BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2016 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 7–18

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

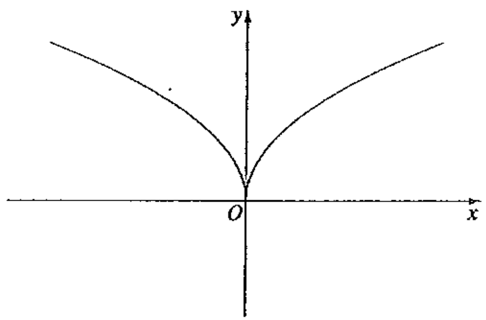
1260

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1 Which equation best represents the following graph?



(A) $y=\sqrt{x}$

(B) $|y|=\sqrt{x}$

(C) $y=\sqrt{|x|}$

(D) $|y|=\sqrt{|x|}$

2 Which polynomial has a multiple root at $x=1$?

(A) $x^5-x^4-x^2+1$

(B) x^5-x^4-x-1

(C) $x^5-x^3-x^2+1$

(D) x^5-x^3-x+1

3 The sum of the eccentricities of two different conics is $\frac{3}{4}$.

Which pair of conics could this be?

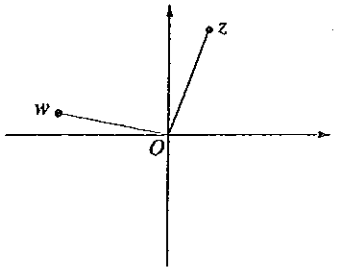
(A) Circle and ellipse

(B) Ellipse and parabola

(C) Parabola and hyperbola

(D) Hyperbola and circle

4 The Argand diagram shows the complex numbers z and w, where z lies in the first quadrant and w lies in the second quadrant.



Which complex number could lie in the 3rd quadrant?

(A) $-w$

(B) $2iz$

(C) $\overline{z}$

(D) $w-z$

5 Multiplying a non-zero complex number by $\frac{1-i}{1+i}$ results in a rotation about the origin on an Argand diagram.

What is the rotation?

(A) Clockwise by $\frac{\pi}{4}$

(B) Clockwise by $\frac{\pi}{2}$

(C) Anticlockwise by $\frac{\pi}{4}$

(D) Anticlockwise by $\frac{\pi}{2}$

6 Let $p(x)=1+x+x^2+x^3+\cdots+x^{12}$.

What is the coefficient of x^8 in the expansion of $p(x+1)$?

(A) 1

(B) 495

(C) 715

(D) 1287

7 The hyperbola with equation $xy=8$ is the hyperbola $x^2-y^2=k$ referred to different axes.

What is the value of k?

(A) 2

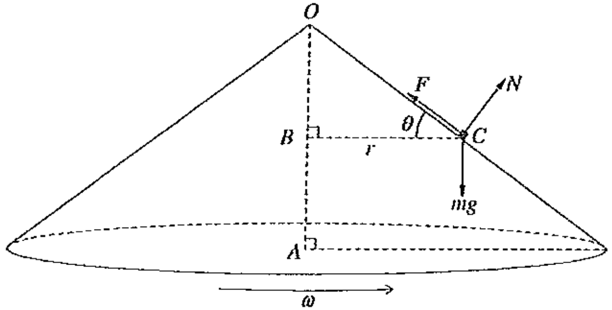
(B) 4

(C) 8

(D) 16

8 A small object of mass m kg sits on a rotating conical surface at C, r metres from the axis OA and with $\angle OCB = \theta$, as shown in the diagram.

The surface is rotating about its axis with angular velocity ω. The forces acting on the object are gravity, a normal reaction force N and a frictional force F, which prevents the object from sliding down the surface.



Which pair of statements is correct?

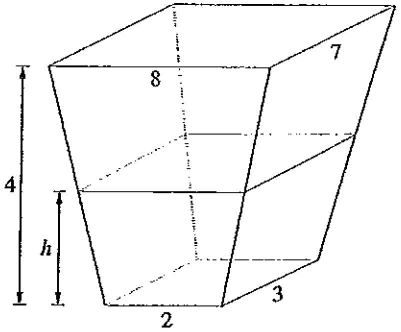
(A) $F\cos\theta + N\sin\theta = mr\omega^2$
$F\sin\theta + N\cos\theta = mg$

(B) $F\cos\theta + N\sin\theta = mr\omega^2$
$F\sin\theta - N\cos\theta = mg$

(C) $F\cos\theta - N\sin\theta = mr\omega^2$
$F\sin\theta + N\cos\theta = mg$

(D) $F\cos\theta - N\sin\theta = mr\omega^2$
$F\sin\theta - N\cos\theta = mg$

9 The diagram shows the dimensions of a polyhedron with parallel base and top. A slice taken at height h parallel to the base is a rectangle.



What is a correct expression for the volume of the polyhedron?

(A) $\displaystyle\int_0^4 (h+3)\left(\frac{3h}{2}+2\right) dh$

(B) $\displaystyle\int_0^4 \left(\frac{5h}{4}+3\right)\left(\frac{3h}{2}+2\right) dh$

(C) $\displaystyle\int_0^4 (h+3)\left(\frac{5h}{4}+2\right) dh$

(D) $\displaystyle\int_0^4 \left(\frac{5h}{4}+3\right)\left(\frac{5h}{4}+2\right) dh$

10 Suppose that $x + \frac{1}{x} = -1$.

What is the value of $x^{2016} + \frac{1}{x^{2016}}$?

(A) 1

(B) 2

(C) $\frac{2\pi}{3}$

(D) $\frac{4\pi}{3}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = \sqrt{3} - i$.

(i) Express z in modulus–argument form. **2**

(ii) Show that z^6 is real. **1**

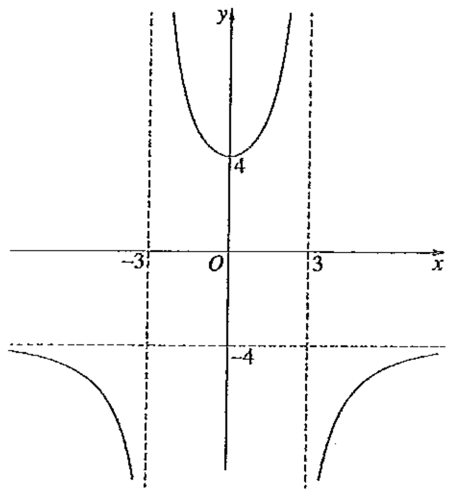
(iii) Find a positive integer n such that z^n is purely imaginary. **1**

(b) Find $\int x e^{-2x}\, dx$. **3**

(c) Find $\frac{dy}{dx}$ for the curve given by $x^3 + y^3 = 2xy$, leaving your answer in terms of x and y. **2**

Question 11 continues on page 8

(d) The diagram shows the graph of $y = f(x)$.



Draw a separate half-page diagram for each of the following functions, showing all asymptotes and intercepts.

(i) $y = \sqrt{f(x)}$ **2**

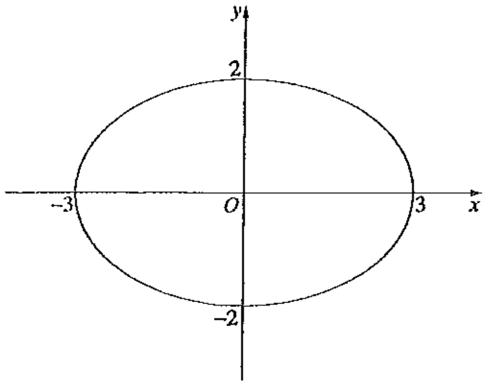
(ii) $y = \dfrac{1}{f(x)}$ **2**

(e) State the domain and range of the function $f(x) = x\sin^{-1}\left(\dfrac{x}{2}\right)$. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows an ellipse.



(i) Write an equation for the ellipse. **1**

(ii) Find the eccentricity of the ellipse. **1**

(iii) Write the coordinates of the foci of the ellipse. **1**

(iv) Write the equations of the directrices of the ellipse. **1**

(b) (i) Differentiate $xf(x) - \int x f'(x)\,dx$. **1**

(ii) Hence, or otherwise, find $\int \tan^{-1} x\,dx$. **2**

Question 12 continues on page 10

(c) Let $z = \cos\theta + i\sin\theta$.

(i) By considering the real part of z^4, show that $\cos 4\theta$ is **2**

$$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

(ii) Hence, or otherwise, find an expression for $\cos 4\theta$ involving only powers of $\cos\theta$. **1**

(d) (i) Show that the equation of the normal to the hyperbola $xy = c^2$, $c \neq 0$, at $P\left(cp, \frac{c}{p}\right)$ is given by $px - \frac{y}{p} = c\left(p^2 - \frac{1}{p^2}\right)$. **2**

(ii) The normal at P meets the hyperbola again at $Q\left(cq, \frac{c}{q}\right)$. **3**

Show that $q = -\frac{1}{p^3}$.

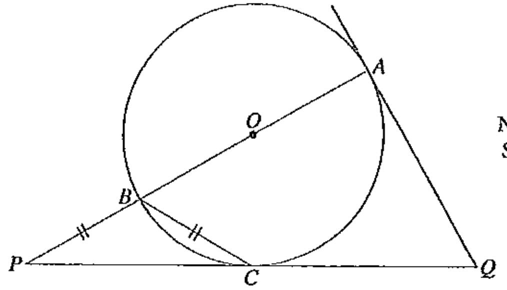
End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The function $f(x) = x^x$ is defined and positive for all $x > 0$. **3**

By differentiating $\ln(f(x))$, find the value of x at which $f(x)$ has a minimum.

(b) The circle centred at O has diameter AB. A point P on AB produced is chosen so that PC is a tangent to the circle at C and $BP = BC$. The tangents to the circle at A and C meet at Q. **4**



NOT TO SCALE

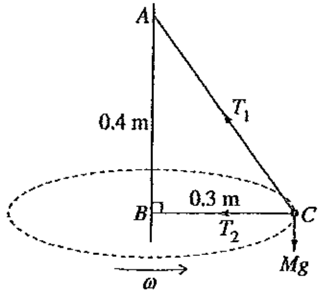
Copy or trace the diagram into your writing booklet.

Prove that $OP = OQ$.

Question 13 continues on page 12

(c) The ends of a string are attached to points A and B, with A directly above B. The points A and B are 0.4 m apart.

An object of mass M kg is fixed to the string at C. The object moves in a horizontal circle with centre B and radius 0.3 m, as shown in the diagram.



The tensions in the string from the object to points A and B are T_1 and T_2 respectively. The object rotates with constant angular velocity ω. You may assume that the acceleration due to gravity is $g = 10\text{ m s}^{-2}$.

(i) Show that $T_2 = 0.3M\left(\omega^2 - 25\right)$. **3**

(ii) For what range of values of ω is $T_2 > T_1$? **1**

(d) Suppose $p(x) = ax^3 + bx^2 + cx + d$ with a, b, c and d real, $a \neq 0$.

(i) Deduce that if $b^2 - 3ac < 0$ then $p(x)$ cuts the x-axis only once. **2**

(ii) If $b^2 - 3ac = 0$ and $p\left(-\dfrac{b}{3a}\right) = 0$, what is the multiplicity of the root $x = -\dfrac{b}{3a}$? **2**

End of Question 13

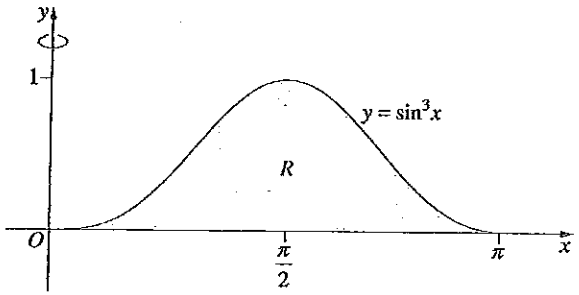
Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\int \sin^3 x\,dx = \frac{1}{3}\cos^3 x - \cos x + C$. **1**

(ii) Using a graphical approach, or otherwise, explain why **1**

$\int_0^{\pi} \cos^{2n-1} x\,dx = 0$, for all positive integers n.

(iii) The diagram shows the region R enclosed by $y = \sin^3 x$ and the x-axis for $0 \le x \le \pi$. **3**



Using the method of cylindrical shells and the results in parts (i) and (ii), find the exact volume of the solid formed when R is rotated about the y-axis.

Question 14 continues on page 14

(b) Let $I_n = \int_0^1 \frac{x^n}{(x^2+1)^2}\,dx$, for $n = 0, 1, 2, \ldots$.

(i) Using a suitable substitution, show that $I_0 = \frac{\pi}{8} + \frac{1}{4}$. **3**

(ii) Show that $I_0 + I_2 = \frac{\pi}{4}$. **1**

(iii) Find I_4. **3**

(c) Show that $x\sqrt{x} + 1 \ge x + \sqrt{x}$, for $x \ge 0$. **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $x^3 - 3x + 1 = 0$ has roots α, β and γ. **2**

Find a cubic equation with integer coefficients that has roots α^2, β^2 and γ^2.

(b) A particle is initially at rest at the point B which is b metres to the right of O. The particle then moves in a straight line towards O.

For $x \neq 0$, the acceleration of the particle is given by $\dfrac{-\mu^2}{x^2}$, where x is the distance from O and μ is a positive constant.

(i) Prove that $\dfrac{dx}{dt} = -\mu\sqrt{2}\sqrt{\dfrac{b-x}{bx}}$. **2**

(ii) Using the substitution $x = b\cos^2\theta$, show that the time taken to reach a distance d metres to the right of O is given by **3**

$$t = \frac{b\sqrt{2b}}{\mu}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \cos^2\theta \, d\theta.$$

It can be shown that $t = \dfrac{1}{\mu}\sqrt{\dfrac{b}{2}}\left(\sqrt{bd-d^2} + b\cos^{-1}\sqrt{\dfrac{d}{b}}\right)$. (Do NOT prove this.)

(iii) What is the limiting time taken for the particle to reach O? **1**

Question 15 continues on page 16

(c) (i) Use partial fractions to show that **2**

$$\frac{3!}{x(x+1)(x+2)(x+3)} = \frac{1}{x} - \frac{3}{x+1} + \frac{3}{x+2} - \frac{1}{x+3}.$$

(ii) Suppose that for n a positive integer **3**

$$\frac{n!}{x(x+1)\dots(x+n)} = \frac{a_0}{x} + \frac{a_1}{x+1} + \dots + \frac{a_k}{x+k} + \dots + \frac{a_n}{x+n}.$$

Show that $a_k = (-1)^k\dbinom{n}{k}$.

(iii) Hence, or otherwise, find the limiting sum of **2**

$$1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \frac{1}{4}\binom{n}{3} + \dots + \frac{(-1)^n}{n+1}.$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) The complex numbers $z = \cos\theta + i\sin\theta$ and $w = \cos\alpha + i\sin\alpha$, where $-\pi < \theta \le \pi$ and $-\pi < \alpha \le \pi$, satisfy **3**

$$1 + z + w = 0.$$

By considering the real and imaginary parts of $1 + z + w$, or otherwise, show that 1, z and w form the vertices of an equilateral triangle in the Argand diagram.

(ii) Hence, or otherwise, show that if the three non-zero complex numbers $2i$, z_1 and z_2 satisfy **2**

$$|2i| = |z_1| = |z_2| \quad \text{AND} \quad 2i + z_1 + z_2 = 0$$

then they form the vertices of an equilateral triangle in the Argand diagram.

(b) (i) The complex numbers 0, u and v form the vertices of an equilateral triangle in the Argand diagram. **2**

Show that $u^2 + v^2 = uv$.

(ii) Give an example of non-zero complex numbers u and v, so that 0, u and v form the vertices of an equilateral triangle in the Argand diagram. **1**

Question 16 continues on page 18

(c) In a group of n people, each has one hat, giving a total of n different hats. They place their hats on a table. Later, each person picks up a hat, not necessarily their own.

A situation in which none of the n people picks up their own hat is called a derangement.

Let $D(n)$ be the number of possible derangements.

(i) Tom is one of the n people. In some derangements Tom finds that he and one other person have each other's hat. **1**

Show that, for $n > 2$, the number of such derangements is $(n-1)D(n-2)$.

(ii) By also considering the remaining possible derangements, show that, for $n > 2$, **2**

$$D(n) = (n-1)\left[D(n-1) + D(n-2)\right].$$

(iii) Hence, show that $D(n) - nD(n-1) = -\left[D(n-1) - (n-1)D(n-2)\right]$, for $n > 2$. **1**

(iv) Given $D(1) = 0$ and $D(2) = 1$, deduce that $D(n) - nD(n-1) = (-1)^n$, for $n > 1$. **1**

(v) Prove by mathematical induction, or otherwise, that for all integers $n \ge 1$, $D(n) = n!\sum_{r=0}^{n} \frac{(-1)^r}{r!}$. **2**

End of paper

2016 Higher School Certificate Solutions Mathematics Extension 2

Section I

Multiple Choice Summary

1 C	2 C	3 A	4 D	5 B
6 C	7 D	8 C	9 A	10 B

Multiple Choice Solutions

1. C

Method 1

Only $y=\sqrt{|x|}$ will have positive and negative values of x and allow only positive values for y.

Method 2

The graph is an even function, constructed by reflecting $y=\sqrt{x}$ in the y-axis. Hence B or D.

But the transformation $|y|=\sqrt{|x|}$ would also reflect in the x-axis. Hence it cannot be D.

2. C

Method 1

For a multiple root at $x=1$ both $P(1)=0$ and $P'(1)=0$.

$P(1)=0$ only for A, C and D

i.e. all have $x=1$ as a root.

For A: $P'(x)=5x^4-4x^3-2x$

$P'(1)\neq 0 \Rightarrow$ not A

For C: $P'(x)=5x^4-3x^2-2x$

$P'(1)=0 \Rightarrow$ C is correct

Method 2

C has the only one with the right pattern of indices to factorise:

$$x^5-x^3-x^2+1=x^3(x^2-1)-1(x^2-1)$$
$$=(x^3-1)(x^2-1)$$
$$=(x+1)(x^2+x+1)(x-1)^2$$

3. A

A circle has $e=0$; an ellipse $0<e<1$; a parabola has $e=1$ and a hyperbola has $e>1$
For the sum of two eccentricities to be a proper fraction the conics must both be ellipses or a circle and an ellipse.

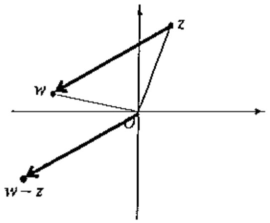
4. D

$-w$ lies in quadrant 4 as it is the reflection of w in both axes.

$2iz$ lies in quadrant 2 as multiplication by $2i$ rotates the vector $\frac{\pi}{2}$ radians anticlockwise about the origin.

$\bar{z}$ lies in quadrant 4 as it is the reflection of z in the real axis.

$w-z$ is illustrated in the diagram.



5. B

Method 1

$$\frac{1-i}{1+i}=\frac{1-i}{1+i}\times\frac{1-i}{1-i}$$
$$=\frac{1-2i-1}{1+1}$$
$$=-i \quad \Rightarrow \text{clockwise rotation of } \tfrac{\pi}{2}$$

Method 2

$$\frac{1-i}{1+i}=\frac{\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)}{\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)}$$
$$=\operatorname{cis}\left(-\tfrac{\pi}{2}\right)$$

6. C

$$p(1+x)=1+(1+x)+(1+x)^2+\ldots+(1+x)^8+(1+x)^9+\ldots+(1+x)^{12}$$

Terms in x^8 only arise from the expansions of $(1+x)^8,(1+x)^9,\ldots,(1+x)^{12}$.

$$\therefore \text{ the coefficient of } x^3=\binom{8}{8}+\binom{9}{8}+\binom{10}{8}+\binom{11}{8}+\binom{12}{8}$$
$$=1+9+45+165+495$$
$$=715$$

7. D

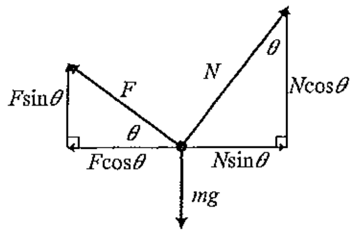
$xy=c^2$ is equivalent to $x^2-y^2=a^2$ by rotation where $c^2=\frac{a^2}{2}$.

But $c^2=8 \Rightarrow a^2=2c^2=16$

$\therefore k=16$

8. C

Resolve both N and F into vertical and horizontal components as illustrated in the diagram.



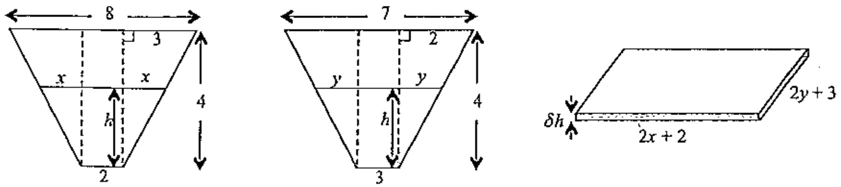
Vertically the object is in equilibrium and so the sum of the vertical components of forces is zero.

$$\therefore F\sin\theta+N\cos\theta-mg=0$$
$$F\sin\theta+N\cos\theta=mg$$

Horizontally there is a net centripetal force directed towards the centre, B.

$$\therefore F\cos\theta-N\sin\theta=mr\omega^2$$

9. A



By similar triangles: $\frac{x}{h}=\frac{3}{4} \Rightarrow x=\frac{3h}{4}$

and $\frac{y}{h}=\frac{2}{4} \Rightarrow y=\frac{h}{2}$

$$\therefore \Delta A=(2x+2)(2y+3)$$
$$=\left(\frac{3h}{2}+2\right)(h+3)$$

10. B

Method 1

$$x+\frac{1}{x}=-1$$
$$x^2+x+1=0$$
$$x=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm i\sqrt{3}}{2}$$
$$\therefore\ x=\text{cis}\left(\pm\tfrac{2\pi}{3}\right)$$

Now $x^{2016}=\left[\text{cis}\left(\pm\tfrac{2\pi}{3}\right)^3\right]^{672}=1$

$$\therefore\ \frac{1}{x^{2016}}=1$$
$$\therefore\ x^{2016}+\frac{1}{x^{2016}}=2$$

Method 2

$$x+\frac{1}{x}=-1$$
$$x^2+x+1=0$$
$$(x-1)(x^2+x+1)=0,\ x\neq 1$$
$$\therefore\ x^3-1=0$$
$$x^3=1$$

i.e. x is a non-real cube root of 1

Now $x^{2016}=(x^3)^{672}=1$

$$\therefore\ \frac{1}{x^{2016}}=1$$
$$\therefore\ x^{2016}+\frac{1}{x^{2016}}=2$$

Section II

Question 11

(a) (i) $|z|=\sqrt{(\sqrt{3})^2+1}=2$ Let $\arg z=\theta$, noting that z lies in quadrant 4.

$\tan\theta=-\frac{1}{\sqrt{3}}$, for $-\frac{\pi}{2}<\theta<0$ i.e. $\theta=-\frac{\pi}{6}$

$$\therefore \sqrt{3}-i=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$$

(ii) $$z^6=\left[2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)\right]^6$$
$$=2^6\left[\cos\left(-\frac{\pi}{6}\times 6\right)+i\sin\left(-\frac{\pi}{6}\times 6\right)\right]\quad\text{[de Moivre's Thm]}$$
$$=64\left(\cos(-\pi)+i\sin(-\pi)\right)$$
$$=-64$$ which is real as required

(iii) $$z^n=\left[2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)\right]^n=2^n\left(\cos\left(-\frac{n\pi}{6}\right)+i\sin\left(-\frac{n\pi}{6}\right)\right)\quad\text{[de Moivre's Thm]}$$

To be purely imaginary then $2^n\cos\left(-\frac{n\pi}{6}\right)=0$ i.e. $\cos\left(-\frac{n\pi}{6}\right)=0$

$$\therefore \frac{n\pi}{6}=2k\pi\pm\frac{\pi}{2},\qquad k\in\mathbb{Z}$$
$$n\pi=12k\pi\pm 3\pi$$
$$n=12k\pm 3$$

$\therefore\ n=3$ is one such integer.

(b) $$\int \underbrace{x}_{u}\underbrace{e^{-2x}}_{dv}\,dx=\underbrace{x}_{u}\times\underbrace{\left(-\frac{e^{-2x}}{2}\right)}_{v}-\int\underbrace{1}_{du}\times\underbrace{\left(-\frac{e^{-2x}}{2}\right)}_{v}dx$$
$$=-\frac{xe^{-2x}}{2}+\int\left(\frac{e^{-2x}}{2}\right)dx$$
$$=-\frac{xe^{-2x}}{2}-\frac{e^{-2x}}{4}+C$$

(c) $x^3+y^3=2xy$

Differentiating with respect to x: $3x^2+3y^2\frac{dy}{dx}=2x\frac{dy}{dx}+2y$

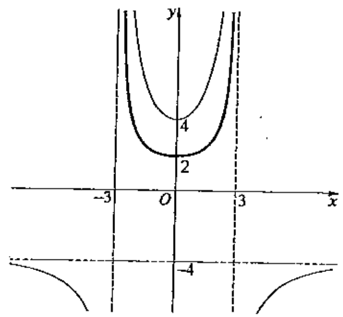
$$\frac{dy}{dx}(3y^2-2x)=2y-3x^2$$
$$\frac{dy}{dx}=\frac{2y-3x^2}{3y^2-2x}$$

(d) (i) $y=\sqrt{f(x)}$ is only defined for $f(x)\geq 0$

$\therefore$ the domain of $y=\sqrt{f(x)}$ is $-3<x<3$.

When $f(x)\geq 1$ then $\sqrt{f(x)}\leq f(x)$.

If $f'(\alpha)=0$, where $f(\alpha)\geq 0$, then there is a stationary point at $y=\sqrt{f(\alpha)}$.



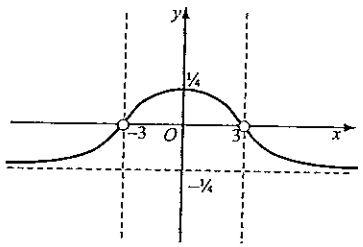
(ii) When taking a reciprocal, the domain is not altered.

For $y=f(x)$ the domain is $x\neq\pm 3$.

$\therefore$ for $y=\frac{1}{f(x)}$ the domain is $x\neq\pm 3$.

The equation of the asymptote changes.

The minimum stationary point becomes a maximum stationary point.



The gradient function of $\frac{1}{f(x)}$ is $\frac{-f'(x)}{[f(x)]^2}$ and hence has the opposite sign to $f(x)$.

Note: Because $y=f(x)$ is even then both $y=\sqrt{f(x)}$ and $y=\frac{1}{f(x)}$ will be even.

(e) $f(x) = x\sin^{-1}\left(\frac{x}{2}\right)$

Domain: $-1 \le \frac{x}{2} \le 1$

i.e. $-2 \le x \le 2$

The product of two odd functions is even.

For $0 \le x \le 2$, then $x \ge 0$ and $\sin^{-1}\left(\frac{x}{2}\right) \ge 0$ and so $f(x) \ge 0$.

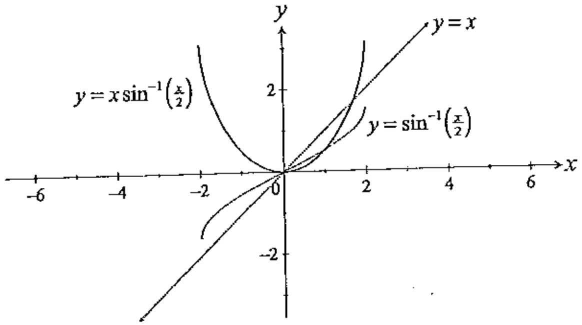
$\therefore f(x) \ge 0$ for $-2 \le x \le 2$.

Now $y = x$ and $y = \sin^{-1}\left(\frac{x}{2}\right)$ are increasing functions and hence the maximum value of $f(x)$ will occur when $x = 2$.

Range: $0 \le f(x) \le 2 \times \frac{\pi}{2}$

i.e. $0 \le y \le \pi$

This is illustrated in the diagram below:



Question 12

(a)

(i) $a = 3, b = 2 \quad \therefore \frac{x^2}{9} + \frac{y^2}{4} = 1$

(ii) $e^2 = 1 - \left(\frac{b}{a}\right)^2$

$= 1 - \frac{4}{9}$

$= \frac{5}{9}$

$e = \frac{\sqrt{5}}{3}$ as $e > 0$

(iii) The foci are at $(\pm ae, 0)$

i.e. $(\pm\sqrt{5}, 0)$.

(iv) The directrices are $x = \pm\frac{a}{e}$

i.e. $x = \pm\frac{9}{\sqrt{5}} = \pm\frac{9\sqrt{5}}{5}$

(b) (i) **Method 1**

$\frac{d}{dx}\left[x\,f(x) - \int x\,f'(x)\,dx\right]$

$= \frac{d}{dx}\left[xf(x)\right] - \frac{d}{dx}\left[\int x\,f'(x)\,dx\right]$

$= x\,f'(x) + f(x) - x\,f'(x)$

$= f(x)$

Method 2

$\int u\,dv = uv - \int v\,du$ where $u = f(x), v = x$

$\therefore xf(x) - \int xf'(x)\,dx = \int f(x).1\,dx$

$\therefore \frac{d}{dx}\left[x\,f(x) - \int x\,f'(x)\,dx\right] = f(x)$

(ii) From (i), $\int f(x)\,dx = x\,f(x) - \int x\,f'(x)\,dx$ and let $f(x) = \tan^{-1}x$:

$\therefore \int \tan^{-1}x\,dx = x\tan^{-1}x - \int x\left(\frac{1}{1+x^2}\right)dx$

$= x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx$

$= x\tan^{-1}x - \frac{1}{2}\ln\left(1+x^2\right) + C$

(c) (i) Let $c = \cos\theta$ and $s = \sin\theta$

$z^4 = (c + is)^4$

$= c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4$

$= c^4 - 6c^2s^2 + s^4 + i\left(4c^3s - 4cs^3\right)$

But by de Moivre's theorem: $z^4 = \cos 4\theta + i\sin 4\theta$

$\therefore \operatorname{Re}\left(z^4\right) = \cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(ii) From part (i): $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$\therefore \cos 4\theta = \cos^4\theta - 6\cos^2\theta\left(1-\cos^2\theta\right) + \left(1-\cos^2\theta\right)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

(d) (i) $xy = c^2 \Rightarrow y = c^2x^{-1}$

$\therefore \frac{dy}{dx} = -\frac{c^2}{x^2}$

At P:

$\frac{dy}{dx} = -\frac{c^2}{c^2p^2}$

$= -\frac{1}{p^2}$

$\therefore$ gradient of the normal at P is p^2

Equation of normal:

$$y-\frac{c}{p}=p^2(x-cp)$$

$$\frac{y}{p}-\frac{c}{p^2}=px-cp^2$$

$$px+\frac{y}{p}=cp^2-\frac{c}{p^2}$$

$$px+\frac{y}{p}=c\left(p^2-\frac{1}{p^2}\right) \quad \text{as required}$$

(ii) **Method 1**

Solving $y=\frac{c^2}{x}$ and $px+\frac{y}{p}=c\left(p^2-\frac{1}{p^2}\right)$ simultaneously:

$$px+\frac{c^2}{px}=c\left(p^2-\frac{1}{p^2}\right)$$

$$p^2x^2-cpx\left(p^2-\frac{1}{p^2}\right)+c^2=0$$

The roots of this equation are the x-values of the points P and Q: cp and cq

Using the product of the roots: **OR** Using the sum of the roots:

$$cpcq=-\frac{c^2}{p^2}$$

$$q=-\frac{1}{p^3}$$

$$cp+cq=\frac{pc\left(p^2-\frac{1}{p^2}\right)}{p^2}$$

$$p+q=p-\frac{1}{p^3}$$

$$q=-\frac{1}{p^3}$$

Method 2

Because the normal meets the hyperbola, $y=\frac{c^2}{x}$, again at $Q\left(cq,\frac{c}{q}\right)$, the coordinates of Q must satisfy the equation of the normal.

$$\therefore\ p(cq)-\frac{\left(\frac{c}{q}\right)}{p}=c\left(p^2-\frac{1}{p^2}\right)$$

$$cpq-\frac{c}{pq}=c\left(p^2-\frac{1}{p^2}\right)$$

$$pq-\frac{1}{pq}=p^2-\frac{1}{p^2}$$

$$pq-p^2=\frac{1}{pq}-\frac{1}{p^2}$$

$$pq-p^2=\frac{p^2-pq}{p^3q}$$

$$\therefore\ p(q-p)=\frac{p(p-q)}{p^3q}$$

$$-1=\frac{1}{p^3q}$$

$$\therefore\ q=-\frac{1}{p^3} \quad \text{as required}$$

Method 3

$$m_{PQ}=\frac{\frac{c}{p}-\frac{c}{q}}{cp-cq}$$

$$=\frac{c(q-p)}{cpq(p-q)}$$

$$=\frac{-1}{pq}$$

But the gradient of the normal $=p^2$

$$\therefore\ p^2=\frac{-1}{pq}$$

$$q=\frac{-1}{p^3}$$

Question 13

(a) $f(x)=x^x$

Then $\ln f(x)=\ln(x^x)$

$$=x\ln x$$

$$\frac{d}{dx}(\ln f(x))=\frac{d}{dx}(x\ln x) \qquad \text{and} \qquad \frac{d^2}{dx^2}(\ln f(x))=\frac{1}{x}$$

$$=x\left(\frac{1}{x}\right)+\ln x$$

$$=1+\ln x$$

Now $\ln x$ is a strictly increasing function and so the the minimum value of $\ln f(x)$ will occur at the same value of x of the minimum of $f(x)$. Hence, to find the value of x that produces the minimum value of $f(x)=x^x$ we need to find the stationary point of $\ln f(x)$.

Stationary points of $\ln f(x)$ occur when $\frac{d}{dx}(\ln f(x))=0$

$$\therefore\ 1+\ln x=0$$

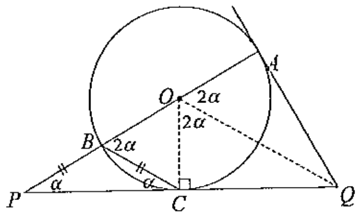
$$\ln x=-1$$

$$x=e^{-1}$$

Substituting into the second derivative: $\frac{1}{e^{-1}}>0$

$\therefore$ there is a minimum of $f(x)$ at $x=e^{-1}$.

(b)



Join OC and OQ

$OA = OB = OC$ [radii of same circle]

$PB = BC$ [given]

$\therefore \triangle BCP$ and $\triangle OBC$ is isosceles [two equal sides]

Let $\angle BPC = \alpha$ ①

$\therefore \angle BCP = \alpha$ [angles opposite equal sides in isosceles $\triangle BCP$]

$\therefore \angle OBC = 2\alpha$ [exterior angle of $\triangle BPC$]

$\therefore \angle OCB = 2\alpha$ [angles opposite equal sides in isosceles $\triangle OBC$]

Now $\angle OCP = 90°$ [angle between tangent and radius]

But $\angle OCP = \angle OCB + \angle BCP$ [adjacent angles]

$= 2\alpha + \alpha$

$= 3\alpha$

$\therefore 3\alpha = 90°$

$\alpha = 30°$

Now $AQ = CQ$ [tangents from external point]

And $OA = OC$ [radii]

$\therefore OAQC$ is a kite [two pairs of adjacent sides equal]

But $\angle AOC = 4\alpha$ [exterior angle $\triangle OBC$]

$\therefore \angle COQ = \angle QOA = 2\alpha$ [OQ diagonal of kite]

Now $\angle OQC + \angle COQ + \angle OCQ = 180°$ [angle sum $\triangle OCQ$]

$\therefore \angle OQC + 2\alpha + 90° = 180°$

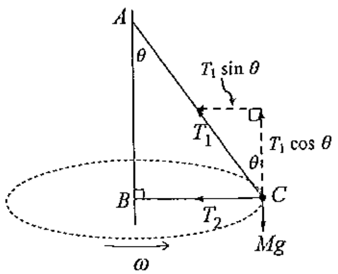
$\angle OQC = \alpha$

But $\angle BPC = \alpha$ [from ① above]

$\therefore \angle OQC = \angle BPC$

$\therefore OP = OQ$ [sides opposite equal angles in $\triangle OPQ$]

(c) (i) Let $\angle BAC = \theta$.



$AC = 0.5$ m [Pythagorean triad]

$\therefore \sin\theta = \dfrac{0.3}{0.5} = 0.6$ and

$\cos\theta = \dfrac{0.4}{0.5} = 0.8$

Resolving vertically: $T_1 \cos\theta = 10M$

$\therefore T_1 = \frac{1}{0.8} \times 10M$

$T_1 = 12.5M$

The net centripetal force is towards B.

Resolving horizontally: $T_1 \sin\theta + T_2 = M\omega^2 \times 0.3$

$$\therefore 12.5M \times 0.6 + T_2 = 0.3M\omega^2$$

$$7.5M + T_2 = 0.3M\omega^2$$

$$T_2 = 0.3M\omega^2 - 7.5M$$

$$\therefore T_2 = 0.3M(\omega^2 - 25) \quad \text{as required}$$

(ii) $T_2 > T_1$

$$\therefore 0.3M(\omega^2 - 25) > 12.5M$$

$$0.3(\omega^2 - 25) > 12.5$$

$$\omega^2 - 25 > \frac{125}{3}$$

$$\omega^2 > \frac{200}{3}$$

$$\therefore \omega > \sqrt{\frac{200}{3}} \quad \text{as } \omega > 0$$

$$\omega > 8.164\ldots$$

$$\omega > 8.2 \quad \text{(2 sig. fig.)}$$

Note: $g \doteqdot 10$ ms^{-2} and so an exact answer is unreasonable.

(d) $p(x) = ax^3 + bx^2 + cx + d$

(i) $p(x)$ is odd and so it will cut the x-axis at least once.

$p'(x) = 3ax^2 + 2bx + c$

For stationary points $p'(x) = 0 \Rightarrow 3ax^2 + 2bx + c = 0$

$$\Delta = 4b^2 - 12ac$$

$$= 4(b^2 - 3ac)$$

If $b^2 - 3ac < 0$ then $\Delta < 0$ and so there would be no real solutions to $p'(x) = 0$.

i.e. the cubic has no stationary points and is always increasing or decreasing for all x.

$\therefore p(x)$ will cut the x-axis only once.

(ii) **Method 1**

If $\Delta = 0$, then $p'(x) = 0$ has a double root at $x = \dfrac{-2b \pm \sqrt{0}}{6a} = -\dfrac{b}{3a}$.

But given that $p\left(-\dfrac{b}{3a}\right) = 0$, then $x = -\dfrac{b}{3a}$ is a triple zero for $y = p(x)$.

i.e. $x = -\dfrac{b}{3a}$ has multiplicity 3.

Method 2

If $\Delta = 0$ then $b^2 = 3ac \Rightarrow c = \dfrac{b^2}{3a}$ and $p(x) = ax^3 + bx^2 + \dfrac{b^2}{3a}x + d$

Given $p\left(-\dfrac{b}{3a}\right) = 0$: $\quad a\left(-\dfrac{b}{3a}\right)^3 + b\left(-\dfrac{b}{3a}\right)^2 + \dfrac{b^2}{3a}\left(-\dfrac{b}{3a}\right) + d = 0$

$$-\frac{ab^3}{27a^3} + \frac{b^3}{9a^2} - \frac{b^3}{9a^2} + d = 0$$

$$d = \frac{b^3}{27a^2}$$

$\therefore\ p(x) = ax^3 + bx^2 + \dfrac{b^2}{3a}x + \dfrac{b^3}{27a^2}$

$27a^2.p(x) = 27a^3x^3 + 27a^2bx^2 + 9ab^2x + b^3 = (3ax+b)^3$

$\therefore\ p(x)$ has a triple root at $x = -\dfrac{b}{3a}$

Question 14

(a) (i)

Method 1

$$\int \sin^3 x\, dx$$

$$= \int \sin x \sin^2 x\, dx$$

$$= \int \sin x\left(1-\cos^2 x\right) dx$$

$$= \int \left(\sin x - \sin x\cos^2 x\right) dx$$

$$= -\cos x + \frac{1}{3}\cos^3 x + C$$

Method 2

$$\frac{d}{dx}\left(\frac{1}{3}\cos^3 x - \cos x + C\right)$$

$$= -\cos^2 x \sin x + \sin x$$

$$= \sin x\left(1-\cos^2 x\right)$$

$$= \sin x \sin^2 x$$

$$= \sin^3 x$$

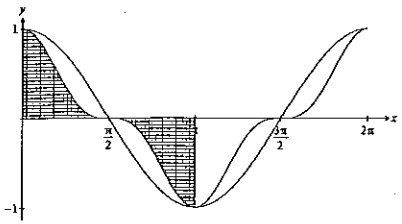
$$\therefore \int \sin^3 x\, dx = \frac{1}{3}\cos^3 x - \cos x + C$$

(ii) $y = \cos x$ for $0 \le x \le \pi$ has point symmetry about $x = \frac{\pi}{2}$. Multiples of odd functions will result in a function that is also odd. Hence $y = \cos^{2n-1} x$ for $0 \le x \le \pi$ also has point symmetry about $x = \frac{\pi}{2}$. This is illustrated in the diagram.

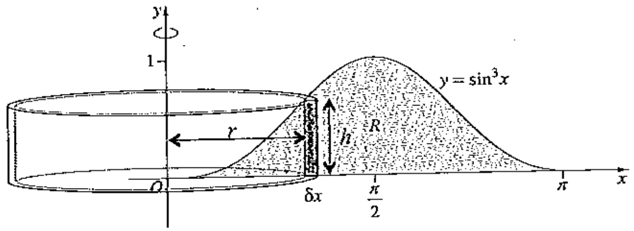
$$\therefore \int_0^{\frac{\pi}{2}} \cos^{2n-1} x\, dx = -\int_{\frac{\pi}{2}}^{\pi} \cos^{2n-1} x\, dx$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^{2n-1} x\, dx + \int_{\frac{\pi}{2}}^{\pi} \cos^{2n-1} x\, dx = 0$$

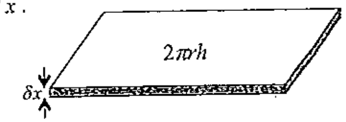
$$\therefore \int_0^{\pi} \cos^{2n-1} x\, dx = 0$$



(iii)



The illustrated shell has $r = x$ and $h = f(x) = \sin^3 x$.



$$\delta V \approx 2\pi rh.\delta x$$

$$= 2\pi x \sin^3 x.\delta x$$

$$\therefore V = \lim_{\delta x \to 0} \sum_{x=0}^{\pi} 2\pi x \sin^3 x\, \delta x$$

$$= \int_0^{\pi} 2\pi x \sin^3 x\, dx$$

$$V = 2\pi \int_0^{\pi} x \frac{d}{dx}\left(\frac{1}{3}\cos^3 x - \cos x\right) dx \quad \text{from (i)}$$

$$= 2\pi\left[x\left(\frac{1}{3}\cos^3 x - \cos x\right)\right]_0^{\pi} - 2\pi\int_0^{\pi}\left(\frac{1}{3}\cos^3 x - \cos x\right) dx \quad \text{by parts}$$

$$= 2\pi\left[\pi\left(\frac{1}{3}\cos^3 \pi - \cos \pi\right) - 0\right] - 2\pi(0) \quad \text{from (ii)}$$

$$= \frac{4\pi^2}{3}$$

$\therefore$ the volume is $\dfrac{4\pi^2}{3}$ unit^3

(b) (i)

$$I_0 = \int_0^1 \frac{1}{\left(x^2+1\right)^2}\, dx$$

Let $x = \tan\theta$
$\therefore dx = \sec^2\theta\, d\theta$
If $x = 0,\ \theta = 1$
If $x = 1,\ \theta = \frac{\pi}{4}$

$$\therefore I_0 = \int_0^{\frac{\pi}{4}} \frac{1}{\left(\tan^2\theta + 1\right)^2} \sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\left(\sec^2\theta\right)^2} \sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos 2\theta)\, d\theta \qquad \left[2\cos^2\theta = 1+\cos 2\theta\right]$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$I_0=\frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\sin\frac{\pi}{2}-0\right)$$

$$=\frac{\pi}{8}+\frac{1}{4} \qquad \text{as required}$$

(ii) $$I_0+I_2=\int_0^1\frac{1}{\left(x^2+1\right)^2}\,dx+\int_0^1\frac{x^2}{\left(x^2+1\right)^2}\,dx$$

$$=\int_0^1\frac{1+x^2}{\left(x^2+1\right)^2}\,dx$$

$$=\int_0^1\frac{1}{x^2+1}\,dx$$

$$=\left[\tan^{-1}x\right]_0^1$$

$$=\frac{\pi}{4} \qquad \text{as required}$$

(iii) $$I_4=\int_0^1\frac{x^4}{\left(x^2+1\right)^2}\,dx \qquad \text{but } x^4=\left(x^2+1\right)^2-2x^2-1$$

$$=\int_0^1\left(\frac{\left(x^2+1\right)^2}{\left(x^2+1\right)^2}-\frac{2x^2}{\left(x^2+1\right)^2}-\frac{1}{\left(x^2+1\right)^2}\right)dx$$

$$=\int_0^1 1\,dx-2I_2-I_0$$

$$=1-2\left(\frac{\pi}{4}-I_0\right)-I_0 \qquad \text{from (ii)}$$

$$=1-\frac{\pi}{2}+I_0$$

$$=1-\frac{\pi}{2}+\frac{\pi}{8}+\frac{1}{4}$$

$$=\frac{5}{4}-\frac{3\pi}{8}$$

(c) To prove: $x\sqrt{x}+1\ge x+\sqrt{x}$ for $x\ge 0$

Consider $$x\sqrt{x}+1-x-\sqrt{x}=x\left(\sqrt{x}-1\right)-\left(\sqrt{x}-1\right)$$

$$=\left(\sqrt{x}-1\right)\left(x-1\right)$$

Method 1

If $x=1$: $\left(\sqrt{x}-1\right)\left(x-1\right)=0$

If $x>1$: $\sqrt{x}-1>0$ and $x-1>0$ $\therefore\left(\sqrt{x}-1\right)\left(x-1\right)>0$

If $0\le x<1$: $x\ge 0$ and $x-1<0$ $\therefore\left(\sqrt{x}-1\right)\left(x-1\right)>0$

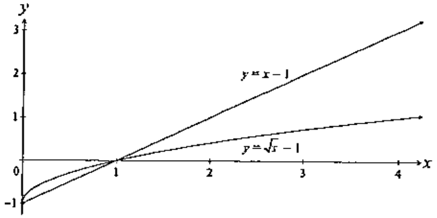
$\therefore\left(\sqrt{x}-1\right)\left(x-1\right)\ge 0$ for all $x\ge 0$

i.e. $x\sqrt{x}+1-x-\sqrt{x}\ge 0$ for all $x\ge 0$

$\therefore x\sqrt{x}+1\ge x+\sqrt{x}$ for all $x\ge 0$

Method 2

Consider the graphs of $y=\sqrt{x}-1$ and $y=x-1$ as illustrated for $x\ge 0$:



By inspection, the product graph of $y=\left(\sqrt{x}-1\right)\left(x-1\right)$ will exist only if $x\ge 0$ and then will always be positive.

$\therefore\left(\sqrt{x}-1\right)\left(x-1\right)\ge 0$ for all $x\ge 0$

i.e. $x\sqrt{x}+1-x-\sqrt{x}\ge 0$ for all $x\ge 0$

$\therefore x\sqrt{x}+1\ge x+\sqrt{x}$ for all $x\ge 0$

Method 3

Consider $$x\sqrt{x}+1-x-\sqrt{x}=\left(x+1\right)\left(\sqrt{x}-1\right)$$

$$=\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)$$

$$=\left(\sqrt{x}-1\right)^2\left(\sqrt{x}+1\right)$$

For $x\ge 0$: $\left(\sqrt{x}-1\right)^2\ge 0$ and $\sqrt{x}+1\ge 0$

$\therefore x\sqrt{x}+1-x-\sqrt{x}=\left(\sqrt{x}-1\right)^2\left(\sqrt{x}+1\right)\ge 0$

$\therefore x\sqrt{x}+1\ge x+\sqrt{x}$ for all $x\ge 0$

Method 4

$\left(a-1\right)^2\ge 0$ for all $a\in\mathbb{R}$

$\therefore a^2-2a+1\ge 0$

$a^2-a+1\ge a$

$\therefore\left(a+1\right)\left(a^2-a+1\right)\ge a\left(a+1\right)$ if $a+1\ge 0$

i.e. $a^3+1\ge a^2+a$

Now if $a=\sqrt{x}$ then $a+1\ge 1$. Substituting this into the line above:

$\therefore x\sqrt{x}+1\ge x+\sqrt{x}$ for all $x\ge 0$

Question 15

(a) **Method 1**

$x^3-3x+1=0$

$\therefore x\left(x^2-3\right)=-1$

$\therefore x^2\left(x^2-3\right)^2=1$

Let $y=x^2$

$\therefore y(y-3)^2=1$ has roots $\alpha^2,\beta^2,\gamma^2$.

i.e. a suitable equation is

$x^3-6x^2+9x-1=0$

Method 2

If $P(x)=0$ has roots α,β,γ

then $\alpha^2,\beta^2,\gamma^2$ are roots of $P\left(\sqrt{x}\right)=0$

$\therefore \left(\sqrt{x}\right)^3-3\sqrt{x}+1=0$

$\sqrt{x}(x-3)=-1$

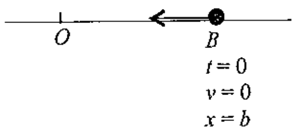
$x(x-3)^2=1$

$x^3-6x^2+9x-1=0$

i.e. a suitable equation is

$x^3-6x^2+9x-1=0$

(b) (i) $t=0,\ x=b(>0),\ v=0,\ \mu>0$



$\ddot{x}=\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

$\therefore \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)=-\dfrac{\mu^2}{x^2}$

$\tfrac{1}{2}v^2=\dfrac{\mu^2}{x}+C$

Substitute $x=b, v=0$: $\quad \therefore 0=\dfrac{\mu^2}{b}+C \Rightarrow C=-\dfrac{\mu^2}{b}$

$\therefore \tfrac{1}{2}v^2=\dfrac{\mu^2}{x}-\dfrac{\mu^2}{b}$

$v^2=2\mu^2\left(\dfrac{b-x}{bx}\right)$

At $x=b, v=0$ but $\ddot{x}=\dfrac{-\mu^2}{x^2}<0$ for all x and hence the motion is always in the negative direction.

The particle will never stop again as $v=0$ only at $x=b$ i.e. $v<0$.

$\therefore v=-\sqrt{2\mu^2\left(\dfrac{b-x}{bx}\right)}$

$\dfrac{dx}{dt}=-\mu\sqrt{2}\sqrt{\dfrac{b-x}{bx}}$ as required

(ii) Let t be the time taken for the particle to reach $x=d\ (<b)$

$\dfrac{dx}{dt}=-\mu\sqrt{2}\sqrt{\dfrac{b-x}{bx}} \Rightarrow \dfrac{dt}{dx}=-\dfrac{1}{\mu\sqrt{2}}\sqrt{\dfrac{bx}{b-x}}$

$\therefore \displaystyle\int_0^t dt=-\frac{1}{\mu\sqrt{2}}\int_b^d\sqrt{\frac{bx}{b-x}}\,dx$

Let $x=b\cos^2\theta$

$dx=-2b\cos\theta\sin\theta\, d\theta$

If $x=b$: $\quad \theta=0$

If $x=d$: $\quad d=b\cos^2\theta$

$\cos^2\theta=\dfrac{d}{b}$

$\cos\theta=\sqrt{\dfrac{d}{b}}$

$\therefore \theta=\cos^{-1}\left(\sqrt{\dfrac{d}{b}}\right)$

$\sqrt{\dfrac{bx}{b-x}}=\sqrt{\dfrac{b^2\cos^2\theta}{b-b\cos^2\theta}}$

$=\sqrt{\dfrac{b^2\cos^2\theta}{b(1-\cos^2\theta)}}$

$=\sqrt{\dfrac{b\cos^2\theta}{\sin^2\theta}}$

$=\dfrac{\sqrt{b}\cos\theta}{\sin\theta}$

Note: If $x=0$, $\theta=\dfrac{\pi}{2}$. The positive square root ensures that $\theta=\cos^{-1}\left(\sqrt{\dfrac{d}{b}}\right)<\dfrac{\pi}{2}$.

$\therefore \displaystyle t=-\frac{1}{\mu\sqrt{2}}\int_0^{\cos^{-1}\left(\sqrt{\frac{d}{b}}\right)}\frac{\sqrt{b}\cos\theta}{\sin\theta}\times-2b\cos\theta\sin\theta\, d\theta$

$\displaystyle =\frac{2b\sqrt{b}}{\mu\sqrt{2}}\int_0^{\cos^{-1}\left(\sqrt{\frac{d}{b}}\right)}\cos^2\theta\, d\theta$

$\displaystyle =\frac{b\sqrt{2b}}{\mu}\int_0^{\cos^{-1}\left(\sqrt{\frac{d}{b}}\right)}\cos^2\theta\, d\theta$

(iii) Given $t=\dfrac{1}{\mu}\sqrt{\dfrac{b}{2}}\left(\sqrt{bd-d^2}+b\cos^{-1}\sqrt{\dfrac{d}{b}}\right)$

As $x\to 0$, $d\to 0$

$\therefore \displaystyle\lim_{d\to 0}t=\lim_{d\to 0}\frac{1}{\mu}\sqrt{\frac{b}{2}}\left(\sqrt{bd-d^2}+b\cos^{-1}\sqrt{\frac{d}{b}}\right)$

$=\dfrac{1}{\mu}\sqrt{\dfrac{b}{2}}\left(0+b\times\dfrac{\pi}{2}\right)$

$=\dfrac{1}{\mu}\sqrt{\dfrac{b}{2}}\left(\dfrac{b\pi}{2}\right)$

$=\dfrac{\pi b\sqrt{2b}}{4\mu}$

$\therefore$ the limiting time is $\dfrac{\pi b\sqrt{2b}}{4\mu}$ sec

(c) (i) $\dfrac{3!}{x(x+1)(x+2)(x+3)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x+2}+\dfrac{D}{x+3}$

$\therefore 3!=A(x+1)(x+2)(x+3)+Bx(x+2)(x+3)+Cx(x+1)(x+3)+Dx(x+1)(x+2)$

If $x=0$: $3!=A(6) \Rightarrow A=1$

If $x=-1$: $3!=B(-1)(1)(2) \Rightarrow B=-3$

If $x=-2$: $3!=C(-2)(-1)(1) \Rightarrow C=3$

If $x=-3$: $3!=D(-3)(-2)(-1) \Rightarrow D=-1$

$$\frac{3!}{x(x+1)(x+2)(x+3)}=\frac{1}{x}-\frac{3}{x+1}+\frac{3}{x+2}-\frac{1}{x+3} \qquad \text{as required}$$

(ii) $$\frac{n!}{x(x+1)\dots(x+n)}=\frac{a_0}{x}+\frac{a_1}{x+1}+\dots+\frac{a_k}{x+k}+\dots+\frac{a_n}{x+n}$$

$$n!=a_0(x+1)..(x+n)+\dots+a_kx(x+1)..(x+k-1)(x+k+1)..(x+n)+\dots+a_nx..(x+n-1)$$

If $x=-k$: $n!=a_k(-k)(-k+1)\dots(-k+k-1)(-k+k+1)\dots(-k+n)$

$$a_k=\frac{n!}{(-k)(-k+1)\dots(-k+k-1)(-k+k+1)\dots(-k+n)}$$

$$=\frac{n!}{\underbrace{(-k)(-k+1)\dots(-1)}_{k\text{ terms}}(1)(2)\dots(-k+n)}$$

$$=\frac{n!}{(-1)^k(k)(k-1)\dots(1)\times(1)(2)\dots(n-k)}$$

$$=\frac{n!}{(-1)^k k!(n-k)!}$$

$$=(-1)^k\binom{n}{k} \qquad \text{as required}$$

(iii) $$\frac{n!}{x(x+1)\dots(x+n)}=\frac{a_0}{x}+\frac{a_1}{x+1}+\dots+\frac{a_k}{x+k}+\dots+\frac{a_n}{x+n}$$

$$=\binom{n}{0}\frac{1}{x}-\binom{n}{1}\frac{1}{x+1}+\dots+(-1)^k\binom{n}{k}\frac{1}{x+k}+\dots+(-1)^n\binom{n}{n}\frac{1}{x+n}$$

If $x=1$: $$\frac{n!}{1(2)\dots n(n+1)}=\binom{n}{0}-\binom{n}{1}\frac{1}{2}+\dots+(-1)^k\binom{n}{k}\frac{1}{k+1}+\dots+(-1)^n\binom{n}{n}\frac{1}{n+1}$$

$$\frac{1}{n+1}=\binom{n}{0}-\binom{n}{1}\frac{1}{2}+\dots+(-1)^k\binom{n}{k}\frac{1}{k+1}+\dots+(-1)^n\binom{n}{n}\frac{1}{n+1}$$

$$\therefore \binom{n}{0}-\binom{n}{1}\frac{1}{2}+\dots+(-1)^k\binom{n}{k}\frac{1}{k+1}+\dots+(-1)^n\binom{n}{n}\frac{1}{n+1}=\frac{1}{n+1}$$

$$\therefore \lim_{n\to\infty}\left[\binom{n}{0}-\binom{n}{1}\frac{1}{2}+\dots+(-1)^k\binom{n}{k}\frac{1}{k+1}+\dots+(-1)^n\binom{n}{n}\frac{1}{n+1}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n+1}$$

$$=0$$

Question 16

(a) (i) $z=\cos\theta+i\sin\theta$ and $w=\cos\alpha+i\sin\alpha$

$|z|=|w|=1$ and so z, w and 1 are located on the unit circle.

Now $\text{Re}(1+z+w)=1+\cos\theta+\cos\alpha$ and $\text{Im}(1+z+w)=\sin\theta+\sin\alpha$

But $z+w+1=0$

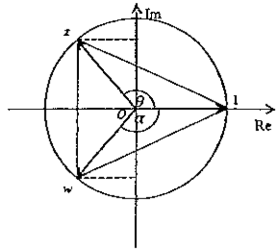
$\Rightarrow \text{Re}(z+w+1)=0$ and $\text{Im}(z+w+1)=0$

Hence $1+\cos\theta+\cos\alpha=0$ ①

and $\sin\theta+\sin\alpha=0$ ②

From ②: $\sin\theta=-\sin\alpha$ but $-\pi<\theta\le\pi$ and $-\pi<\alpha\le\pi$

$\Rightarrow \theta=-\alpha$ ③



Hence z and w are conjugates as illustrated.

Note that z and w can be interchanged.

Now substituting ③ into ①:

$$1+\cos\theta+\cos(-\theta)=0$$

$$1+\cos\theta+\cos\theta=0$$

$$\therefore \cos\theta=-\tfrac{1}{2}$$

$$\theta=\tfrac{2\pi}{3}$$

$$\therefore \alpha=-\tfrac{2\pi}{3}$$

Hence 1, z and w are equally spaced around the unit circle, with O as the centroid.

$\therefore$ 1, z and w are the vertices of an equilateral triangle.

(ii) Now $|2i|=|z_1|=|z_2|=2$ so $2i, z_1$ and z_2 are located on a circle of radius 2.

Also $2i+z_1+z_2=0$

$$\therefore 1+\frac{z_1}{2i}+\frac{z_2}{2i}=0$$

Let $z=\frac{z_1}{2i}$ and $w=\frac{z_2}{2i}$, then $|z|=|w|=1$ and so $\frac{z_1}{2i}$, $\frac{z_2}{2i}$ and 1 are the vertices of an equilateral triangle from part (i).

Now $2i=1\times 2i$, $z_1=2iz$ and $z_2=2iw$

i.e. In each case, $2i$, z_1 and z_2 are located by stretching the vectors 1, z and w on the Argand diagram by a factor of 2 and then rotating each 90° anti-clockwise.

In effect this has enlarged the triangle formed by 1, z and w by a scale factor of 2 and rotating the result anti-clockwise about O through 90°.

Since 1, z and w are the vertices of an equilateral triangle then so are $2i$, z_1 and z_2.

(b) (i)

Method 1

Since 0, u and v form the vertices of an equilateral triangle then $v = u\operatorname{cis}\left(\frac{\pi}{3}\right)$ without loss of generalisation.

$$\therefore v^2 = u^2\operatorname{cis}\left(\tfrac{2\pi}{3}\right)$$

$$\begin{aligned}\therefore u^2+v^2 &= u^2+u^2\operatorname{cis}\left(\tfrac{2\pi}{3}\right)\\ &= u^2\left(1+\operatorname{cis}\left(\tfrac{2\pi}{3}\right)\right)\\ &= u^2\left[1+\left(-\tfrac{1}{2}+i\tfrac{\sqrt{3}}{2}\right)\right]\\ &= u^2\left(\tfrac{1}{2}+i\tfrac{\sqrt{3}}{2}\right)\\ &= u\times u\left(\tfrac{1}{2}+i\tfrac{\sqrt{3}}{2}\right)\\ &= u\times u\operatorname{cis}\left(\tfrac{\pi}{3}\right)\\ &= uv\end{aligned}$$

Method 2

$$v = \left(\cos\tfrac{\pi}{3}+i\sin\tfrac{\pi}{3}\right)u$$

$$\therefore v^3 = \left(\cos\tfrac{\pi}{3}+i\sin\tfrac{\pi}{3}\right)^3 u^3$$

$$v^3 = \left(\cos\pi + i\sin\pi\right)u^3 \qquad \text{[by de Moivre]}$$

$$v^3 = -u^3$$

$$\therefore v^3+u^3 = 0$$

$$\therefore (v+u)\left(v^2-vu+u^2\right)=0$$

But $v \neq -u \Rightarrow v^2-vu+u^2=0$

$$\therefore v^2+u^2 = vu$$

(ii) Take $v = 1$, then $u = \operatorname{cis}\left(\frac{\pi}{3}\right) = \frac{1}{2}+i\frac{\sqrt{3}}{2}$.

Note: 1 is a complex number that is purely real.

(c) (i) There are $(n-1)$ different people with whom Tom could directly exchange hats.

If Tom and his friend, Mot, have their own hats, there are $(n-2)$ hats remaining for derangement. This can be done in $D(n-2)$ ways.

Now if Tom and Mot exchange hats then everyone has a different hat.

$\therefore$ the number of derangements $=(n-1)D(n-2)$ as required.

(ii) Note: The situations in part (i) and part (ii) are mutually exclusive and together account for all possible derangements of n hats.

Method 1

If we remove the situation in part (i) from all possible derangements then the remaining possible derangements are $D(n)-(n-1)D(n-2)$.

i.e. Tom does not make a direct exchange of hats with anyone in the group.

Let everyone else derange their hats first. This produces $D(n-1)$ derangements and no one will have their own hat except for Tom.

If Tom now chooses a hat, the person with whom he exchanges hats will not give him their own hat and so there will not be a direct exchange of hats.

Tom has $(n-1)$ different hats to choose from.

$\therefore$ this situation results in $(n-1)D(n-1)$ derangements.

$$\begin{aligned}\therefore D(n)-(n-1)D(n-2) &= (n-1)D(n-1)\\ D(n) &= (n-1)D(n-1)+(n-1)D(n-2)\\ D(n) &= (n-1)\left[D(n-1)+D(n-2)\right] \quad \text{as required}\end{aligned}$$

Method 2

Ignoring Tom, there are $D(n-1)$ derangements of the remaining hats. If Tom now exchanges his hat with any of the $(n-1)$ others, we have a suitable derangement.

$\therefore$ this situation results in $(n-1)D(n-1)$ derangements.

Tom either exchanges his hat with one person, which is the situation in part (i) or he doesn't, which is the situation just calculated.

$$\begin{aligned}\therefore D(n) &= (n-1)D(n-1)+(n-1)D(n-2)\\ &= (n-1)\left[D(n-1)+D(n-2)\right]\end{aligned}$$

(iii)

$$D(n) = (n-1)\left[D(n-2)+D(n-1)\right]$$

$$\therefore D(n) = (n-1)D(n-2)+nD(n-1)-D(n-1)$$

$$\begin{aligned}\therefore D(n)-nD(n-1) &= (n-1)D(n-2)-D(n-1)\\ &= -D(n-1)+(n-1)D(n-2)\\ &= -\left[D(n-1)-(n-1)D(n-2)\right]\end{aligned}$$

(iv) $D(n)-nD(n-1) = -\left[D(n-1)-(n-1)D(n-2)\right]$ for $n>1$

$n=2$: $D(2)-2D(1) = 1-2\times 0 = 1 = (-1)^2$

$n=3$: $D(3)-3D(2) = -\left[D(2)-2D(1)\right] = -(1-0) = -1 = (-1)^3$

Applying the recursive formula:

$$\begin{aligned}D(n)-nD(n-1) &= (-1)^1\left[D(n-1)-(n-1)D(n-2)\right]\\ &= -1\times -\left[D(n-2)-(n-2)D(n-3)\right]\\ &= (-1)^2\left[D(n-2)-(n-2)D(n-3)\right]\\ &= (-1)^2\times -\left[D(n-3)-(n-3)D(n-4)\right]\\ &= (-1)^3\left[D(n-3)-(n-3)D(n-4)\right]\\ &\;\vdots\\ &= (-1)^{n-2}\times\left[D(n-(n-2))-((n-(n-2))D(n-(n-2)-1)\right]\\ &= (-1)^{n-2}\times\left[\underbrace{D(2)-2D(1)}_{n=2}\right]\\ &= (-1)^{n-2}\times(-1)^2\\ &= (-1)^n \qquad \text{as required}\end{aligned}$$

(v) Using the recurrence relation $D(n)-nD(n-1)=(-1)^n$, strong induction is not needed.

To prove: $D(n) = k! \times \sum_{r=0}^{k} \frac{(-1)^r}{r!}$

Test $n=1$: LHS $= D(1) = 0$ from part (iv)

$$\text{RHS} = 1! \times \sum_{r=0}^{1} \frac{(-1)^r}{r!}$$
$$= 1 \times \left(\frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} \right)$$
$$= 1 \times (1-1)$$
$$= 0$$
$$= \text{LHS}$$

$\therefore$ true for $n=1$

Let $n=k$ be a value for which the result is true.

i.e. $D(k) = k! \times \sum_{r=0}^{k} \frac{(-1)^r}{r!}$ is a true statement.

Need to prove the result true for $n=k+1$.

i.e. To prove $D(k+1) = (k+1)! \times \sum_{r=0}^{k+1} \frac{(-1)^r}{r!}$

From (iv): $D(n) - nD(n-1) = (-1)^n$

$$\therefore D(k+1) - (k+1)D(k) = (-1)^{k+1}$$
$$D(k+1) = (-1)^{k+1} + (k+1)D(k)$$

Now $D(k+1) = (-1)^{k+1} + (k+1)D(k)$

$$= (-1)^{k+1} + (k+1) \times k! \times \sum_{r=0}^{k} \frac{(-1)^r}{r!}$$
$$= (-1)^{k+1} + (k+1)! \times \sum_{r=0}^{k} \frac{(-1)^r}{r!}$$
$$= (k+1)! \times \frac{(-1)^{k+1}}{(k+1)!} + (k+1)! \times \sum_{r=0}^{k} \frac{(-1)^r}{r!}$$
$$= (k+1)! \times \sum_{r=0}^{k+1} \frac{(-1)^r}{r!}$$

$\therefore$ $D(k+1)$ is true if $D(k)$ is true.

$\therefore$ by the principle of mathematical induction the formula is true for all integers n where $n \geq 1$.

Note: If the recurrence relation in part (ii) had been used and not that in part (iv), then strong induction would be required.

End of solutions