BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2015 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 7–18

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

2610

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which conic has eccentricity $\dfrac{\sqrt{13}}{3}$?

(A) $\dfrac{x^2}{3}+\dfrac{y^2}{2}=1$

(B) $\dfrac{x^2}{3^2}+\dfrac{y^2}{2^2}=1$

(C) $\dfrac{x^2}{3}-\dfrac{y^2}{2}=1$

(D) $\dfrac{x^2}{3^2}-\dfrac{y^2}{2^2}=1$

2 What value of z satisfies $z^2=7-24i$?

(A) $4-3i$

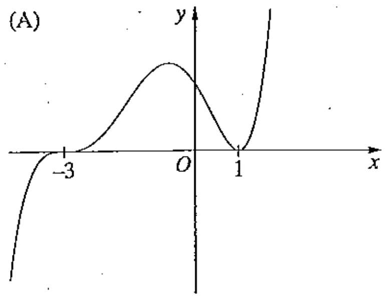
(B) $-4-3i$

(C) $3-4i$

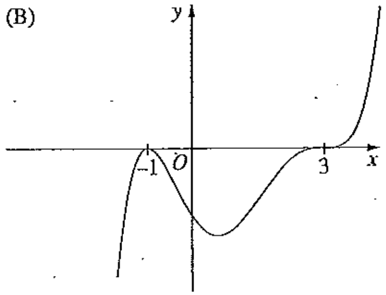
(D) $-3-4i$

3 Which graph best represents the curve $y=(x-1)^2(x+3)^5$?

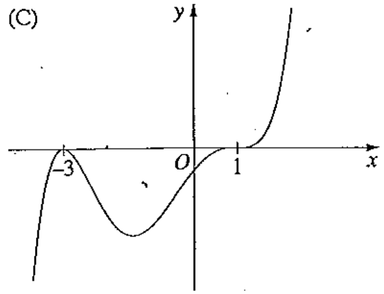
(A)



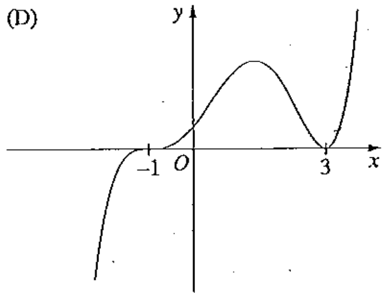
(B)



(C)



(D)



4 The polynomial x^3+x^2-5x+3 has a double root at $x=\alpha$.

What is the value of α?

(A) $-\dfrac{5}{3}$

(B) -1

(C) 1

(D) $\dfrac{5}{3}$

5 Given that $z = 1 - i$, which expression is equal to z^3?

(A) $\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right)$

(B) $2\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right)$

(C) $\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)$

(D) $2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)$

6 Which expression is equal to $\int x^2 \sin x \, dx$?

(A) $-x^2\cos x - \int 2x\cos x\, dx$

(B) $-2x\cos x + \int x^2 \cos x\, dx$

(C) $-x^2\cos x + \int 2x\cos x\, dx$

(D) $-2x\cos x - \int x^2\cos x\, dx$

7 The numbers $1, 2, \ldots n$, for $n \ge 4$, are randomly arranged in a row.

What is the probability that the number 1 is somewhere to the left of the number 2?

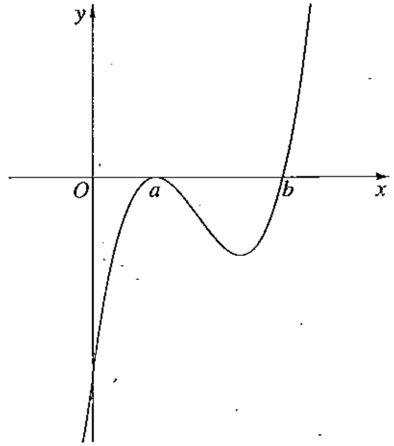
(A) $\frac{1}{2}$

(B) $\frac{1}{n}$

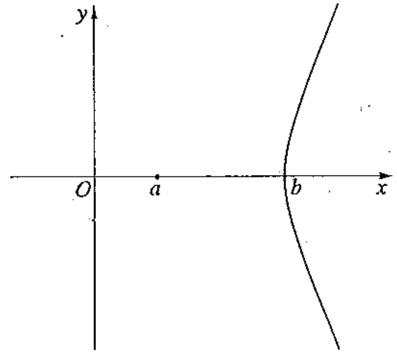
(C) $\frac{1}{2(n-2)!}$

(D) $\frac{1}{2(n-1)!}$

8 The graph of the function $y = f(x)$ is shown.



A second graph is obtained from the function $y = f(x)$.



Which equation best represents the second graph?

(A) $y^2 = |f(x)|$

(B) $y^2 = f(x)$

(C) $y = \sqrt{f(x)}$

(D) $y = f\left(\sqrt{x}\right)$

9 The complex number z satisfies $|z-1|=1$.

What is the greatest distance that z can be from the point i on the Argand diagram?

(A) 1

(B) $\sqrt{5}$

(C) $2\sqrt{2}$

(D) $\sqrt{2}+1$

10 Consider the expansion of

$$\left(1+x+x^2+\cdots+x^n\right)\left(1+2x+3x^2+\cdots+(n+1)x^n\right).$$

What is the coefficient of x^n when $n = 100$?

(A) 4950

(B) 5050

(C) 5151

(D) 5253

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Express $\dfrac{4+3i}{2-i}$ in the form $x+iy$, where x and y are real. **2**

(b) Consider the complex numbers $z=-\sqrt{3}+i$ and $w=3\left(\cos\dfrac{\pi}{7}+i\sin\dfrac{\pi}{7}\right)$.

(i) Evaluate $|z|$. **1**

(ii) Evaluate $\arg(z)$. **1**

(iii) Find the argument of $\dfrac{z}{w}$. **1**

(c) Find A, B and C such that $\dfrac{1}{x\left(x^2+2\right)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+2}$. **2**

(d) Sketch $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$ indicating the coordinates of the foci. **2**

(e) Find the value of $\dfrac{dy}{dx}$ at the point $(2,-1)$ on the curve $x+x^2y^3=-2$. **3**

(f) (i) Show that $\cot\theta+\operatorname{cosec}\theta=\cot\left(\dfrac{\theta}{2}\right)$. **2**

(ii) Hence, or otherwise, find $\displaystyle\int(\cot\theta+\operatorname{cosec}\theta)\,d\theta$. **1**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The complex number z is such that $|z| = 2$ and $\arg(z) = \frac{\pi}{4}$.

Plot each of the following complex numbers on the same half-page Argand diagram.

(i) z 1

(ii) $u = z^2$ 1

(iii) $v = z^2 - \overline{z}$ 1

(b) The polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ has roots $a + ib$ and $a + 2ib$ where a and b are real and $b \neq 0$.

(i) By evaluating a and b, find all the roots of $P(x)$. 3

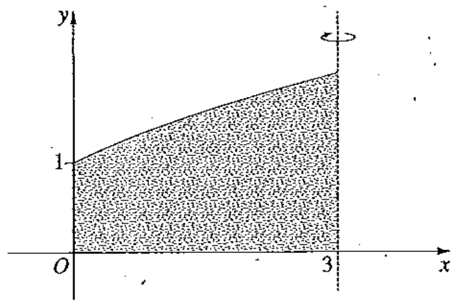
(ii) Hence, or otherwise, find one quadratic polynomial with real coefficients that is a factor of $P(x)$. 1

(c) (i) By writing $\frac{(x-2)(x-5)}{x-1}$ in the form $mx + b + \frac{a}{x-1}$, find the equation of the oblique asymptote of $y = \frac{(x-2)(x-5)}{x-1}$. 2

(ii) Hence sketch the graph $y = \frac{(x-2)(x-5)}{x-1}$, clearly indicating all intercepts and asymptotes. 2

Question 12 continues on page 9

(d) The diagram shows the graph $y = \sqrt{x+1}$ for $0 \le x \le 3$. The shaded region is rotated about the line $x = 3$ to form a solid. 4



Use the method of cylindrical shells to find the volume of the solid.

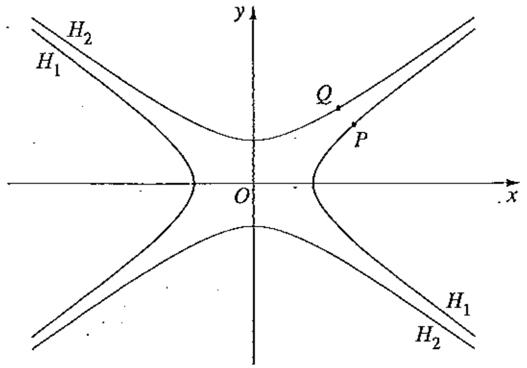
End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The hyperbolas $H_1: \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $H_2: \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$ are shown in the diagram.

Let $P(a\sec\theta, b\tan\theta)$ lie on H_1 as shown on the diagram.

Let Q be the point $(a\tan\theta, b\sec\theta)$.



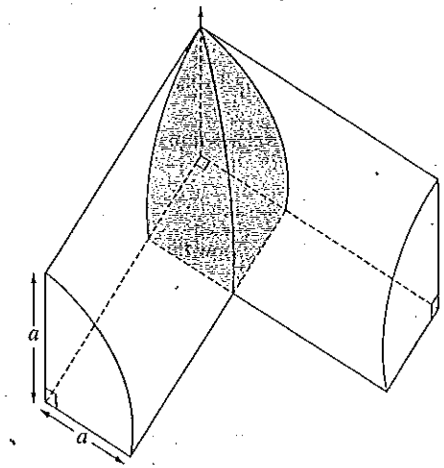
(i) Verify that the coordinates of $Q(a\tan\theta, b\sec\theta)$ satisfy the equation for H_2. **1**

(ii) Show that the equation of the line PQ is $bx + ay = ab(\tan\theta + \sec\theta)$. **2**

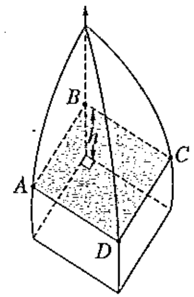
(iii) Prove that the area of $\triangle OPQ$ is independent of θ. **3**

Question 13 continues on page 11

(b) Two quarter cylinders, each of radius a, intersect at right angles to form the shaded solid.



A horizontal slice $ABCD$ of the solid is taken at height h from the base. You may assume that $ABCD$ is a square, and is parallel to the base.



(i) Show that $AB = \sqrt{a^2 - h^2}$. **1**

(ii) Find the volume of the solid. **2**

Question 13 continues on page 12

(c) A small spherical balloon is released and rises into the air. At time t seconds, it has radius r cm, surface area $S = 4\pi r^2$ and volume $V = \frac{4}{3}\pi r^3$.

As the balloon rises it expands, causing its surface area to increase at a rate of $\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$ cm^2 s^{-1}. As the balloon expands it maintains a spherical shape.

(i) By considering the surface area, show that $\frac{dr}{dt} = \frac{1}{8\pi r}\left(\frac{4}{3}\pi\right)^{\frac{1}{3}}$. **2**

(ii) Show that $\frac{dV}{dt} = \frac{1}{2}V^{\frac{1}{3}}$. **2**

(iii) When the balloon is released its volume is 8000 cm^3. When the volume of the balloon reaches 64 000 cm^3 it will burst. **2**

How long after it is released will the balloon burst?

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Differentiate $\sin^{n-1}\theta\cos\theta$, expressing the result in terms of $\sin\theta$ only. **2**

(ii) Hence, or otherwise, deduce that $\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta = \frac{(n-1)}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\, d\theta$, **2**

for $n > 1$.

(iii) Find $\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta$. **1**

(b) The cubic equation $x^3 - px + q = 0$ has roots α, β and γ.

It is given that $\alpha^2 + \beta^2 + \gamma^2 = 16$ and $\alpha^3 + \beta^3 + \gamma^3 = -9$.

(i) Show that $p = 8$. **1**

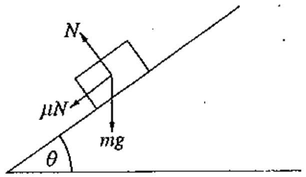
(ii) Find the value of q. **2**

(iii) Find the value of $\alpha^4 + \beta^4 + \gamma^4$. **2**

Question 14 continues on page 14

Question 14 (continued)

(c) A car of mass m is driven at speed v around a circular track of radius r. The track is banked at a constant angle θ to the horizontal, where $0 < \theta < \frac{\pi}{2}$. At the speed v there is a tendency for the car to slide up the track. This is opposed by a frictional force μN, where N is the normal reaction between the car and the track, and $\mu > 0$. The acceleration due to gravity is g.



(i) Show that $v^2 = rg\left(\dfrac{\tan\theta + \mu}{1 - \mu\tan\theta}\right)$. **3**

(ii) At the particular speed V, where $V^2 = rg$, there is still a tendency for the car to slide up the track. **2**

Using the result from part (i), or otherwise, show that $\mu < 1$.

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) A particle A of unit mass travels horizontally through a viscous medium. When $t = 0$, the particle is at point O with initial speed u. The resistance on particle A due to the medium is kv^2, where v is the velocity of the particle at time t and k is a positive constant.

When $t = 0$, a second particle B of equal mass is projected vertically upwards from O with the same initial speed u through the same medium. It experiences both a gravitational force and a resistance due to the medium. The resistance on particle B is kw^2, where w is the velocity of the particle B at time t. The acceleration due to gravity is g.

(i) Show that the velocity v of particle A is given by $\dfrac{1}{v} = kt + \dfrac{1}{u}$. **2**

(ii) By considering the velocity w of particle B, show that **3**

$$t = \frac{1}{\sqrt{gk}}\left(\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) - \tan^{-1}\left(w\sqrt{\frac{k}{g}}\right)\right).$$

(iii) Show that the velocity V of particle A when particle B is at rest is given by **1**

$$\frac{1}{V} = \frac{1}{u} + \sqrt{\frac{k}{g}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right).$$

(iv) Hence, if u is very large, explain why $V \approx \dfrac{2}{\pi}\sqrt{\dfrac{g}{k}}$. **1**

Question 15 continues on page 16

Question 15 (continued)

(b) Suppose that $x \geq 0$ and n is a positive integer.

(i) Show that $1 - x \leq \dfrac{1}{1+x} \leq 1$. **2**

(ii) Hence, or otherwise, show that $1 - \dfrac{1}{2n} \leq n \ln\left(1 + \dfrac{1}{n}\right) \leq 1$. **2**

(iii) Hence, explain why $\lim\limits_{n \to \infty}\left(1 + \dfrac{1}{n}\right)^n = e$. **1**

(c) For positive real numbers x and y, $\sqrt{xy} \leq \dfrac{x+y}{2}$. (Do NOT prove this.)

(i) Prove $\sqrt{xy} \leq \sqrt{\dfrac{x^2 + y^2}{2}}$, for positive real numbers x and y. **1**

(ii) Prove $\sqrt[4]{abcd} \leq \sqrt{\dfrac{a^2 + b^2 + c^2 + d^2}{4}}$, for positive real numbers a, b, c and d. **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) A table has 3 rows and 5 columns, creating 15 cells as shown. **2**

Counters are to be placed randomly on the table so that there is one counter in each cell. There are 5 identical black counters and 10 identical white counters.

Show that the probability that there is exactly one black counter in each column is $\dfrac{81}{1001}$.

(ii) The table is extended to have n rows and q columns. There are nq counters, where q are identical black counters and the remainder are identical white counters. The counters are placed randomly on the table with one counter in each cell. **2**

Let P_n be the probability that each column contains exactly one black counter.

Show that $P_n = \dfrac{n^q}{\binom{nq}{q}}$.

(iii) Find $\lim\limits_{n \to \infty} P_n$. **2**

Question 16 continues on page 18

Question 16 (continued)

(b) Let n be a positive integer.

(i) By considering $(\cos\alpha + i\sin\alpha)^{2n}$, show that **2**

$$\cos(2n\alpha) = \cos^{2n}\alpha - \binom{2n}{2}\cos^{2n-2}\alpha\sin^2\alpha + \binom{2n}{4}\cos^{2n-4}\alpha\sin^4\alpha - \cdots$$

$$+ \cdots + (-1)^{n-1}\binom{2n}{2n-2}\cos^2\alpha\,\sin^{2n-2}\alpha + (-1)^n\sin^{2n}\alpha.$$

Let $T_{2n}(x) = \cos\left(2n\cos^{-1}x\right)$, for $-1 \le x \le 1$.

(ii) Show that **2**

$$T_{2n}(x) = x^{2n} - \binom{2n}{2}x^{2n-2}\left(1-x^2\right) + \binom{2n}{4}x^{2n-4}\left(1-x^2\right)^2 + \cdots + (-1)^n\left(1-x^2\right)^n.$$

(iii) By considering the roots of $T_{2n}(x)$, find the value of **3**

$$\cos\left(\frac{\pi}{4n}\right)\cos\left(\frac{3\pi}{4n}\right)\cdots\cos\left(\frac{(4n-1)\pi}{4n}\right).$$

(iv) Prove that **2**

$$1 - \binom{2n}{2} + \binom{2n}{4} - \binom{2n}{6} + \cdots + (-1)^n\binom{2n}{2n} = 2^n\cos\left(\frac{n\pi}{2}\right).$$

End of paper

2015 Higher School Certificate Solutions Mathematics Extension 2

SECTION I

Summary

1 D	3 A	5 B	7 A	9 D
2 A	4 C	6 C	8 B	10 C

SECTION I

1 (D) $e=\frac{\sqrt{13}}{3}>1$

$\therefore$ hyperbola thus it is (C) or (D)

$b^2=a^2\left(e^2-1\right)$

$\frac{b^2}{a^2}=e^2-1$

$e^2=\frac{b^2}{a^2}+1$

$\left(\frac{\sqrt{13}}{3}\right)^2=\frac{b^2}{a^2}+1$

$\frac{13}{9}=\frac{b^2}{a^2}+1$

$\frac{4}{9}=\frac{b^2}{a^2}$

$b=2,\quad a=3$

$\frac{x^2}{3^2}-\frac{y^2}{2^2}=1$

$\therefore$ (D)

2 (A) Let $z=a+ib$

$z^2=a^2-b^2+2iab$

But $z^2=7-24i$

$\therefore\ 7=a^2-b^2$

$-24=2ab$

By inspection: $a=\pm4,\quad b=\mp3$

$\therefore z=4-3i.$

3 (A) $y=(x-1)^2(x+3)^5$

A double root at $x=1$.

A quintuple root at $x=-3$.

$\therefore$ (A).

4 (C) $f(x)=x^3+x^2-5x+3$

$f'(x)=3x^2+2x-5$

$=(3x+5)(x-1)$

Try $x=1$:

$f(1)=(1)^3+(1)^2-5(1)+3=0$

$f'(x)=3(1)^2+2(1)-5=0$

$\therefore$ Double root at $x=1$.

5 (B) $z=1-i$

$=\sqrt{2}\,cis\left(-\frac{\pi}{4}\right)$

$z^3=2\sqrt{2}\,cis\left(-\frac{3\pi}{4}\right).$

6 (C) Integration by parts, where:

$u=x^2 \qquad v=-\cos x$

$du=2x\,dx \qquad dv=\sin x\,dx$

$\int x^2\sin x\,dx=2x.(-\cos x)-\int x^2.-\cos x\,dx$

$=-2x\cos x+\int x^2\cos x\,dx.$

7 (A) *Method 1:*

Once the 2 is in place, the 1 can be either to the left or to the right of the 2. Because the 2 can be placed anywhere, the problem is symmetrical.

$\therefore\ P(\text{left of }2)=P(\text{right of }2)=0.5.$

OR

Method 2:

For $n=4$, consider the probability of the 2, then consider the probability of a favourable position for the 1:

2 _ _ _	$\frac{1}{4}\times0=0$
_ 2 _ _	$\frac{1}{4}\times\frac{1}{3}=\frac{1}{12}$
_ _ 2 _	$\frac{1}{4}\times\frac{2}{3}=\frac{1}{6}$
_ _ _ 2	$\frac{1}{4}\times1=\frac{1}{4}$

Total probability $=\frac{1}{4}+\frac{1}{6}+\frac{1}{12}$

$=\frac{1}{2}$

Similarly, with n numbers, the probability is:

$\frac{1}{n}\cdot\frac{0}{n-1}+\frac{1}{n}\cdot\frac{1}{n-1}+\frac{1}{n}\cdot\frac{2}{n-1}+\ldots+\frac{1}{n}\cdot\frac{n-1}{n-1}$

$=\frac{1}{n}\cdot\frac{1}{n-1}(0+1+2+\ldots+(n-1))$

$=\frac{1}{n}\cdot\frac{1}{n-1}\times\frac{n}{2}(0+n-1)$

$=\frac{1}{n}\cdot\frac{1}{n-1}\cdot\frac{n}{2}(n-1)$

$=\frac{1}{2}$

8 (B) The new function exists for $x=a$ and $x\geq b$.

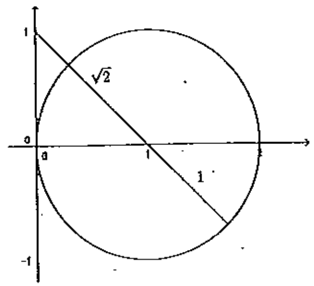
Thus it is $y=\sqrt{f(x)}$.

But within this domain, it also exists for negative y.

$\therefore\ y=\pm\sqrt{f(x)}$

$y^2=f(x)$.

9 (D)



Distance $=\sqrt{1^2+1^2}+1$

$=\sqrt{2}+1.$

10 (C) The terms in x^{100} are:

$1\times101x^{100}+x\times100x^{99}+x^2\times99x^{98}+\ldots+x^{100}\times1$

The coefficients of each term with x^{100} are: $1+2+3+\ldots+101$

$a=1,\quad \ell=101,\quad n=101$

$S_{101}=\frac{101}{2}(1+101)$

$=5151.$

SECTION II

Question 11

(a) $\frac{4+3i}{2-i}=\frac{4+3i}{2-i}\times\frac{2+i}{2+i}$

$=\frac{8+4i+6i+3(-1)}{4+1}$

$=\frac{8-3+i(4+6)}{4+1}$

$=\frac{5+10i}{5}$

$=1+2i.$

(b) (i) $|z|=\left|-\sqrt{3}+i\right|$

$=\sqrt{\left(\sqrt{3}\right)^2+1^2}$

$=2.$

(ii) *Method 1:*

$\arg(z)=\alpha$

where $\tan\alpha=\frac{1}{-\sqrt{3}}$

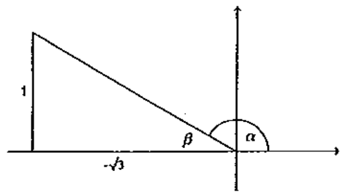
and z is in the second quadrant.

$\arg(z)=\tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$

$=\pi-\frac{\pi}{6}$

$=\frac{5\pi}{6}.$

OR

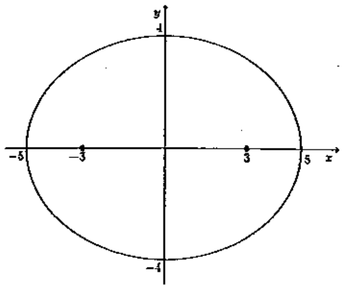


$$\tan\beta = \frac{1}{\sqrt{3}}$$
$$\beta = \frac{\pi}{6}$$
$$\arg(z) = \pi - \frac{\pi}{6}$$
$$= \frac{5\pi}{6}.$$

(iii) $$\arg\left(\frac{z}{w}\right) = \frac{5\pi}{6} - \frac{\pi}{7}$$
$$= \frac{29\pi}{42}.$$

(c) $$\frac{1}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$
$$1 = A(x^2+2) + (Bx+C)x$$
$$= (A+B)x^2 + Cx + 2A$$
By equating coefficients:
$$2A = 1$$
$$A = \frac{1}{2}$$
$$C = 0$$
$$A + B = 0$$
$$B = -A$$
$$= -\frac{1}{2}$$
$$\therefore A = \frac{1}{2},\quad B = -\frac{1}{2},\quad C = 0.$$

(d) $$b^2 = a^2(1-e^2)$$
$$e^2 = 1 - \frac{16}{25} = \frac{9}{25}, \therefore e = \frac{3}{5}.$$
The foci $(\pm ae, 0)$ are $(\pm 3, 0)$:



(e) $$x + x^2y^3 = -2$$
$$1 + 2xy^3 + x^2.3y^2.\frac{dy}{dx} = 0$$
$$3x^2y^2\frac{dy}{dx} = -1 - 2xy^3$$
$$\frac{dy}{dx} = \frac{-1-2xy^3}{3x^2y^2}$$
At $(2,1)$:
$$\frac{dy}{dx} = \frac{-1-2xy^3}{3x^2y^2}$$
$$= \frac{-1-2(2)(-1)^3}{3(2)^2(-1)^2}$$
$$= \frac{-1+4}{12}$$
$$= \frac{1}{4}.$$

(f) (i) *Method 1*
$$\cot\theta + \operatorname{cosec}\theta = \frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}$$
$$= \frac{\cos\theta + 1}{\sin\theta}$$
$$= \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$
$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$
$$= \cot\frac{\theta}{2}.$$

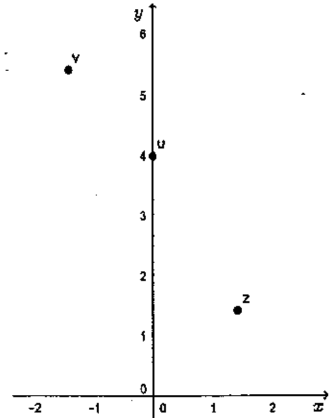
OR

Method 2
Let $t = \tan\frac{\theta}{2}$:
$$\cot\theta + \operatorname{cosec}\theta = \frac{1}{\tan\theta} + \frac{1}{\sin\theta}$$
$$= \frac{1-t^2}{2t} + \frac{1+t^2}{2t}$$
$$= \frac{2}{2t}$$
$$= \frac{1}{t}$$
$$= \cot\frac{\theta}{2}.$$

(ii) $$\int(\cot\theta + \operatorname{cosec}\theta)\,d\theta = \int\cot\frac{\theta}{2}\,d\theta$$
$$\int\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,d\theta$$
$$= 2\int\frac{\frac{1}{2}\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,d\theta$$
$$= 2\ln\left(\sin\frac{\theta}{2}\right) + C.$$

Question 12

(a) $$z = 2\operatorname{cis}\frac{\pi}{4} = \sqrt{2} + \sqrt{2}i$$
$$u = z^2 = 4\operatorname{cis}\frac{\pi}{2} = 4i$$
$$v = z^2 - \bar{z}$$
$$= 4i - (\sqrt{2} - \sqrt{2}i)$$
$$= -\sqrt{2} + (4 + \sqrt{2})i$$



(b) (i) The roots will be in conjugate pairs.
$\therefore$ The roots are $a \pm ib$ and $a \pm 2ib$.
$$\Sigma\alpha = (a+ib) + (a-ib) + (a+2ib) + (a-2ib)$$
$$4 = 4a$$
$$a = 1$$
$$\alpha\beta\gamma\delta = (a+ib)(a-ib)(a+2ib)(a-2ib)$$
$$10 = (a^2+b^2)(a^2+4b^2)$$
$$= (1^2+b^2)(1^2+4b^2)$$
$$= 1 + b^2 + 4b^2 + 4b^4$$
$$= 1 + 5b^2 + 4b^4$$
$$0 = 4b^4 + 5b^2 - 9$$
$$= (b^2-1)(4b^2+9)$$
$$b = \pm 1,\quad \pm\frac{3}{2}i$$
But b is real.
$\therefore a = 1,\quad b = \pm 1$
$\therefore$ The roots are $1 \pm i$ and $1 \pm 2i$.

(ii) *Method 1:*
$$P(x) = (x-(1+i))(x-(1-i)) \times (x-(1+2i))(x-(1-2i))$$
$$= (x^2 - 2x + 2)(x^2 - 2x + 5)$$
$\therefore$ $x^2 - 2x + 2$ or $x^2 - 2x + 5$ is a quadratic factor of $P(x)$.

OR

Method 2:
Let the roots be $\alpha = 1+i$ and $\beta = 1-i$.

$$\alpha + \beta = 2$$
$$\alpha\beta = 2$$

$\therefore\ x^2 - 2x + 2$ is a quadratic factor of $P(x)$.

OR

Method 3:
Let the roots be $\alpha = 1+2i$ and $\beta = 1-2i$.

$$\alpha + \beta = 2$$
$$\alpha\beta = 5$$

$\therefore\ x^2 - 2x + 5$ is a quadratic factor of $P(x)$.

(c) (i) $\dfrac{(x-2)(x-5)}{x-1} = \dfrac{x^2-7x+10}{x-1}$

$$\begin{array}{r} x-6 \\ x-1\overline{)\,x^2-7x+10} \\ \underline{x^2-x} \\ -6x+10 \\ \underline{-6x+6} \\ 4 \end{array}$$

$$\frac{(x-2)(x-5)}{x-1} = \frac{x^2-7x+10}{x-1}$$
$$= x - 6 + \frac{4}{x-1}$$

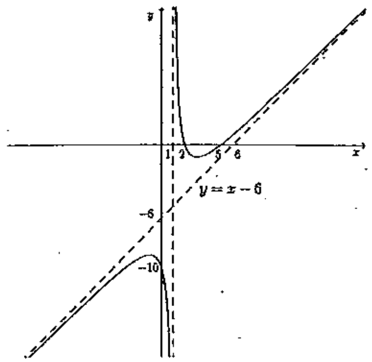
Equation of the oblique asymptote is $y = x - 6$.

(ii) Vertical asymptote: $x = 1$

Oblique asymptote: $y = x - 6$

x-intercepts: $x = 2,\ \ 5$

y-intercept: $y = -10$



(d) $\delta V = 2\pi r h \delta x$

where $r = 3 - x,\ h = y,\ y = \sqrt{x+1}$

$$\delta V = 2\pi(3-x)y\,\delta x$$
$$= 2\pi(3-x)\sqrt{x+1}\,\delta x$$

$$V = 2\pi\int_0^3 (3-x)\sqrt{x+1}\ dx$$

$$u^2 = x+1 \quad \Rightarrow x = u^2 - 1$$
$$3 - x = 4 - u^2$$

Let $2u\,du = dx$

when $x = 3,\quad u = 4$

when $x = 0,\quad u = 1$

$$V = 2\pi\int_0^3 (3-x)\sqrt{x+1}\ dx$$
$$= 2\pi\int_1^2 \left(4-u^2\right)u\,2u\ du$$
$$= 4\pi\int_1^2 \left(4-u^2\right)u^2\ du$$
$$= 4\pi\int_1^2 \left(4u^2-u^4\right)\ du$$
$$= 4\pi\left[\frac{4u^3}{3} - \frac{u^5}{5}\right]_1^2$$
$$= 4\pi\left(\frac{32}{3} - \frac{32}{5} - \left(\frac{4}{3} - \frac{1}{5}\right)\right)$$

$$V = 4\pi\left(\frac{28}{3} - \frac{31}{5}\right)$$
$$= 4\pi\left(\frac{140}{15} - \frac{93}{15}\right)$$
$$= 4\pi \times \left(\frac{47}{15}\right)$$
$$= \frac{188\pi}{15}\ \text{units}^3.$$

Question 13

(a) (i) $LHS = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

$$= \frac{(a\tan\theta)^2}{a^2} - \frac{(b\sec\theta)^2}{b^2}$$
$$= \tan^2\theta - \sec^2\theta$$
$$= \tan^2\theta - \left(1 + \tan^2\theta\right)$$
$$= -1$$
$$= RHS$$

$\therefore Q$ lies on H_2.

(ii) $m = \dfrac{b\sec\theta - b\tan\theta}{a\tan\theta - a\sec\theta}$

$$= \frac{b(\sec\theta - \tan\theta)}{a(\tan\theta - \sec\theta)}$$
$$= \frac{-b(\tan\theta - \sec\theta)}{a(\tan\theta - \sec\theta)}$$
$$= -\frac{b}{a}$$

$$y - y_1 = m(x - x_1)$$
$$y - b\sec\theta = -\frac{b}{a}(x - a\tan\theta)$$
$$ay - ab\sec\theta = -bx + ab\tan\theta$$
$$bx + ay = ab\tan\theta + ab\sec\theta$$
$$bx + ay = ab(\tan\theta + \sec\theta).$$

(iii) Use PQ as the base of the triangle:

$$PQ = \sqrt{(a\sec\theta - a\tan\theta)^2 + (b\tan\theta - b\sec\theta)^2}$$
$$= \sqrt{a^2(\sec\theta - \tan\theta)^2 + (-b)^2(\sec\theta - \tan\theta)^2}$$
$$= |\sec\theta - \tan\theta|\sqrt{a^2+b^2}$$

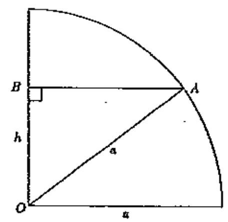
Let h be the perpendicular distance from O to PQ.

$$h = \left|\frac{b(0) + a(0) - ab(\tan\theta + \sec\theta)}{\sqrt{a^2+b^2}}\right|$$
$$= \left|\frac{ab(\tan\theta + \sec\theta)}{\sqrt{a^2+b^2}}\right|$$

$$A_{\Delta OPQ} = \frac{1}{2} \times |\sec\theta - \tan\theta|\sqrt{a^2+b^2} \times \left|\frac{ab(\tan\theta + \sec\theta)}{\sqrt{a^2+b^2}}\right|$$
$$= \frac{1}{2} \times ab\left|-\tan^2\theta + \sec^2\theta\right|$$
$$= \frac{1}{2} \times ab \times 1$$
$$= \frac{ab}{2}$$

This area is independent of θ.

(b) (i) The view from the side where A and B are situated is:



$$OA^2 = AB^2 + OB^2$$
$$AB^2 = OA^2 - OB^2$$
$$= a^2 - h^2$$
$$AB = \sqrt{a^2 - h^2}.$$

(ii) $V=\int_0^a \left(\sqrt{a^2-h^2}\right)^2\,dh$

$=\int_0^a \left(a^2-h^2\right)\,dh$

$=\left[a^2h-\frac{h^3}{3}\right]_0^a$

$=a^3-\frac{a^3}{3}$

$=\frac{2a^3}{3}$ units^3.

(c) (i) $\frac{dS}{dt}=\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$ (given)

$S=4\pi r^2$

$\frac{dS}{dr}=8\pi r$

$\frac{dS}{dt}=\frac{dS}{dr}\times\frac{dr}{dt}$

$\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}=8\pi r\times\frac{dr}{dt}$

$\frac{dr}{dt}=\frac{1}{8\pi r}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}.$

(ii) $V=\frac{4}{3}\pi r^3$

$\frac{dV}{dr}=4\pi r^2$

$\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$

$=4\pi r^2\times\frac{1}{8\pi r}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$

$=\frac{r}{2}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$

$=\frac{1}{2}\left(\frac{4\pi r^3}{3}\right)^{\frac{1}{3}}$

$=\frac{1}{2}V^{\frac{1}{3}}.$

(iii) $\frac{dV}{dt}=\frac{1}{2}V^{\frac{1}{3}}$

$2\,dV=V^{\frac{1}{3}}\,dt$

$2V^{-\frac{1}{3}}\,dV=1\,dt$

$\int 2V^{-\frac{1}{3}}\,dV=\int dt$

$\int dt=\int 2V^{-\frac{1}{3}}\,dV$

$t=3V^{\frac{2}{3}}+C$

When $t=0,\ V=8000$:

$t=3V^{\frac{2}{3}}+C$

$0=3(8000)^{\frac{2}{3}}+C$

$0=3\times 400+C$

$C=-1200$

$t=3V^{\frac{2}{3}}-1200$

When $V=64\,000$:

$t=3V^{\frac{2}{3}}-1200$

$=3(64\,000)^{\frac{2}{3}}-1200$

$=3(40)^2-1200$

$=3600$ seconds

$=1$ hour.

Question 14

(a) (i) $P(\theta)=\sin^{n-1}\theta\cos\theta$

$P'(\theta)=(n-1)\sin^{n-2}\theta.\cos\theta.\cos\theta$
$\quad-\sin^{n-1}\theta\sin\theta$

$=(n-1)\sin^{n-2}\theta\cos^2\theta-\sin^n\theta$

$=(n-1)\sin^{n-2}\theta\left(1-\sin^2\theta\right)-\sin^n\theta$

$=(n-1)\sin^{n-2}\theta-(n-1)\sin^n\theta-\sin^n\theta$

$=(n-1)\sin^{n-2}\theta-\sin^n\theta\left((n-1)+1\right)$

$=(n-1)\sin^{n-2}\theta-n\sin^n\theta$

(ii) From part (i):
If $y=\sin^{n-1}\theta\cos\theta$, then

$\frac{dy}{d\theta}=(n-1)\sin^{n-2}\theta-n\sin^n\theta$

$dy=\left((n-1)\sin^{n-2}\theta-n\sin^n\theta\right)d\theta$

Integrate both sides:

$\int_0^{\frac{\pi}{2}} dy=\int_0^{\frac{\pi}{2}}(n-1)\sin^{n-2}\theta\,d\theta-\int_0^{\frac{\pi}{2}}n\sin^n\theta\,d\theta$

$[y]_0^{\frac{\pi}{2}}=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta\,d\theta-n\int_0^{\frac{\pi}{2}}\sin^n\theta\,d\theta$

Substitute $y=\sin^{n-1}\theta\cos\theta$:

$\left[\sin^{n-1}\theta\cos\theta\right]_0^{\frac{\pi}{2}}=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta\,d\theta$
$\quad-n\int_0^{\frac{\pi}{2}}\sin^n\theta\,d\theta$

The LHS is 0:

$0=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta\,d\theta-n\int_0^{\frac{\pi}{2}}\sin^n\theta\,d\theta$

Rearranging:

$n\int_0^{\frac{\pi}{2}}\sin^n\theta\,d\theta=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta\,d\theta$

$\int_0^{\frac{\pi}{2}}\sin^n\theta\,d\theta=\frac{(n-1)}{n}\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta\,d\theta.$

(iii) Let $I_4=\int_0^{\frac{\pi}{2}}\sin^4\theta\,d\theta$

Using the result from part (ii):

$I_4=\frac{3}{4}I_2$

$=\frac{3}{4}\cdot\frac{1}{2}I_0$

$=\frac{3}{8}I_0$

$I_4=\frac{3}{8}\int_0^{\frac{\pi}{2}}d\theta=\frac{3}{8}\left[\,\theta\,\right]_0^{\frac{\pi}{2}}=\frac{3}{8}\cdot\frac{\pi}{2}$

$\therefore\ I_4=\frac{3\pi}{16}.$

(b) (i) $(\alpha+\beta+\gamma)^2=\Sigma\alpha^2+2(\Sigma\alpha\beta)$

$0=16+2(-p)$

$2p=16$

$p=8.$

(ii) $x^3-px+q=0$
Let the roots be α,β,γ:

$\alpha^3-p\alpha+q=0$

$\beta^3-p\beta+q=0$

$\gamma^3-p\gamma+q=0$

Combining these:

$\Sigma\alpha^3-p(\Sigma\alpha)+3q=0$

$-9-p(0)+3q=0$

$3q=9$

$q=3.$

(iii) $x^3=px-q$

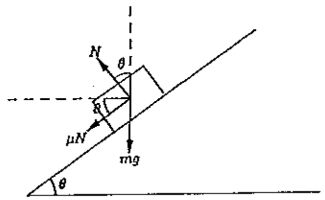
$x^4=px^2-qx$

$\Sigma\alpha^4=p\left(\Sigma\alpha^2\right)-q(\Sigma\alpha)$

$=8(16)-3(0)$

$=128.$

(c) (i)



Resolving the forces:

Vertical: $N\cos\theta-\mu N\sin\theta=mg$ ①

Horizontal: $N\sin\theta+\mu N\cos\theta=\frac{mv^2}{r}$ ②

② ÷ ①:

$\frac{mv^2}{r}\div mg=\frac{N\sin\theta+\mu N\cos\theta}{N\cos\theta-\mu N\sin\theta}$

$\frac{v^2}{rg}=\frac{\sin\theta+\mu\cos\theta}{\cos\theta-\mu\sin\theta}$

$$\frac{v^2}{rg}=\frac{\tan\theta+\mu}{1-\mu\tan\theta}$$

$$v^2=rg\left(\frac{\tan\theta+\mu}{1-\mu\tan\theta}\right).$$

(ii) If $V^2=rg$ then

$$v^2=rg\left(\frac{\tan\theta+\mu}{1-\mu\tan\theta}\right)$$

$$rg=rg\left(\frac{\tan\theta+\mu}{1-\mu\tan\theta}\right)$$

$$1=\left(\frac{\tan\theta+\mu}{1-\mu\tan\theta}\right)$$

$$1-\mu\tan\theta=\tan\theta+\mu$$

$$\mu+\mu\tan\theta=1-\tan\theta$$

$$\mu(1+\tan\theta)=1-\tan\theta$$

$$\mu=\frac{1-\tan\theta}{1+\tan\theta}$$

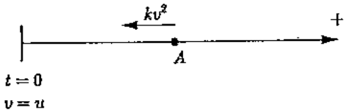
For $0<\theta<\frac{\pi}{2}$ then $\tan\theta>0$:

$$\frac{1-\tan\theta}{1+\tan\theta}<1$$

$$\mu<1.$$

Question 15

(a) (i) For particle A:



$$\frac{dv}{dt}=-kv^2$$

$$\frac{dv}{v^2}=-k.dt$$

$$\int\frac{1}{v^2}\,dv=-k\int dt$$

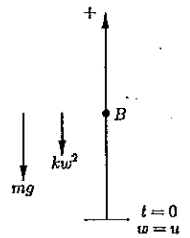
$$-v^{-1}=-kt+C$$

When $t=0$, $v=u$ $\quad\therefore C=-u^{-1}$

$$-v^{-1}=-kt-u^{-1}$$

$$\frac{1}{v}=kt+\frac{1}{u}.$$

(ii) For particle B:



$$\frac{dw}{dt}=-kw^2-g$$

$$\int\frac{1}{g+kw^2}\,dw=-\int dt$$

$$\frac{1}{\sqrt{gk}}\tan^{-1}\left(w\sqrt{\frac{k}{g}}\right)=-t+C$$

When $t=0$, $w=u$

$$\therefore C=\frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

$$t=\frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)-\frac{1}{\sqrt{gk}}\tan^{-1}\left(w\sqrt{\frac{k}{g}}\right)$$

$$=\frac{1}{\sqrt{gk}}\left(\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)-\tan^{-1}\left(w\sqrt{\frac{k}{g}}\right)\right).$$

(iii) When B is at rest, $w=0$:

$$t=\frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

Use part (i) with $v=V$:

$$\frac{1}{V}=kt+\frac{1}{u}$$

$$=k\times\frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)+\frac{1}{u}$$

$$=\frac{1}{u}+\frac{\sqrt{k^2}}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

$$=\frac{1}{u}+\sqrt{\frac{k}{g}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right).$$

(iv) When $u\to\infty$, $\frac{1}{u}\to 0$

and $\tan^{-1}u\sqrt{\frac{k}{g}}\to\frac{\pi}{2}$

$$\therefore\frac{1}{V}\to\sqrt{\frac{k}{g}}\times\frac{\pi}{2}$$

$$V=\sqrt{\frac{g}{k}}\times\frac{2}{\pi}$$

$$=\frac{2}{\pi}\sqrt{\frac{g}{k}}.$$

(b) (i) For $x\geq 0$:

$$1-x^2\leq 1$$

$$(1-x)(1+x)\leq 1$$

$$1-x\leq\frac{1}{1+x}\quad\text{since } 1+x>0$$

Also $\frac{1}{1+x}\leq 1$

since the denominator > numerator.

Combining the two parts:

$$1-x\leq\frac{1}{1+x}\leq 1.$$

(ii)

$$\int_0^{\frac{1}{n}}1-x\,dx\leq\int_0^{\frac{1}{n}}\frac{1}{1+x}\,dx\leq\int_0^{\frac{1}{n}}dx$$

$$\left[x-\frac{x^2}{2}\right]_0^{\frac{1}{n}}\leq\left[\ln(1+x)\right]_0^{\frac{1}{n}}\leq[x]_0^{\frac{1}{n}}$$

$$\frac{1}{n}-\frac{1}{2n^2}\leq\ln\left(1+\frac{1}{n}\right)\leq\frac{1}{n}$$

Multiply by n ($n>0$):

$$1-\frac{1}{2n}\leq n\ln\left(1+\frac{1}{n}\right)\leq 1.$$

(iii) $n\to\infty$, $\frac{1}{2n}\to 0$

From part (ii):

$$1\leq n\ln\left(1+\frac{1}{n}\right)\leq 1$$

$$\therefore n\ln\left(1+\frac{1}{n}\right)\to 1$$

$$n\ln\left(1+\frac{1}{n}\right)=1$$

$$\ln\left(1+\frac{1}{n}\right)^n=\ln e$$

$$\left(1+\frac{1}{n}\right)^n=e.$$

(c) (i) *Method 1:*

$$(x-y)^2\geq 0$$

$$x^2-2xy+y^2\geq 0$$

$$x^2+y^2\geq 2xy$$

$$xy\leq\frac{x^2+y^2}{2}\quad ①$$

$$\sqrt{xy}\leq\sqrt{\frac{x^2+y^2}{2}}.$$

OR

Method 2:

Given $x>0$ and $y>0$:

$$x+y=\sqrt{(x+y)^2}$$

$$=\sqrt{x^2+y^2+2xy}$$

$$\leq\sqrt{x^2+y^2}$$

$$\therefore\frac{x+y}{2}\leq\frac{\sqrt{x^2+y^2}}{2}$$

$$\sqrt{xy}\leq\frac{x+y}{2}\quad\text{(given)}$$

$$\leq\frac{\sqrt{x^2+y^2}}{2}\quad\text{(above)}$$

$$\leq\sqrt{\frac{x^2+y^2}{2}}$$

$$\therefore\sqrt{xy}\leq\sqrt{\frac{x^2+y^2}{2}}.$$

(ii) *Method 1:*

From ① above:

$$ab\leq\frac{a^2+b^2}{2}$$

and $cd\leq\frac{c^2+d^2}{2}$

$$\therefore ab+cd \le \frac{a^2+b^2}{2}+\frac{c^2+d^2}{2}$$
$$\le \frac{a^2+b^2+c^2+d^2}{2} \quad ②$$

$$\sqrt{abcd}=\sqrt{(ab)+(cd)}$$
$$\le \frac{ab+cd}{2} \quad \text{(by given inequality)}$$
$$\le \frac{a^2+d^2+c^2+d^2}{4} \quad \text{from ②}$$

Taking square roots:

$$\sqrt[4]{abcd} \le \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}.$$

OR

Method 2:

From part (i):

$$\sqrt{ab} \le \sqrt{\frac{a^2+b^2}{2}}$$
$$\sqrt{cd} \le \sqrt{\frac{c^2+d^2}{2}}$$
$$\sqrt{\sqrt{ab}\sqrt{cd}} \le \sqrt{\frac{\left(\sqrt{\frac{a^2+b^2}{2}}\right)^2+\left(\sqrt{\frac{c^2+d^2}{2}}\right)^2}{2}}$$
$$\le \sqrt{\frac{\frac{a^2+b^2}{2}+\frac{c^2+d^2}{2}}{2}}$$
$$\sqrt[4]{abcd} \le \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}.$$

Question 16

(a) (i) There is one black counter in each column so there are 3 ways of placing a black counter in a column, and then there are 5 columns giving 3^5 ways.

The total number of ways of placing 5 black counters is:

$${}^{15}C_5 \times {}^{10}C_{10} = {}^{15}C_5$$

$$P(\text{black in each column}) = \frac{3^5}{{}^{15}C_5}$$
$$= \frac{243}{3003}$$
$$= \frac{81}{1001}.$$

(ii) Using similar logic to part (i): There are n ways of placing a black counter in a column, and then there are q columns giving n^q ways.

The total number of ways of placing q black counters and nq-q white counters in nq cells is:

$${}^{nq}C_q \times {}^{nq-q}C_{nq-q} = {}^{nq}C_q$$

$$P(\text{black in each column}) = \frac{n^q}{{}^{nq}C_q}$$
$$= \frac{n^q}{\binom{nq}{q}}.$$

(iii)
$$P_n = \frac{n^q}{\binom{nq}{q}}$$
$$= \frac{n^q}{\frac{(nq)!}{(nq-q)!\,q!}}$$
$$= \frac{n^q (nq-q)!\,q!}{(nq)!}$$
$$= \frac{n^q q!(nq-q)!}{(nq)(nq-1)\ldots(nq-q+1)(nq-q)!}$$
$$= \frac{n^q q!}{(nq)(nq-1)\ldots(nq-q+1)}$$
$$= \frac{n^q q!}{n^q\left(q\left(q-\frac{1}{n}\right)\left(q-\frac{2}{n}\right)\ldots\left(q-\frac{q-1}{n}\right)\right)}$$
$$= \frac{q!}{q\left(q-\frac{1}{n}\right)\left(q-\frac{2}{n}\right)\ldots\left(q-\frac{q-1}{n}\right)}$$

$$\lim_{n\to\infty} P_n = \frac{q!}{q^q}$$

(b) (i) By De Moivre's theorem:

$$(\cos\alpha+i\sin\alpha)^{2n}=\cos 2n\alpha+i\sin 2n\alpha$$

By expanding:

$$(\cos\alpha+i\sin\alpha)^{2n}=\cos^{2n}\alpha+\binom{2n}{1}\cos^{2n-1}\alpha(i\sin\alpha)+\binom{2n}{2}\cos^{2n-2}\alpha(i\sin\alpha)^2+\binom{2n}{3}\cos^{2n-3}\alpha(i\sin\alpha)^3+\ldots+\binom{2n}{2n-1}\cos\alpha(i\sin\alpha)^{2n-1}+(i\sin\alpha)^{2n}$$

Consider the real parts (allowing for even powers of i, $i^2=-1$, $i^4=1$):

$$\cos 2n\alpha=\cos^{2n}\alpha-\binom{2n}{2}\cos^{2n-2}\alpha\sin^2\alpha+\binom{2n}{4}\cos^{2n-4}\alpha\sin^4\alpha\ldots(-1)^{n-1}\binom{2n}{2n-2}\cos^2\alpha\sin^{2n-2}\alpha+(-1)^n\sin^{2n}\alpha.$$

(ii)
$$T_{2n}(x)=\cos\left(2n\cos^{-1}x\right)$$
$$\therefore \alpha=\cos^{-1}x$$
and $x=\cos\alpha$

Also $\sin^2\alpha=1-\cos^2\alpha$
$$=1-x^2$$

$$\cos 2n\left(\cos^{-1}x\right)=\cos^{2n}\left(\cos^{-1}x\right)-\binom{2n}{2}\cos^{2n-2}\left(\cos^{-1}x\right)\sin^2\alpha+\binom{2n}{4}\cos^{2n-4}\left(\cos^{-1}x\right)\sin^4\alpha\ldots+(-1)^n\sin^{2n}\alpha.$$

This then becomes:

$$T_{2n}(x)=x^{2n}-\binom{2n}{2}x^{2n-2}\left(1-x^2\right)+\binom{2n}{4}x^{2n-4}\left(1-x^2\right)^2\ldots+(-1)^n\left(1-x^2\right)^n.$$

(iii) Let $\cos(2n\cos^{-1}x)=0$

$$2n\cos^{-1}x=\frac{\pi}{2},\frac{3\pi}{2},\ldots,\frac{(2k-1)\pi}{2},\quad k=1,2,3,\ldots 2n$$
$$\cos^{-1}x=\frac{\pi}{4n},\frac{3\pi}{4n},\ldots,\frac{(2k-1)\pi}{4n},\quad k=1,2,3,\ldots 2n$$
$$x=\cos\left(\frac{\pi}{4n}\right),\cos\left(\frac{3\pi}{4n}\right),\ldots,\cos\left(\frac{(4n-1)\pi}{4n}\right)$$

These are the roots of $T_{2n}(x)$.

The product of the roots:

$$\cos\left(\frac{\pi}{4n}\right)\cos\left(\frac{3\pi}{4n}\right)\ldots\cos\left(\frac{(4n-1)\pi}{4n}\right) \quad ①$$

From part (ii), setting $T_{2n}(x)=0$:

$$0=x^{2n}-\binom{2n}{2}x^{2n-2}\left(1-x^2\right)+\binom{2n}{4}x^{2n-4}\left(1-x^2\right)^2\ldots+(-1)^n\left(1-x^2\right)^n$$

The constant term will be:

$$\frac{(-1)^{2n}(-1)^n}{\text{coefficient of } x^{2n}}=\frac{(-1)^n}{1+\binom{2n}{2}+\binom{2n}{4}+\ldots+\binom{2n}{2n}}$$

To find the value of the denominator:

$$(1+x)^{2n}=1+\binom{2n}{1}x+\binom{2n}{2}x^2+\ldots+\binom{2n}{2n}x^{2n}$$

For $x=1$:

$$(1+1)^{2n}=1+\binom{2n}{1}1+\binom{2n}{2}1^2+\ldots+\binom{2n}{2n}1^{2n}$$
$$2^{2n}=1+\binom{2n}{1}+\binom{2n}{2}+\ldots+\binom{2n}{2n}$$

For $x=-1$:

$$(1-1)^{2n}=1+\binom{2n}{1}(-1)+\binom{2n}{2}(-1)^2+\ldots+\binom{2n}{2n}(-1)^{2n}$$
$$0=1-\binom{2n}{1}+\binom{2n}{2}+\ldots+(-1)^n\binom{2n}{2n}$$

Adding the 2 previous results:

$$2^{2n} = 2\left(1+\binom{2n}{2}+\binom{2n}{4}+\ldots+\binom{2n}{2n}\right)$$

Divide by 2:

$$2^{2n-1} = 1+\binom{2n}{2}+\binom{2n}{4}+\ldots+\binom{2n}{2n} \quad ③$$

Combining ①, ②, ③:

$$\cos\left(\frac{\pi}{4n}\right)\cos\left(\frac{3\pi}{4n}\right)\ldots\cos\left(\frac{(4n-1)\pi}{4n}\right)$$

$$= \frac{(-1)^n}{1+\binom{2n}{2}+\binom{2n}{4}+\ldots+\binom{2n}{2n}}$$

$$= \frac{(-1)^n}{2^{2n-1}}.$$

(iv) From part (ii):

$$\cos 2n\left(\cos^{-1} x\right) = x^{2n} - \binom{2n}{2}x^{2n-2}\left(1-x^2\right)$$
$$+\binom{2n}{4}x^{2n-4}\left(1-x^2\right)^2$$
$$\ldots$$
$$+(-1)^n\left(1-x^2\right)^n$$

For $x = \frac{1}{\sqrt{2}}$ and using part (ii):

$$\cos 2n\left(\cos^{-1}\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)^{2n}$$
$$-\binom{2n}{2}\left(\frac{1}{\sqrt{2}}\right)^{2n-2}\left(1-\left(\frac{1}{\sqrt{2}}\right)^2\right)$$
$$+\binom{2n}{4}\left(\frac{1}{\sqrt{2}}\right)^{2n-4}\left(1-\left(\frac{1}{\sqrt{2}}\right)^2\right)^2$$
$$\ldots$$
$$+(-1)^n\left(1-\left(\frac{1}{\sqrt{2}}\right)^2\right)^n$$

$$\cos\left(2n\frac{\pi}{4}\right) = \left(\frac{1}{2}\right)^n$$
$$-\binom{2n}{2}\left(\frac{1}{2}\right)^{n-1}\left(1-\left(\frac{1}{2}\right)\right)$$
$$+\binom{2n}{4}\left(\frac{1}{2}\right)^{n-2}\left(1-\left(\frac{1}{2}\right)\right)^2$$
$$\ldots$$
$$+(-1)^n\left(1-\left(\frac{1}{2}\right)\right)^n$$

$$\cos\left(\frac{n\pi}{2}\right) = \left(\frac{1}{2}\right)^n - \binom{2n}{2}\left(\frac{1}{2}\right)^n + \binom{2n}{4}\left(\frac{1}{2}\right)^n$$
$$\ldots + (-1)^n\binom{2n}{2n}\left(\frac{1}{2}\right)^n$$
$$= \frac{1}{2^n}\left[1-\binom{2n}{2}+\binom{2n}{4}+\ldots+(-1)^n\binom{2n}{2n}\right]$$

Multiply by 2^n:

$$\therefore 1-\binom{2n}{2}+\binom{2n}{4}+\ldots+(-1)^n\binom{2n}{2n} = 2^n\cos\left(\frac{n\pi}{2}\right)$$

End of Mathematics Extension 2 solutions