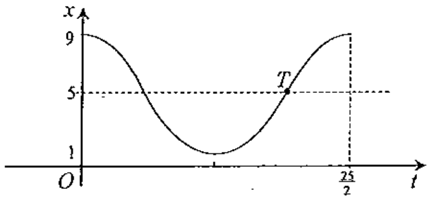
(a) (i) The tide oscillates between 1 and 9

$\therefore$ the centre of motion = 5 and the amplitude of the motion = 4

There are 2 full periods every 25 hours $\Rightarrow$ period = $\frac{25}{2}$

but period $=\frac{2\pi}{n}$ $\qquad$ $\therefore \frac{25}{2}=\frac{2\pi}{n} \Rightarrow n=\frac{4\pi}{25}$

If $t=0$ at high tide, the graph of the function will look like this:



This is a cosine function, namely $x=5+4\cos\left(\frac{4\pi}{25}t\right)$.

(ii) Let 2 am be equivalent to $t=0$.
The tide is moving in simple harmonic motion and hence the speed is fastest at the centre of motion. We also need the tide to be increasing so this occurs at the point marked T in the diagram above.

Method 1

Using the t-axis scale, at T:

$$t=\frac{3}{4}\left(\frac{25}{4}\right)=9.375$$

Now $0.375\times 60\text{ min}=22.5\text{ min}$

$\therefore$ $t=9.375$ is equivalent to 9 h 22.5 min after 2 am.

i.e. the earliest the tide increases at the fastest rate is at 11:22:30 am tomorrow

Method 2

At T, $x=5$: $\quad \therefore\ 5=5+4\cos\frac{4\pi}{25}t$

$$4\cos\frac{4\pi}{25}t=0$$

$$\frac{4\pi}{25}t=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\ldots$$

$$t=\frac{25}{8},\frac{75}{8},\frac{125}{8},\ldots$$

We need the second of these solutions

$$t=\frac{75}{8}=9.375$$

(b) (i) $x=ut\cos\theta$ and $y=ut\sin\theta-5t^2$

The greatest height is attained then the vertical velocity is zero.

$$\frac{dy}{dt}=u\sin\theta-10t$$

$$\therefore\ 0=u\sin\theta-10t$$

$$10t=u\sin\theta$$

$$t=\frac{u\sin\theta}{10}$$

When $t=\frac{u\sin\theta}{10}$: $\quad y=u\left(\frac{u\sin\theta}{10}\right)\sin\theta-5\left(\frac{u\sin\theta}{10}\right)^2$

$$y=\frac{u^2\sin^2\theta}{10}-\frac{u^2\sin^2\theta}{120}$$

$$y=\frac{u^2\sin^2\theta}{20}\quad\text{as required}$$

(ii) For the greatest height above the point of projection, substitute $u=30$, $\theta=30°$ into the formula in (i).

Greatest height above projection $=\frac{30^2\sin^2 30°}{20}=\frac{45}{4}$

But the point of projection is 20 m above ground.

$\therefore$ height above ground $=20+\frac{45}{4}=\frac{125}{4}$

i.e. the ball hits the wall $\frac{125}{4}$ m above the ground

(iii) At the instant the ball rebounds it travels horizontally. $\therefore$ $u=10$, $\theta=0°$

Also take the new point of projection as $y=0$.

The vertical motion is now given by: $y=10t\sin 0°-5t^2$

$$\therefore\ y=-5t^2$$

We need to find t when the ball has fallen to the ground. i.e. $y=-\frac{125}{4}$

$$\therefore\ -5t^2=-\frac{125}{4}$$

$$t^2=\frac{25}{4}$$

$$t=\frac{5}{2}\quad\text{as } t\geq 0$$

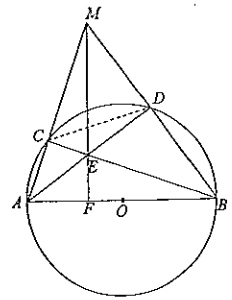
$\therefore$ it will take 2.5 sec for the ball to hit the ground

(iv) Take the new point of projection as $x=0$.

Then $x=ut\cos\theta$ where $u=10$, $\theta=0°$, $t=2.5$

$\therefore$ $x=10(2.5)\cos 0°=25$

$\therefore$ the ball will land 25 m from the wall



(c) (i)

$\angle ACB=90°$	[angle in a semicircle]
$\therefore$ $\angle MCB=90°$	[straight angle]
Similarly $\angle MDA=90°$	
Now $\angle MCB+\angle MDA=180°$	
But these are opposite angles of quadrilateral $CMDE$	
$\therefore$ $CMDE$ is cyclic	[one pair of opposite angles supplementary]

(ii) Join CD

In $\triangle MCE$ and $\triangle EBF$

1. $\angle MEC=\angle BEF$	[vertically opposite]
2. $\angle CME=\angle CDE$	[angles in the same segment on CE]
$=\angle CBA$	[angles in the same segment on AC]
$=\angle FBE$	[same angle]
$\therefore$ $\triangle MCE \mid\mid\mid \triangle BFE$	[equiangular]
$\therefore$ $\angle MCE=\angle BFE$	[matching angles in similar triangles]
But $\angle MCE=\angle MBE=90°$	[proven in (i)]
$\therefore$ $\angle BFE=90°$	
$\therefore$ $MF\perp AB$ as required	

Question 14

(a) (i) $(n+1)(4n^2+14n+9)=n(4n^2+14n+9)+1(4n^2+14n+9)$

$=4n^3+14n^2+9n+4n^2+14n+9$

$=4n^3+18n^2+23n+9$ as required

(ii) To prove $1\times3+3\times5+5\times7+...+(2n-1)(2n+1)=\frac{1}{3}n(4n^2+6n-1)$ for $n\geq1$.

Test $n=1$:

LHS $=(2\times1-1)(2\times1+1)$
$=1\times3$
$=3$

RHS $=\frac{1}{3}\times1\times(4\times1^2+6\times1-1)$
$=\frac{1}{3}\times9$
$=3$

LHS = RHS

$\therefore$ true for $n=1$

Let $n=k$ be a value for which the result is true.

i.e. $1\times3+3\times5+5\times7+...+(2k-1)(2k+1)=\frac{1}{3}k(4k^2+6k-1)$ is true ①

Test if the result is true for $n=k+1$:
i.e. To prove

$1\times3+3\times5+5\times7+...+[2(k+1)-1][2(k+1)+1]=\frac{1}{3}(k+1)\left[4(k+1)^2+6(k+1)-1\right]$

i.e. $1\times3+3\times5+5\times7+...+(2k+1)(2k+3)=\frac{1}{3}(k+1)\left[4(k^2+2k+1)+6k+6-1\right]$

$=\frac{1}{3}(k+1)(4k^2+14k+9)$

LHS $=1\times3+3\times5+5\times7+...(2k-1)(2k+1)+(2k+1)(2k+3)$

$=\frac{1}{3}k(4k^2+6k-1)+(2k+1)(2k+3)$

$=\frac{1}{3}\left[k(4k^2+6k-1)+3(2k+1)(2k+3)\right]$ using ①

$=\frac{1}{3}\left[4k^3+6k^2-k+3(4k^2+8k+3)\right]$

$=\frac{1}{3}\left[4k^3+6k^2-k+12k^2+24k+9\right]$

$=\frac{1}{3}(4k^3+18k^2+23k+9)$

$=\frac{1}{3}(k+1)(4k^2+14k+9)$ using the result in part (i)

$=$ RHS

$\therefore$ true for $n=k+1$ when true for $n=k$

$\therefore$ by induction the result is true for all integers $n\geq1$

(b) (i) $(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\cdots+\binom{n}{n}x^n$

Substitute $x=1$:

$(1+1)^n=\binom{n}{0}+\binom{n}{1}(1)+\binom{n}{2}(1)^2+\binom{n}{3}(1)^3+\cdots+\binom{n}{n}(1)^n$

$\therefore\ 2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}\cdots+\binom{n}{n}$ as required

(ii) Differentiate both sides with respect to x:

$\frac{d}{dx}(1+x)^n=\frac{d}{dx}\left[\binom{n}{1}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3\cdots+\binom{n}{n}x^n\right]$

$\therefore\ n(1+x)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2\cdots+n\binom{n}{n}x^{n-1}$

Substitute $x=1$:

$n(1+1)^{n-1}=\binom{n}{1}+2\binom{n}{2}(1)+3\binom{n}{3}(1)^2\cdots+n\binom{n}{n}(1)^{n-1}$

$n2^{n-1}=\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}\cdots+n\binom{n}{n}$ as required

(iii) **Method 1**

$\sum_{r=1}^{n}\binom{n}{r}(2r-n)=\binom{n}{1}(2\times1-n)+\binom{n}{2}(2\times2-n)+\binom{n}{3}(2\times3-n)+...+\binom{n}{n}(2\times n-n)$

$=\binom{n}{1}(2-n)+\binom{n}{2}(4-n)+\binom{n}{3}(6-n)+...+\binom{n}{n}(2n-n)$

$=2\left[\binom{n}{1}(1)+\binom{n}{2}(2)+\binom{n}{3}(3)+...+\binom{n}{n}(n)\right]-n\left[\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+...+\binom{n}{n}\right]$

$=2\left[n2^{n-1}\right]-n\left[2^n-1\right]$ using the results proved in parts (i) and (ii)

$=n2^n-n2^n+n$

$=n$ as required

Method 2

$\sum_{r=1}^{n}\binom{n}{r}(2r-n)=2\sum_{r=1}^{n}\binom{n}{r}r-n\sum_{r=1}^{n}\binom{n}{r}$

$=2\sum_{r=1}^{n}\binom{n}{r}r-n\left[\sum_{r=1}^{n}\binom{n}{r}+\binom{n}{0}-1\right]$

$=2\sum_{r=1}^{n}\binom{n}{r}r-n\left[\sum_{r=0}^{n}\binom{n}{r}-1\right]$

$=2n2^{n-1}-n(2^n-1)$

$=n$ as required

(c) (i) The tangent at $T(2at, at^2)$ is $y = tx - at^2$ and at D, $y = -a$.

$$\therefore\ -a = tx - at^2$$
$$tx = at^2 - a$$
$$x = at - \frac{a}{t} \qquad \therefore\ D \text{ has coordinates } \left(at - \frac{a}{t}, -a\right)$$

(ii) The normal at $T(2at, at^2)$ is $x + ty = at^3 + 2at$.

At R, $x = at - \frac{a}{t}$: $\quad \therefore\ at - \frac{a}{t} + ty = at^3 + 2at$

$$ty = at^3 + at + \frac{a}{t}$$
$$y = a\left(t^2 + 1 + \frac{1}{t^2}\right)$$

$\therefore$ R has coordinates $\left(at - \frac{a}{t}, a\left(t^2 + 1 + \frac{1}{t^2}\right)\right)$ and the parametric equations of R are:

$$x = at - \frac{a}{t} \qquad ①$$
$$y = a\left(t^2 + 1 + \frac{1}{t^2}\right) \qquad ②$$

From ①: $x = a\left(t - \frac{1}{t}\right) \Rightarrow t - \frac{1}{t} = \frac{x}{a}$ $\qquad ③$

From ②: $y = a\left(t^2 - 2 + \frac{1}{t^2}\right) + 3a$

$$= a\left(t - \frac{1}{t}\right)^2 + 3a$$
$$\therefore\ y = a\left(\frac{x}{a}\right)^2 + 3a \qquad \text{from ③}$$
$$\therefore\ y = \frac{x^2}{a} + 3a \quad \text{which is a parabola}$$

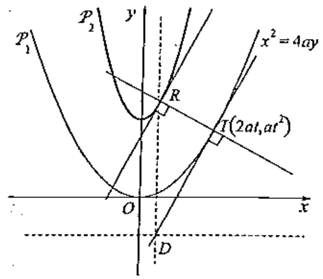
$\therefore$ the locus of R lies on another parabola, $\mathcal{P}_2$

(iii) $\mathcal{P}_2$: $y = \frac{x^2}{a} + 3a$

$$ay = x^2 + 3a^2$$
$$x^2 = ay - 3a^2$$
$$\therefore\ x^2 = a(y - 3a)$$
$$\therefore\ x^2 = 4\left(\frac{a}{4}\right)(y - 3a)$$

$\therefore$ the focal length of $\mathcal{P}_2$ is $\frac{a}{4}$

(iv) If the normal to $\mathcal{P}_1$ at T is also the normal to $\mathcal{P}_2$ at R then the gradient of the tangent to $\mathcal{P}_1$ at T and the gradient of the tangent to $\mathcal{P}_2$ at R must be equal.



For $\mathcal{P}_1$: $y = \frac{x^2}{4a} \Rightarrow \frac{dy}{dx} = \frac{x}{2a}$

$\therefore$ the gradient of the tangent to $\mathcal{P}_1$ at $T(2at, at^2)$ is $m_1 = t$.

For $\mathcal{P}_2$: $y = \frac{x^2}{a} + 3a \Rightarrow \frac{dy}{dx} = \frac{2x}{a}$

$\therefore$ the gradient of the tangent to $\mathcal{P}_2$ at $R\left(at - \frac{a}{t}, at^2 + 1 + \frac{a}{t^2}\right)$ is $m_2 = \frac{2}{a}\left(at - \frac{a}{t}\right)$

Now $m_1 = m_2$: $\quad \therefore\ t = \frac{2}{a}\left(at - \frac{a}{t}\right)$

$$t = 2t - \frac{2}{t}$$
$$t^2 = 2t^2 - 2$$
$$t^2 = 2$$
$$t = \pm\sqrt{2}$$

$\therefore$ when $t = \pm\sqrt{2}$ the distance between R and T is a minimum.

End of solutions