BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2015 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–13

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the remainder when $x^3 - 6x$ is divided by $x + 3$?

(A) -9

(B) 9

(C) $x^2 - 2x$

(D) $x^2 - 3x + 3$

2 Given that $N = 100 + 80e^{kt}$, which expression is equal to $\dfrac{dN}{dt}$?

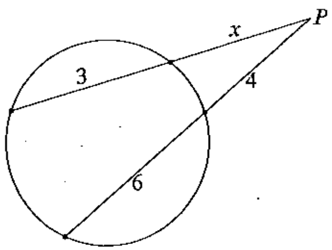
(A) $k(100 - N)$

(B) $k(180 - N)$

(C) $k(N - 100)$

(D) $k(N - 180)$

3 Two secants from the point P intersect a circle as shown in the diagram.



NOT TO SCALE

What is the value of x?

(A) 2

(B) 5

(C) 7

(D) 8

4 A rowing team consists of 8 rowers and a coxswain.

The rowers are selected from 12 students in Year 10.

The coxswain is selected from 4 students in Year 9.

In how many ways could the team be selected?

(A) ${}^{12}C_8 + {}^{4}C_1$

(B) ${}^{12}P_8 + {}^{4}P_1$

(C) ${}^{12}C_8 \times {}^{4}C_1$

(D) ${}^{12}P_8 \times {}^{4}P_1$

5 What are the asymptotes of $y = \dfrac{3x}{(x+1)(x+2)}$?

(A) $y = 0,\ x = -1,\ x = -2$

(B) $y = 0,\ x = 1,\ x = 2$

(C) $y = 3,\ x = -1,\ x = -2$

(D) $y = 3,\ x = 1,\ x = 2$

6 What is the domain of the function $f(x) = \sin^{-1}(2x)$?

(A) $-\pi \le x \le \pi$

(B) $-2 \le x \le 2$

(C) $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$

(D) $-\dfrac{1}{2} \le x \le \dfrac{1}{2}$

7 What is the value of k such that $\int_0^k \frac{1}{\sqrt{4-x^2}}\,dx = \frac{\pi}{3}$?

(A) 1

(B) $\sqrt{3}$

(C) 2

(D) $2\sqrt{3}$

8 What is the value of $\lim_{x \to 3} \frac{\sin(x-3)}{(x-3)(x+2)}$?

(A) 0

(B) $\frac{1}{5}$

(C) 5

(D) Undefined

9 Two particles oscillate horizontally. The displacement of the first is given by $x = 3\sin 4t$ and the displacement of the second is given by $x = a\sin nt$. In one oscillation, the second particle covers twice the distance of the first particle, but in half the time.

What are the values of a and n?

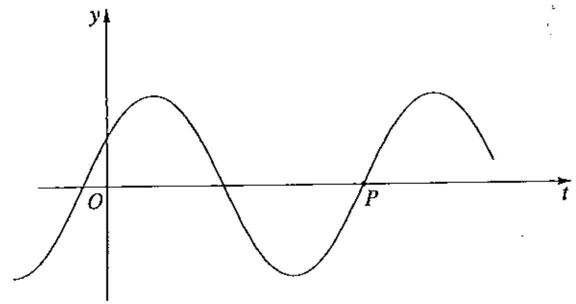
(A) $a = 1.5, \quad n = 2$

(B) $a = 1.5, \quad n = 8$

(C) $a = 6, \quad n = 2$

(D) $a = 6, \quad n = 8$

10 The graph of the function $y = \cos\left(2t - \frac{\pi}{3}\right)$ is shown below.



What are the coordinates of the point P?

(A) $\left(\frac{5\pi}{12}, 0\right)$

(B) $\left(\frac{2\pi}{3}, 0\right)$

(C) $\left(\frac{11\pi}{12}, 0\right)$

(D) $\left(\frac{7\pi}{6}, 0\right)$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \sin^2 x \, dx$. **2**

(b) Calculate the size of the acute angle between the lines $y = 2x + 5$ and $y = 4 - 3x$. **2**

(c) Solve the inequality $\frac{4}{x+3} \geq 1$. **3**

(d) Express $5\cos x - 12\sin x$ in the form $A\cos(x + \alpha)$, where $0 \leq \alpha \leq \frac{\pi}{2}$. **2**

(e) Use the substitution $u = 2x - 1$ to evaluate $\int_1^2 \frac{x}{(2x-1)^2} dx$. **3**

(f) Consider the polynomials $P(x) = x^3 - kx^2 + 5x + 12$ and $A(x) = x - 3$.

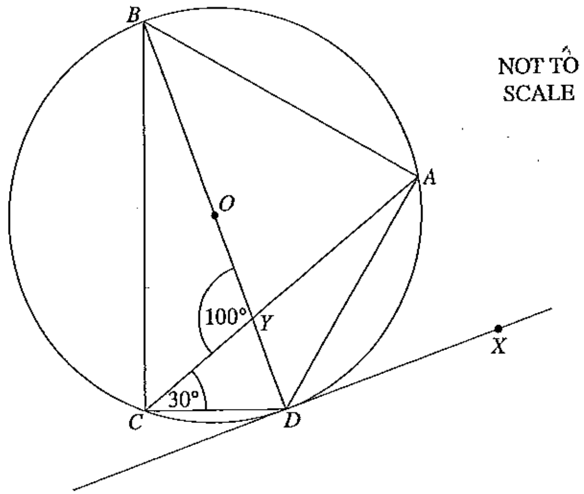
(i) Given that $P(x)$ is divisible by $A(x)$, show that $k = 6$. **1**

(ii) Find all the zeros of $P(x)$ when $k = 6$. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram, the points A, B, C and D are on the circumference of a circle, whose centre O lies on BD. The chord AC intersects the diameter BD at Y. The tangent at D passes through the point X.

It is given that $\angle CYB = 100°$ and $\angle DCY = 30°$.



Copy or trace the diagram into your writing booklet.

(i) What is the size of $\angle ACB$? **1**

(ii) What is the size of $\angle ADX$? **1**

(iii) Find, giving reasons, the size of $\angle CAB$. **2**

Question 12 continues on page 8

(b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

The equation of the chord PQ is given by $(p+q)x - 2y - 2apq = 0$. (Do NOT prove this.)

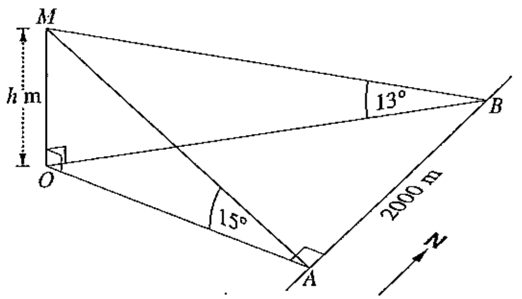
(i) Show that if PQ is a focal chord then $pq = -1$. **1**

(ii) If PQ is a focal chord and P has coordinates $(8a, 16a)$, what are the coordinates of Q in terms of a? **2**

(c) A person walks 2000 metres due north along a road from point A to point B. The point A is due east of a mountain OM, where M is the top of the mountain. The point O is directly below point M and is on the same horizontal plane as the road. The height of the mountain above point O is h metres.

From point A, the angle of elevation to the top of the mountain is 15°.

From point B, the angle of elevation to the top of the mountain is 13°.

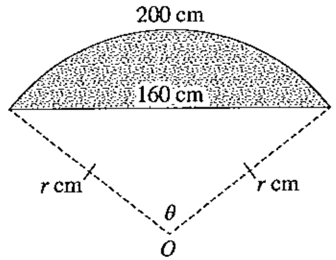


(i) Show that $OA = h\cot 15°$. **1**

(ii) Hence, find the value of h. **2**

Question 12 continues on page 9

(d) A kitchen bench is in the shape of a segment of a circle. The segment is bounded by an arc of length 200 cm and a chord of length 160 cm. The radius of the circle is r cm and the chord subtends an angle θ at the centre O of the circle.



(i) Show that $160^2 = 2r^2(1 - \cos\theta)$. **1**

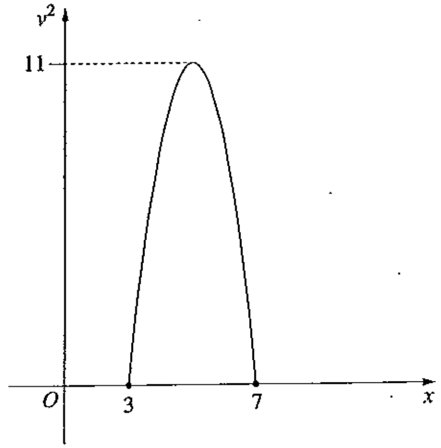
(ii) Hence, or otherwise, show that $8\theta^2 + 25\cos\theta - 25 = 0$. **2**

(iii) Taking $\theta_1 = \pi$ as a first approximation to the value of θ, use one application of Newton's method to find a second approximation to the value of θ. Give your answer correct to two decimal places. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is moving along the x-axis in simple harmonic motion. The displacement of the particle is x metres and its velocity is $v\ \mathrm{m\,s^{-1}}$. The parabola below shows v^2 as a function of x.



(i) For what value(s) of x is the particle at rest? **1**

(ii) What is the maximum speed of the particle? **1**

(iii) The velocity v of the particle is given by the equation **3**

$$v^2 = n^2\left(a^2 - (x-c)^2\right)$$

where a, c and n are positive constants.

What are the values of a, c and n?

Question 13 continues on page 11

(b) Consider the binomial expansion

$$\left(2x + \frac{1}{3x}\right)^{18} = a_0x^{18} + a_1x^{16} + a_2x^{14} + \cdots$$

where $a_0, a_1, a_2, \ldots$ are constants.

(i) Find an expression for a_2. **2**

(ii) Find an expression for the term independent of x. **2**

(c) Prove by mathematical induction that for all integers $n \geq 1$, **3**

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

(d) Let $f(x) = \cos^{-1}(x) + \cos^{-1}(-x)$, where $-1 \leq x \leq 1$.

(i) By considering the derivative of $f(x)$, prove that $f(x)$ is constant. **2**

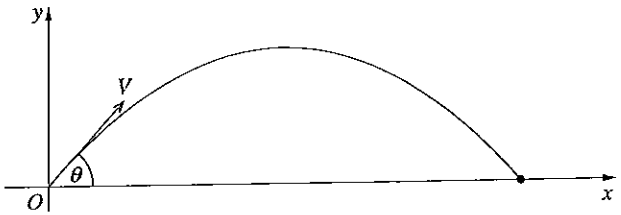
(ii) Hence deduce that $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) A projectile is fired from the origin O with initial velocity $V \text{ m s}^{-1}$ at an angle θ to the horizontal. The equations of motion are given by

$$x = Vt\cos\theta, \quad y = Vt\sin\theta - \frac{1}{2}gt^2. \quad \text{(Do NOT prove this.)}$$



(i) Show that the horizontal range of the projectile is $\dfrac{V^2 \sin 2\theta}{g}$. **2**

A particular projectile is fired so that $\theta = \dfrac{\pi}{3}$.

(ii) Find the angle that this projectile makes with the horizontal when $t = \dfrac{2V}{\sqrt{3}\,g}$. **2**

(iii) State whether this projectile is travelling upwards or downwards when $t = \dfrac{2V}{\sqrt{3}\,g}$. Justify your answer. **1**

Question 14 continues on page 13

(b) A particle is moving horizontally. Initially the particle is at the origin O moving with velocity 1 m s^{-1}.

The acceleration of the particle is given by $\ddot{x} = x - 1$, where x is its displacement at time t.

(i) Show that the velocity of the particle is given by $\dot{x} = 1 - x$. **3**

(ii) Find an expression for x as a function of t. **2**

(iii) Find the limiting position of the particle. **1**

(c) Two players A and B play a series of games against each other to get a prize. In any game, either of the players is equally likely to win.

To begin with, the first player who wins a total of 5 games gets the prize.

(i) Explain why the probability of player A getting the prize in exactly 7 games is $\dbinom{6}{4}\left(\dfrac{1}{2}\right)^7$. **1**

(ii) Write an expression for the probability of player A getting the prize in at most 7 games. **1**

(iii) Suppose now that the prize is given to the first player to win a total of $(n+1)$ games, where n is a positive integer. **2**

By considering the probability that A gets the prize, prove that

$$\binom{n}{n}2^n + \binom{n+1}{n}2^{n-1} + \binom{n+2}{n}2^{n-2} + \cdots + \binom{2n}{n} = 2^{2n}.$$

End of paper

2015 Higher School Certificate Solutions Mathematics Extension 1

SECTION I

Summary

1 A	3 B	5 A	7 B	9 D
2 C	4 C	6 D	8 B	10 C

SECTION I

1 (A) $f(x)=x^3-6x$

$$f(-3)=(-3)^3-6(-3)$$
$$=-27+18$$
$$=-9.$$

2 (C) $N=100+80e^{kt}$

$$\frac{dN}{dt}=80ke^{kt}$$
$$=k\left(80e^{kt}\right)$$
$$=k(N-100).$$

3 (B) $x(x+3)=4(4+6)$

$$x^2+3x-40=0$$

For $(x-5)(x+8)=0$

$$x=5,\ -8$$
$$x=5\quad(\text{since } x>0).$$

4 (C) Select 8 from 12 in ${}^{12}C_8$ ways.

Select 1 from 4 in ${}^{4}C_1$ ways.

Thus the answer is ${}^{12}C_8\times{}^{4}C_1$.

5 (A) $y\to\infty$ as $x\to-1^{\pm}$ and $x\to-2^{\pm}$

$x\to 0^{\pm}$ as $y\to 0^{\pm}$

$\therefore\ x=-1,\quad x=-2,\quad y=0.$

6 (D) Using the standard integrals with $a=2$:

$$-1\le 2x\le 1$$
$$-\frac{1}{2}\le x\le\frac{1}{2}.$$

7 (B) $\displaystyle\int_0^k\frac{1}{\sqrt{4-x^2}}\,dx=\left[\sin^{-1}\frac{x}{2}\right]_0^k$

$$\frac{\pi}{3}=\sin^{-1}\frac{k}{2}$$
$$\frac{k}{2}=\frac{\sqrt{3}}{2}$$
$$k=\sqrt{3}.$$

8 (B) $\displaystyle\lim_{x\to3}\frac{\sin(x-3)}{(x-3)(x+2)}=\lim_{x\to3}\left(\frac{\sin(x-3)}{(x-3)}\times\frac{1}{(x+2)}\right)$

$$=1\times\lim_{x\to3}\frac{1}{(x+2)}$$
$$=\frac{1}{5}.$$

9 (D) $a=2\times3=6$

$$T=\frac{1}{2}\times\frac{2\pi}{4}=\frac{2\pi}{8}\ \Rightarrow\ n=8$$

$\therefore\ a=6,\quad n=8.$

10 (C) $\cos\left(2t-\frac{\pi}{3}\right)=0$

$$2t-\frac{\pi}{3}=\frac{\pi}{2}\pm n\pi$$
$$2t=\frac{5\pi}{6}\pm n\pi$$
$$t=\frac{5\pi}{6},\quad\frac{11\pi}{12},\ldots$$

$\therefore$ P is $\left(\frac{11\pi}{12},0\right)$

SECTION II

Question 11

(a)
$$\int\sin^2x\,dx=\int\frac{1}{2}(1-\cos 2x)\,dx$$
$$=\frac{x}{2}-\frac{\sin 2x}{4}+C.$$

(b) $m_1=2,\quad m_2=-3$

$$\tan\alpha=\left|\frac{2-(-3)}{1+2\times(-3)}\right|$$
$$=1$$
$$\alpha=45^\circ.$$

(c)
$$\frac{4}{x+3}\ge1,\quad x\ne-3$$
$$4(x+3)\ge(x+3)^2$$
$$4(x+3)-(x+3)^2\ge0$$
$$(x+3)(4-(x+3))\ge0$$
$$(x+3)(1-x)\ge0$$
$$-3<x\le1.$$

(d)
$$A\cos(x+\alpha)=A\cos x\cos\alpha-A\sin x\sin\alpha$$
$$\therefore\ A\cos\alpha=5\quad\text{and}\quad A\sin\alpha=12$$
$$A^2\cos^2\alpha+A^2\sin^2\alpha=5^2+12^2$$
$$A^2=169$$
$$A=13$$
$$\therefore\ \cos\alpha=\frac{5}{13}\quad\text{and}\quad\sin\alpha=\frac{12}{13}$$

and $\alpha=\tan^{-1}\left(\frac{12}{5}\right)\approx1.176$

$$5\cos x-12\sin x=13\cos\left(x+\tan^{-1}\frac{12}{5}\right)$$
$$=13\cos(x+1.176).$$

(e) $u=2x-1\qquad du=2\,dx$

$$x=\frac{1}{2}(u+1)$$

when $x=1,\quad u=1$

$x=2,\quad u=3$

$$\int_1^2\frac{x}{(2x-1)^2}\,dx=\frac{1}{2}\int_1^3\frac{(u+1)}{u^2}\cdot\frac{1}{2}du$$
$$=\frac{1}{4}\int_1^3\frac{1}{u}+\frac{1}{u^2}\,du$$
$$=\frac{1}{4}\left[\ln u-\frac{1}{u}\right]_1^3$$
$$=\frac{1}{4}\left(\ln3-\frac{1}{3}-(\ln1-1)\right)$$
$$=\frac{1}{4}\left(\ln3-\frac{1}{3}+1\right)$$
$$=\frac{1}{4}\left(\ln3+\frac{2}{3}\right)$$
$$=\frac{3\ln3+2}{12}.$$

(f) (i) $P(x)=x^3-kx^2+5x+12$

$$P(3)=(3)^3-k(3)^2+5(3)+12$$
$$0=27-9k+15+12$$
$$9k=54$$
$$k=6.$$

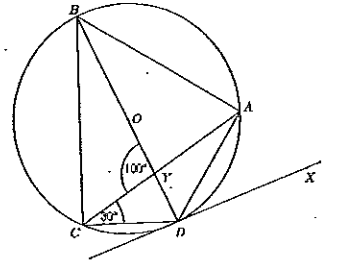
(ii) $P(x)=x^3-6x^2+5x+12$

$$=(x-3)(x^2-3x-4)$$
$$=(x-3)(x-4)(x+1)$$
$$x=3,\ 4,\ -1$$

$\therefore$ the zeros are $4,3,-1$.

Question 12

(a)



(i) $\angle BCD = 90°$ (angle in a semicircle)
$\angle ACB = 90° - 30°$
$= 60°$.

(ii) $\angle ADX = \angle ACD$ (Alt. Seg. Theorem)
$= 30°$.

(iii) $\angle BYC = \angle YCD + \angle CDB$ (ext. angle of triangle)
$100° = 30° + \angle CDB$
$\angle CDB = 70°$
$\angle CAB = \angle CDB$ (angles in the same segment)
$= 70°$.

(b) (i) If PQ is a focal chord then $(0, a)$ should satisfy the equation:

$$(p+q)x - 2y - 2apq = 0$$
$$(p+q)(0) - 2(a) - 2apq = 0$$
$$-2a - 2apq = 0$$
$$-2apq = 2a$$
$$\therefore pq = -1.$$

(ii) P is $(2ap, ap^2) \equiv (8a, 16a)$ $\therefore p = 4$

$$pq = -1$$
$$q = \frac{-1}{p}$$
$$= -\frac{1}{4}$$

Q is $(2aq, aq^2) \equiv \left(2a\left(-\frac{1}{4}\right), a\left(-\frac{1}{4}\right)^2\right)$

$$= \left(-\frac{a}{2}, \frac{a}{16}\right).$$

(c) (i) In $\triangle MOA$:

$$\tan 15° = \frac{h}{OA}$$
$$OA = \frac{h}{\tan 15°}$$
$$= h \cot 15°.$$

(ii) Similarly $OB = h \cot 13°$

Using Pythagoras' Theorem:

$$OB^2 = OA^2 + AB^2$$
$$AB^2 = OB^2 - OA^2$$
$$(2000)^2 = (h \cot 13°)^2 - (h \cot 15°)^2$$
$$2000^2 = h^2 \cot^2 13° - h^2 \cot^2 15°$$
$$= h^2 (\cot^2 13° - \cot^2 15°)$$

$$h^2 = \frac{2000^2}{\cot^2 13° - \cot^2 15°}$$
$$h = \frac{2000}{\sqrt{\cot^2 13° - \cot^2 15°}}$$
$$= 909.7038\ldots$$
$$= 909.7 \text{ m} \quad (1 \text{ d.p.})$$

(d) (i) By the cosine rule:

$$160^2 = r^2 + r^2 - 2 \times r \times r \times \cos\theta$$
$$= 2r^2 - 2r^2 \cos\theta$$
$$= 2r^2 (1 - \cos\theta).$$

(ii) The arc length is:

$$200 = r\theta$$
$$r = \frac{200}{\theta}$$

$$160^2 = 2r^2(1 - \cos\theta)$$
$$= 2 \times \left(\frac{200}{\theta}\right)^2 \times (1 - \cos\theta)$$
$$\theta^2 = \frac{2 \times 200^2}{160^2}(1 - \cos\theta)$$
$$= \frac{2 \times 5^2}{4^2}(1 - \cos\theta)$$
$$8\theta^2 = 25(1 - \cos\theta)$$
$$8\theta^2 = 25 - 25\cos\theta$$
$$\therefore 8\theta^2 + 25\cos\theta - 25 = 0.$$

(iii)
$$f(\theta) = 8\theta^2 + 25\cos\theta - 25$$
$$f'(\theta) = 16\theta - 25\sin\theta$$
$$\theta_1 = \pi$$
$$\theta_2 = \pi - \frac{8\pi^2 + 25\cos\pi - 25}{16\pi - 25\sin\pi}$$
$$= \pi - \frac{8\pi^2 - 25 - 25}{16\pi - 0}$$
$$= \pi - \frac{8\pi^2 - 50}{16\pi}$$
$$= 2.56551\ldots$$
$$= 2.57 \quad (2 \text{ d.p.}).$$

Question 13

(a) (i) From the graph when $v = 0$:
$x = 3$ m or 7 m.

(ii) Maximum when $v^2 = 11$:
$v = \sqrt{11}$ ms^{-1}.

(iii) Amplitude: $2a = 7 - 3$
$a = 2$

Centre of motion: $c = \frac{3+7}{2}$
$= 5$

When $x = 5, v^2 = 11$:

$$v^2 = n^2\left(a^2 - (x-c)^2\right)$$
$$11 = n^2\left(2^2 - (5-5)^2\right)$$
$$11 = 4n^2$$
$$n^2 = \frac{11}{4}$$
$$n = \frac{\sqrt{11}}{2}$$
$$\therefore a = 2, c = 5, n = \frac{\sqrt{11}}{2}.$$

(b) (i) For $\left(2x + \frac{1}{3x}\right)^{18}$:

$$T_{r+1} = {}^{18}C_r\, (2x)^{18-r} \left(\frac{1}{3}x^{-1}\right)^r$$
$$= Ax^{18-r}x^{-r}$$
$$= Ax^{18-2r}$$
$$x^{14} = x^{18-2r}$$
$$14 = 18 - 2r$$
$$r = 2$$
$$\therefore T_3 = {}^{18}C_2\, (2x)^{16} \left(\frac{1}{3}x^{-1}\right)^2$$
$$= {}^{18}C_2 \times 2^{16} \times \frac{1}{3}^2 \times x^{14}$$
$$= \frac{{}^{18}C_2 \times 2^{16}}{3^2} x^{14}$$
$$\therefore a_2 = \frac{{}^{18}C_2 \times 2^{16}}{3^2}.$$

(ii)
$$x^0 = x^{18-2r}$$
$$0 = 18 - 2r$$
$$r = 9$$
$$T_{10} = {}^{18}C_9\, (2x)^9 \left(\frac{1}{3}x^{-1}\right)^9$$
$$= {}^{18}C_9 \times 2^9 \times \frac{1}{3}^9 \times x^0$$
$$= \frac{{}^{18}C_9 \times 2^9}{3^9} x^0$$

The term independent of x is $\frac{{}^{18}C_9 \times 2^9}{3^9}$.

(c) Let $P(n)$ be the proposition that

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

for all integers $n \geq 1$.

$$P(1): \frac{1}{2!} = 1 - \frac{1}{(1+1)!}$$

$$LHS = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{(1+1)!} = RHS$$

$\therefore P(1)$ is true.

Assume $P(k)$ is true for integer $k \geq 1$.

i.e. $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{k}{(k+1)!}=1-\frac{1}{(k+1)!}$

$P(k+1)$:

$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{k+1}{(k+2)!}=1-\frac{1}{(k+2)!}$

$$LHS=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{k+1}{(k+2)!}$$
$$=1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}$$
$$=1-\frac{k+2}{(k+2)!}+\frac{k+1}{(k+2)!}$$
$$=1-\frac{(k+2)-(k+1)}{(k+2)!}$$
$$=1-\frac{1}{(k+2)!}$$
$$=RHS$$

$\therefore P(k+1)$ is true assuming $P(k)$ is true.

$\therefore P(n)$ is true by Mathematical Induction.

(d) (i) $f(x)=\cos^{-1}(x)+\cos^{-1}(-x)$

$$f'(x)=\frac{-1}{\sqrt{1-x^2}}+\frac{-1}{\sqrt{1-(-x)^2}}\times -1$$
$$=\frac{-1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^2}}$$
$$=0$$

The derivative is 0.

$\therefore f(x)$ is a constant.

(ii) Since $f(x)$ is a constant. It must be constant for any value. Let $x=0$:

$$f(0)=\cos^{-1}(0)+\cos^{-1}(0)$$
$$=\frac{\pi}{2}+\frac{\pi}{2}$$
$$=\pi$$

$\therefore \cos^{-1}(x)+\cos^{-1}(-x)=\pi$

$\therefore \cos^{-1}(-x)=\pi-\cos^{-1}(x)$.

Question 14

(a) (i) When $y=0$:

$$Vt\sin\theta-\frac{1}{2}gt^2=0$$
$$t\left(V\sin\theta-\frac{1}{2}gt\right)=0$$
$$V\sin\theta-\frac{1}{2}gt=0 \quad \text{or} \quad t=0$$
$$\frac{1}{2}gt=V\sin\theta$$
$$t=\frac{2V\sin\theta}{g}$$

Substitute this in $x=Vt\cos\theta$:

$$x=V\left(\frac{2V\sin\theta}{g}\right)t\cos\theta$$
$$=\frac{2V^2\sin\theta\cos\theta}{g}$$
$$=\frac{V^2\sin 2\theta}{g}.$$

(ii) $x=Vt\cos\theta$

$$\dot{x}=V\cos\theta$$
$$=V\cos\frac{\pi}{3}$$
$$=\frac{V}{2}$$

$$y=Vt\sin\theta-\frac{1}{2}gt^2$$
$$\dot{y}=V\sin\theta-gt$$
$$=V\sin\frac{\pi}{3}-g\left(\frac{2V}{\sqrt{3}g}\right)$$
$$=V\frac{\sqrt{3}}{2}-V\frac{2}{\sqrt{3}}$$
$$=V\left(\frac{\sqrt{3}}{2}-\frac{2\sqrt{3}}{3}\right)$$
$$=V\left(\frac{3\sqrt{3}-4\sqrt{3}}{6}\right)$$
$$=-\frac{\sqrt{3}}{6}V$$

$$\frac{\dot{y}}{\dot{x}}=-\frac{\sqrt{3}}{6}V\div\frac{V}{2}$$
$$=-\frac{\sqrt{3}}{6}\times\frac{2}{1}$$
$$=-\frac{\sqrt{3}}{3}$$
$$\tan\theta=-\frac{\sqrt{3}}{3}$$
$$\theta=-\frac{\pi}{6}.$$

(iii) From part (ii), the negative sign on the angle indicates that the particle is travelling downwards.

(b) (i)

$$\ddot{x}=x-1$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=x-1$$
$$\frac{1}{2}\dot{x}^2=\frac{1}{2}x^2-x+C$$

When $x=0$, $\dot{x}=1$:

$$\frac{1}{2}(1)^2=\frac{1}{2}(0)^2-(0)+C$$
$$C=\frac{1}{2}$$
$$\frac{1}{2}\dot{x}^2=\frac{1}{2}x^2-x+\frac{1}{2}$$
$$\dot{x}^2=x^2-2x+1$$
$$=(x-1)^2$$
$$\dot{x}=\pm(x-1)$$

When $x=0$, $\dot{x}=1$:

$$\dot{x}=\pm(x-1)$$
$$1=\pm(0-1)$$
$$\therefore \dot{x}=-(x-1)$$
$$=1-x.$$

(ii) $\dot{x}=1-x$

$$\frac{dx}{dt}=1-x$$
$$\frac{dt}{dx}=\frac{1}{1-x}$$
$$t=-\ln(1-x)+C$$

When $t=0$, $x=0$ $\therefore C=0$

$$t=-\ln(1-x)$$
$$-t=\ln(1-x)$$
$$e^{-t}=1-x$$
$$x=1-e^{-t}.$$

(iii) When $t\to\infty$, $e^{-t}\to 0$

$x=1-e^{-t}$

$\therefore x=1$ is the limiting position.

(c) (i) $P(5\text{ wins})=P(4\text{ wins in 6 games})\times P(\text{win in last game})$

$$=\binom{6}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2\times\left(\frac{1}{2}\right)$$
$$=\binom{6}{4}\left(\frac{1}{2}\right)^7.$$

(ii) From part (i)

$$P(5\text{ wins})=P(5\text{ wins in }5)+P(5\text{ wins in }6)+P(5\text{ wins in }7)$$
$$=\binom{4}{4}\left(\frac{1}{2}\right)^5+\binom{5}{4}\left(\frac{1}{2}\right)^6+\binom{6}{4}\left(\frac{1}{2}\right)^7.$$

(iii)

$$P(n+1\text{ wins})=\binom{n}{n}\left(\frac{1}{2}\right)^{n+1}+\binom{n+1}{n}\left(\frac{1}{2}\right)^{n+2}+\binom{n+2}{n}\left(\frac{1}{2}\right)^{n+3}+\ldots+\binom{2n}{n}\left(\frac{1}{2}\right)^{2n+1}$$
$$=\frac{1}{2}$$

$$\binom{n}{n}\left(\frac{1}{2}\right)^{n+1}+\binom{n+1}{n}\left(\frac{1}{2}\right)^{n+2}+\ldots+\binom{2n}{n}\left(\frac{1}{2}\right)^{2n+1}=\frac{1}{2}$$

Multiply throughout by 2^{2n+1}:

$$\binom{n}{n}2^n+\binom{n+1}{n}2^{n-1}+\ldots+\binom{2n}{n}=2^{2n}.$$

End of Mathematics Extension 1 solutions