BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2015 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 7–17

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is 0.005 233 59 written in scientific notation, correct to 4 significant figures?

(A) 5.2336×10^{-2}

(B) 5.234×10^{-2}

(C) 5.2336×10^{-3}

(D) 5.234×10^{-3}

2 What is the slope of the line with equation $2x - 4y + 3 = 0$?

(A) -2

(B) $-\frac{1}{2}$

(C) $\frac{1}{2}$

(D) 2

3 The first three terms of an arithmetic series are 3, 7 and 11.

What is the 15th term of this series?

(A) 59

(B) 63

(C) 465

(D) 495

4 The probability that Mel's soccer team wins this weekend is $\frac{5}{7}$.

The probability that Mel's rugby league team wins this weekend is $\frac{2}{3}$.

What is the probability that neither team wins this weekend?

(A) $\frac{2}{21}$

(B) $\frac{10}{21}$

(C) $\frac{13}{21}$

(D) $\frac{19}{21}$

5 Using the trapezoidal rule with 4 subintervals, which expression gives the approximate area under the curve $y = xe^x$ between $x = 1$ and $x = 3$?

(A) $\frac{1}{4}\left(e^1 + 6e^{1.5} + 4e^2 + 10e^{2.5} + 3e^3\right)$

(B) $\frac{1}{4}\left(e^1 + 3e^{1.5} + 4e^2 + 5e^{2.5} + 3e^3\right)$

(C) $\frac{1}{2}\left(e^1 + 6e^{1.5} + 4e^2 + 10e^{2.5} + 3e^3\right)$

(D) $\frac{1}{2}\left(e^1 + 3e^{1.5} + 4e^2 + 5e^{2.5} + 3e^3\right)$

6 What is the value of the derivative of $y = 2\sin 3x - 3\tan x$ at $x = 0$?

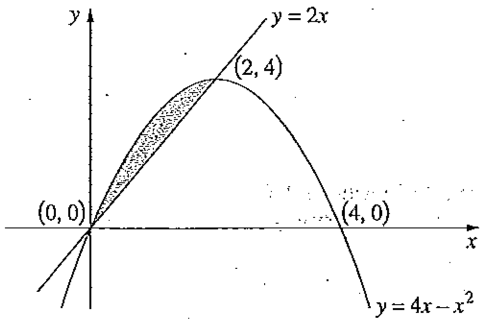
(A) -1

(B) 0

(C) 3

(D) -9

7 The diagram shows the parabola $y = 4x - x^2$ meeting the line $y = 2x$ at $(0, 0)$ and $(2, 4)$.



Which expression gives the area of the shaded region bounded by the parabola and the line?

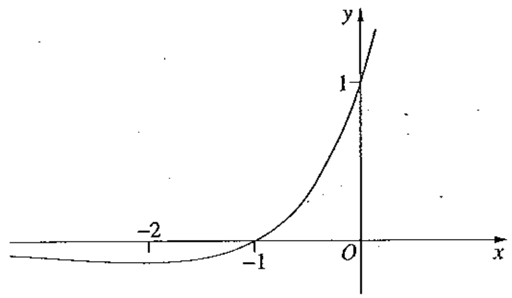
(A) $\int_0^2 x^2 - 2x\,dx$

(B) $\int_0^2 2x - x^2\,dx$

(C) $\int_0^4 x^2 - 2x\,dx$

(D) $\int_0^4 2x - x^2\,dx$

8 The diagram shows the graph of $y = e^x(1 + x)$.



How many solutions are there to the equation $e^x(1 + x) = 1 - x^2$?

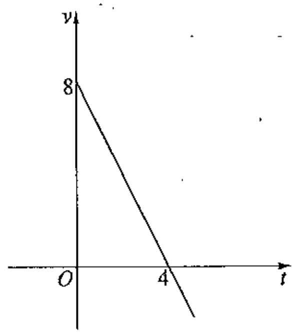
(A) 0

(B) 1

(C) 2

(D) 3

9 A particle is moving along the x-axis. The graph shows its velocity v metres per second at time t seconds.



When $t = 0$ the displacement x is equal to 2 metres.

What is the maximum value of the displacement x?

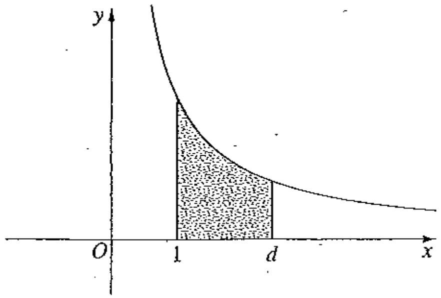
(A) 8 m

(B) 14 m

(C) 16 m

(D) 18 m

10 The diagram shows the area under the curve $y=\frac{2}{x}$ from $x=1$ to $x=d$.



What value of d makes the shaded area equal to 2?

(A) e

(B) $e+1$

(C) $2e$

(D) e^2

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Simplify $4x-(8-6x)$. **1**

(b) Factorise fully $3x^2-27$. **2**

(c) Express $\frac{8}{2+\sqrt{7}}$ with a rational denominator. **2**

(d) Find the limiting sum of the geometric series $1-\frac{1}{4}+\frac{1}{16}-\frac{1}{64}+\cdots$. **2**

(e) Differentiate $(e^x+x)^5$. **2**

(f) Differentiate $y=(x+4)\ln x$. **2**

(g) Evaluate $\int_0^{\frac{\pi}{4}} \cos 2x \, dx$. **2**

(h) Find $\int \frac{x}{x^2-3}\, dx$. **2**

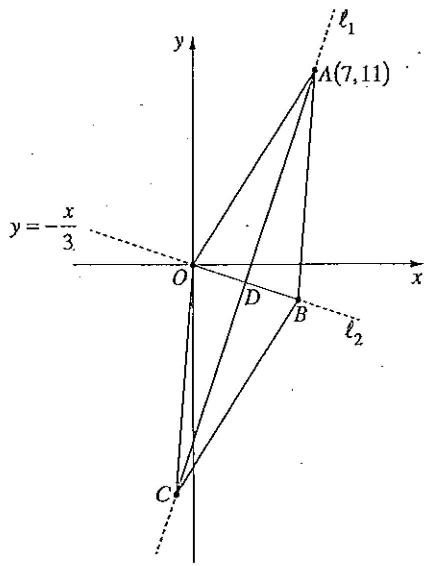
Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) Find the solutions of $2\sin\theta = 1$ for $0 \le \theta \le 2\pi$. **2**

(b) The diagram shows the rhombus $OABC$.

The diagonal from the point $A(7, 11)$ to the point C lies on the line ℓ_1.

The other diagonal, from the origin O to the point B, lies on the line ℓ_2 which has equation $y = -\frac{x}{3}$.



(i) Show that the equation of the line ℓ_1 is $y = 3x - 10$. **2**

(ii) The lines ℓ_1 and ℓ_2 intersect at the point D. **2**

Find the coordinates of D.

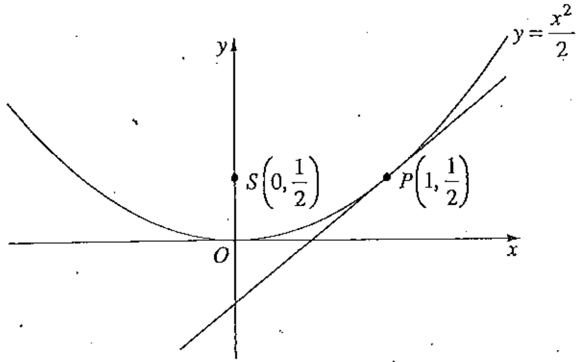
Question 12 continues on page 9

Question 12 (continued)

(c) Find $f'(x)$, where $f(x) = \frac{x^2 + 3}{x - 1}$. **2**

(d) For what values of k does the quadratic equation $x^2 - 8x + k = 0$ have real roots? **2**

(e) The diagram shows the parabola $y = \frac{x^2}{2}$ with focus $S\left(0, \frac{1}{2}\right)$. A tangent to the parabola is drawn at $P\left(1, \frac{1}{2}\right)$.



(i) Find the equation of the tangent at the point P. **2**

(ii) What is the equation of the directrix of the parabola? **1**

(iii) The tangent and directrix intersect at Q.

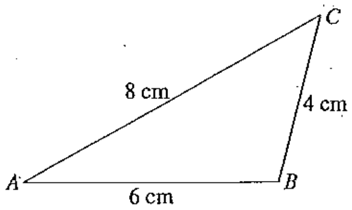
Show that Q lies on the y-axis. **1**

(iv) Show that $\triangle PQS$ is isosceles. **1**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) The diagram shows $\triangle ABC$ with sides $AB = 6$ cm, $BC = 4$ cm and $AC = 8$ cm.



(i) Show that $\cos A = \frac{7}{8}$. **1**

(ii) By finding the exact value of $\sin A$, determine the exact value of the area of $\triangle ABC$. **2**

(b) (i) Find the domain and range for the function $f(x) = \sqrt{9 - x^2}$. **2**

(ii) On a number plane, shade the region where the points (x, y) satisfy both of the inequalities $y \le \sqrt{9 - x^2}$ and $y \ge x$. **2**

(c) Consider the curve $y = x^3 - x^2 - x + 3$.

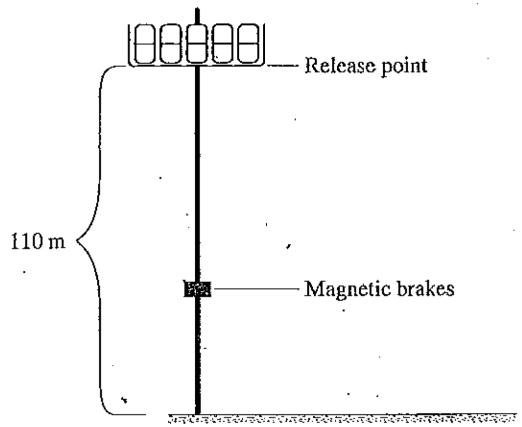
(i) Find the stationary points and determine their nature. **4**

(ii) Given that the point $P\left(\frac{1}{3}, \frac{70}{27}\right)$ lies on the curve, prove that there is a point of inflexion at P. **2**

(iii) Sketch the curve, labelling the stationary points, point of inflexion and y-intercept. **2**

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) In a theme park ride, a chair is released from a height of 110 metres and falls vertically. Magnetic brakes are applied when the velocity of the chair reaches –37 metres per second.



The height of the chair at time t seconds is x metres. The acceleration of the chair is given by $\ddot{x} = -10$. At the release point, $t = 0$, $x = 110$ and $\dot{x} = 0$.

(i) Using calculus, show that $x = -5t^2 + 110$. **2**

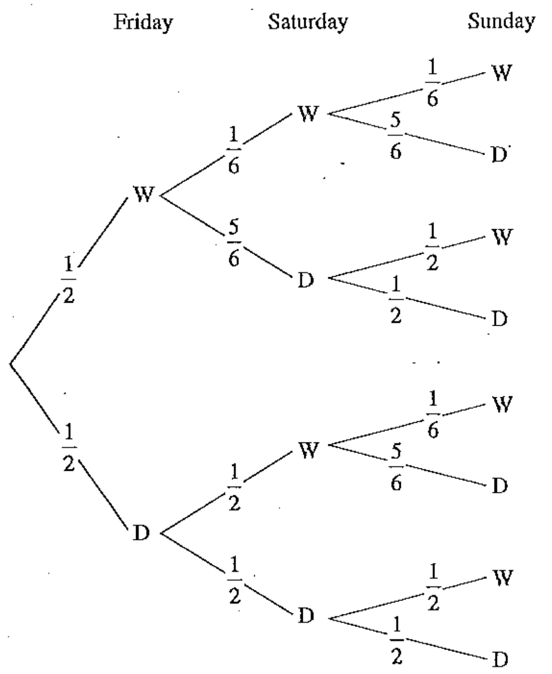
(ii) How far has the chair fallen when the magnetic brakes are applied? **2**

Question 14 continues on page 12

(b) Weather records for a town suggest that:

- if a particular day is wet (W), the probability of the next day being dry is $\frac{5}{6}$
- if a particular day is dry (D), the probability of the next day being dry is $\frac{1}{2}$.

In a specific week Thursday is dry. The tree diagram shows the possible outcomes for the next three days: Friday, Saturday and Sunday.



(i) Show that the probability of Saturday being dry is $\frac{2}{3}$. **1**

(ii) What is the probability of both Saturday and Sunday being wet? **2**

(iii) What is the probability of at least one of Saturday and Sunday being dry? **1**

Question 14 continues on page 13

(c) Sam borrows \$100 000 to be repaid at a reducible interest rate of 0.6% per month. Let $\$A_n$ be the amount owing at the end of n months and $\$M$ be the monthly repayment.

(i) Show that $A_2 = 100\,000(1.006)^2 - M(1 + 1.006)$. **1**

(ii) Show that $A_n = 100\,000(1.006)^n - M\left(\frac{(1.006)^n - 1}{0.006}\right)$. **2**

(iii) Sam makes monthly repayments of \$780. **1**

Show that after making 120 monthly repayments the amount owing is \$68 500 to the nearest \$100.

(iv) Immediately after making the 120th repayment, Sam makes a one-off payment, reducing the amount owing to \$48 500. The interest rate and monthly repayment remain unchanged. **3**

After how many more months will the amount owing be completely repaid?

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) The amount of caffeine, C, in the human body decreases according to the equation

$$\frac{dC}{dt} = -0.14C,$$

where C is measured in mg and t is the time in hours.

(i) Show that $C = Ae^{-0.14t}$ is a solution to $\frac{dC}{dt} = -0.14C$, where A is a constant. 1

When $t = 0$, there are 130 mg of caffeine in Lee's body.

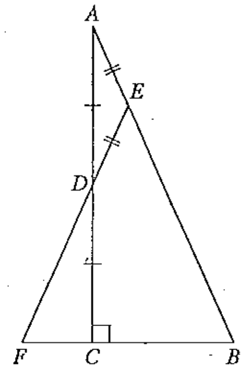
(ii) Find the value of A. 1

(iii) What is the amount of caffeine in Lee's body after 7 hours? 1

(iv) What is the time taken for the amount of caffeine in Lee's body to halve? 2

Question 15 continues on page 15

(b) The diagram shows $\triangle ABC$ which has a right angle at C. The point D is the midpoint of the side AC. The point E is chosen on AB such that $AE = ED$. The line segment ED is produced to meet the line BC at F.



Copy or trace the diagram into your writing booklet.

(i) Prove that $\triangle ACB$ is similar to $\triangle DCF$. 2

(ii) Explain why $\triangle EFB$ is isosceles. 1

(iii) Show that $EB = 3AE$. 2

(c) Water is flowing in and out of a rock pool. The volume of water in the pool at time t hours is V litres. The rate of change of the volume is given by

$$\frac{dV}{dt} = 80\sin(0.5t).$$

At time $t = 0$, the volume of water in the pool is 1200 litres and is increasing.

(i) After what time does the volume of water first start to decrease? 2

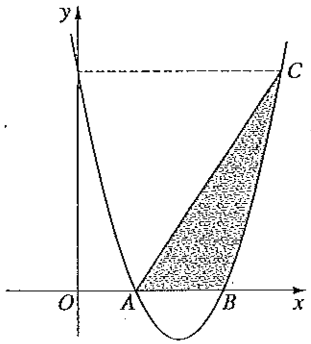
(ii) Find the volume of water in the pool when $t = 3$. 2

(iii) What is the greatest volume of water in the pool? 1

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet.

(a) The diagram shows the curve with equation $y = x^2 - 7x + 10$. The curve intersects the x-axis at points A and B. The point C on the curve has the same y-coordinate as the y-intercept of the curve.



(i) Find the x-coordinates of points A and B. **1**

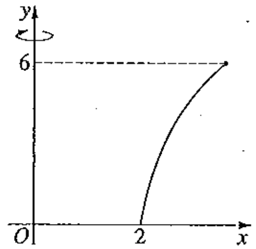
(ii) Write down the coordinates of C. **1**

(iii) Evaluate $\int_0^2 \left(x^2 - 7x + 10\right) dx$. **1**

(iv) Hence, or otherwise, find the area of the shaded region. **2**

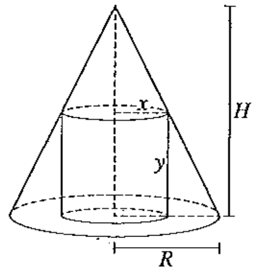
Question 16 continues on page 17

(b) A bowl is formed by rotating the curve $y = 8\log_e(x-1)$ about the y-axis for $0 \le y \le 6$. **3**



Find the volume of the bowl. Give your answer correct to 1 decimal place.

(c) The diagram shows a cylinder of radius x and height y inscribed in a cone of radius R and height H, where R and H are constants.



The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.

The volume of a cylinder of radius r and height h is $\pi r^2 h$.

(i) Show that the volume, V, of the cylinder can be written as $V = \frac{H}{R}\pi x^2(R - x)$. **3**

(ii) By considering the inscribed cylinder of maximum volume, show that the volume of any inscribed cylinder does not exceed $\frac{4}{9}$ of the volume of the cone. **4**

End of paper

2015 Higher School Certificate Solutions Mathematics

SECTION I

Summary

1 D	3 A	5 B	7 B	9 D
2 C	4 A	6 C	8 C	10 A

SECTION I

1 (D) $0.00523359 = 0.005234$ (4 s.f.)

$= 5.234 \times 10^{-3}$.

2 (C) $2x - 4y + 3 = 0$

$4y = 2x + 3$

$y = \frac{1}{2}x + \frac{3}{4}$

$\therefore$ the slope is $\frac{1}{2}$.

3 (A) $3, 7, 11, \ldots$

$a = 4, \quad d = 4, \quad n = 15$

$T_n = a + (n-1)d$

$T_{15} = 3 + (15-1)4$

$= 59$.

4 (A) $P(LL) = \left(1 - \frac{5}{7}\right)\left(1 - \frac{2}{3}\right)$

$= \frac{2}{7} \times \frac{1}{3}$

$= \frac{2}{21}$.

5 (B)

x	1	1.5	2	2.5	3
xe^x	e^1	$1.5e^{1.5}$	$2e^2$	$2.5e^{2.5}$	$3e^3$
weights	1	2	2	2	1

$h = \frac{3-1}{4} = 0.5$

Area $\approx \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right]$

$\approx \frac{0.5}{2}\left[e^1 + 2(1.5e^{1.5} + 2e^2 + 2.5e^{2.5}) + 3e^3\right]$

$\approx \frac{1}{4}\left[e^1 + 3e^{1.5} + 4e^2 + 5e^{2.5} + 3e^3\right]$.

6 (C) $y = 2\sin 3x - 3\tan x$

$\frac{dy}{dx} = 6\cos 3x - 3\sec^2 x$

At $x = 0$:

$\frac{dy}{dx} = 6\cos 3(0) - 3\sec^2(0)$

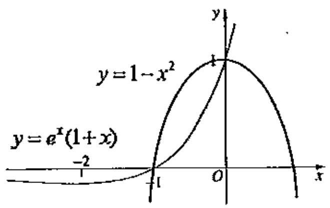
$= 6 \times 1 - 3$

$= 3$.

7 (B) $A = \int_0^2 \left(4x - x^2\right) - 2x \, dx$

$= \int_0^2 2x - x^2 \, dx$.

8 (C)



From the graphs, there are two points of intersection, so there are two solutions.

9 (D) The area under a velocity time graph gives the distance travelled.

Distance $= \frac{1}{2} \times 4 \times 8 = 16$

Since the particle starts at 2, then the maximum displacement is $16 + 2 = 18$ m.

10 (A) $\int_1^d \frac{2}{x} \, dx = 2$

$2\left[\ln x\right]_1^d = 2$

$\left[\ln x\right]_1^d = 1$

$\ln d - \ln 1 = 1$

$\ln d = 1$

$d = e^1$

$d = e$.

SECTION II

Question 11

(a) $4x - (8 - 6x) = 4x - 8 + 6x$

$= 10x - 8$.

(b) $3x^2 - 27 = 3(x^2 - 9)$

$= 3(x+3)(x-3)$.

(c) $\frac{8}{2+\sqrt{7}} = \frac{8}{2+\sqrt{7}} \times \frac{2-\sqrt{7}}{2-\sqrt{7}}$

$= \frac{8(2-\sqrt{7})}{4-7}$

$= \frac{8(2-\sqrt{7})}{-3}$

$= \frac{8(\sqrt{7}-2)}{3}$.

(d) $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \ldots$

$a = 1, \quad r = -\frac{1}{4}$

$S = \frac{a}{1-r}$

$= \frac{1}{1-\left(-\frac{1}{4}\right)}$

$= \frac{1}{\frac{5}{4}}$

$= \frac{4}{5}$.

(e) $y = \left(e^x + x\right)^5$

$\frac{dy}{dx} = 5\left(e^x + x\right)^4 \times \frac{d}{dx}\left(e^x + x\right)$

$= 5\left(e^x + x\right)^4\left(e^x + 1\right)$.

(f) $\frac{d}{dx}(uv) = uv' + vu'$

$u = x + 4 \qquad v = \ln x$

$u' = 1 \qquad v' = \frac{1}{x}$

$y = (x+4)\ln x$

$\frac{dy}{dx} = (x+4) \times \frac{1}{x} + \ln x \times 1$

$= 1 + \frac{4}{x} + \ln x$.

(g) $\int_0^{\frac{\pi}{4}} \cos 2x \, dx = \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{4}\right) - \sin 2(0)\right]$

$= \frac{1}{2}\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right]$

$= \frac{1}{2}[1 - 0]$

$= \frac{1}{2}$.

(h) $\int \frac{x}{x^2 - 3} \, dx = \frac{1}{2}\int \frac{2x}{x^2 - 3} \, dx$

$= \frac{1}{2}\ln\left(x^2 - 3\right) + C$.

Question 12

(a) $2\sin\theta = 1, \quad 0 \le \theta \le 2\pi$

$\sin\theta = \frac{1}{2} \quad (1^{st}, 2^{nd} \text{ quadrants})$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}.$

(b) (i) $\ell_1 \perp \ell_2$ (diagonals of a rhombus)

Gradient of ℓ_2 is $-\frac{1}{3}$.

$\therefore$ Gradient of ℓ_1 is 3.

Equation of ℓ_1:

$y - y_1 = m(x - x_1)$

$y - 11 = 3(x - 7)$

$y - 11 = 3x - 21$

$y = 3x - 10.$

(ii) Solve simultaneously:

$y = -\frac{x}{3}$ ①

$y = 3x - 10$ ②

Substitute ① in ②:

$-\frac{x}{3} = 3x - 10$

$-x = 9x - 30$

$30 = 10x$

$x = 3$

Substitute $x = 3$ in ①:

$y = -\frac{3}{3}$

$= -1$

$\therefore D$ is $(3, -1)$.

(c) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

$u = x^2 + 3 \qquad v = x - 1$

$u' = 2x \qquad v' = 1$

$f(x) = \frac{x^2 + 3}{x - 1}$

$$f'(x) = \frac{(x-1)\times 2x - (x^2+3)\times 1}{(x-1)^2}$$

$$= \frac{2x^2 - 2x - x^2 - 3}{(x-1)^2}$$

$$= \frac{x^2 - 2x - 3}{(x-1)^2}$$

$$= \frac{(x+1)(x-3)}{(x-1)^2}.$$

(d) For real roots $\Delta \ge 0$:

$\Delta = b^2 - 4ac$

$= (-8)^2 - 4 \times 1 \times k$

$= 64 - 4k$

$\therefore 64 - 4k \ge 0$

$64 \ge 4k$

$4k \le 64$

$k \le 16.$

(e) (i) $y = \frac{x^2}{2}$

$\frac{dy}{dx} = \frac{2x}{2}$

$= x$

When $x = 1$, $\frac{dy}{dx} = 1$

$\therefore m = 1$

Equation of tangent at P is:

$y - y_1 = m(x - x_1)$

$y - \frac{1}{2} = 1(x - 1)$

$y - \frac{1}{2} = x - 1$

$y = x - \frac{1}{2}.$

(ii) Focal length is $\frac{1}{2}$.

Vertex is $(0, 0)$.

$\therefore$ the directrix is $y = -\frac{1}{2}$.

(iii) Solve simultaneously:

$y = x - \frac{1}{2}$ ①

$y = -\frac{1}{2}$ ②

Substitute ② in ①:

$-\frac{1}{2} = x - \frac{1}{2}$

$x = 0$

$\therefore Q = \left(0, -\frac{1}{2}\right)$

This shows that the x-coordinate of the point of intersection is 0.

$\therefore Q$ lies on the y-axis.

(iv) For ΔPQS to be isosceles, two sides must be equal.

Using $P\left(1, \frac{1}{2}\right)$ and $S\left(0, \frac{1}{2}\right)$:

$PS = 1 - 0 = 1$

Using $Q\left(0, -\frac{1}{2}\right)$ and $S\left(0, \frac{1}{2}\right)$:

$QS = \frac{1}{2} + \frac{1}{2} = 1$

Thus $PS = QS$.

$\therefore \Delta PQS$ is isosceles.

Question 13

(a) (i) Using the cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{8^2 + 6^2 - 4^2}{2 \times 8 \times 6}$$

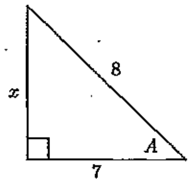
$$= \frac{64 + 36 - 16}{96}$$

$$= \frac{84}{96}$$

$$= \frac{7}{8}.$$

(ii) *Method 1:*

Consider the right angle triangle with $\cos A = \frac{7}{8}$



$x^2 = 8^2 - 7^2$

$x^2 = 64 - 49$

$x = \sqrt{15}$

From the triangle:

$\sin A = \frac{\sqrt{15}}{8}$

OR

Method 2:

$\sin^2 A + \cos^2 A = 1$ and A is acute

$\sin^2 A + \left(\frac{7}{8}\right)^2 = 1$

$\sin^2 A = 1 - \frac{49}{64}$

$= \frac{15}{64}$

$\sin A = \frac{\sqrt{15}}{8}$

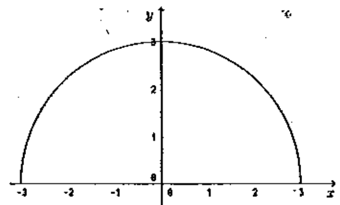
Having found $\sin A = \frac{\sqrt{15}}{8}$ via either method:

Area $= \frac{1}{2} bc \sin A$

$= \frac{1}{2} \times 8 \times 6 \times \frac{\sqrt{15}}{8}$

$= 3\sqrt{15}$ cm².

(b) (i)



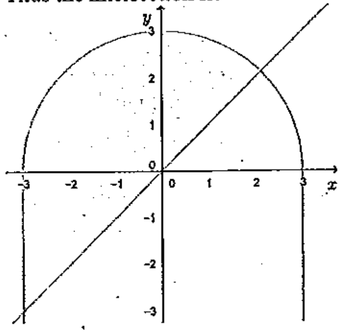
Domain: $-3 \le x \le 3$

Range: $0 \le f(x) \le 3$

(ii) $y \le \sqrt{9-x^2}$ represents all points directly below $y=\sqrt{9-x^2}$.

$y \ge x$ represents all points above $y = x$.

Thus the intersection is:



(c) (i) $y = x^3 - x^2 - x + 3$

For stationary points $\frac{dy}{dx} = 0$:

$$\frac{dy}{dx} = 3x^2 - 2x - 1$$
$$= (3x+1)(x-1)$$

For $(3x+1)(x-1) = 0$

$$3x+1=0 \quad \text{or} \quad x-1=0$$
$$x = -\frac{1}{3} \qquad x = 1$$

$$\frac{d^2y}{dx^2} = 6x - 2$$

When $x = -\frac{1}{3}$:

$$y = \left(-\frac{1}{3}\right)^3 - \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) + 3$$
$$= \frac{86}{27}$$

The value of $\frac{d^2y}{dx^2}$ determines the nature:

$$\frac{d^2y}{dx^2} = 6\left(-\frac{1}{3}\right) - 2$$
$$= -4$$
$$< 0$$

$\therefore \left(-\frac{1}{3}, \frac{86}{27}\right)$ is a maximum turning point.

When $x = 1$:

$$y = (1)^3 - (1)^2 - (1) + 3$$
$$= 2$$

The value of $\frac{d^2y}{dx^2}$ determines the nature:

$$\frac{d^2y}{dx^2} = 6(1) - 2$$
$$= 4$$
$$> 0$$

$\therefore (1, 2)$ is a minimum turning point.

(ii) At a point of inflexion $\frac{d^2y}{dx^2} = 0$:

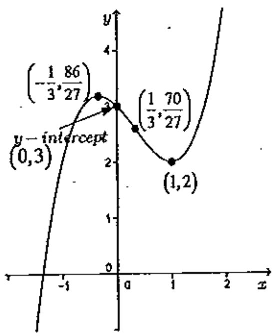
$$\frac{d^2y}{dx^2} = 6x - 2$$
$$6x - 2 = 0$$
$$x = \frac{1}{3},\ y = \left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right) + 3$$
$$= \frac{70}{27}$$

Testing for a concavity change:

x	0	$\frac{1}{3}$	1
$\frac{d^2y}{dx^2}$	−2	0	4

There is a change of sign either side of $x = \frac{1}{3}$ indicating a concavity change and thus there is a point of inflexion at P.

(iii)



Question 14

(a) (i) $\ddot{x} = -10$

$$\dot{x} = -10t + C$$

When $t = 0, \dot{x} = 0 \quad \therefore C = 0$

$$\therefore \dot{x} = -10t$$

$$x = -5t^2 + D$$

When $t = 0, x = 110$:

$$110 = -5(0)^2 + D$$
$$D = 110$$
$$x = -5t^2 + 110.$$

(ii)
$$\dot{x} = -37$$
$$-37 = -10t$$
$$t = 3.7$$

then $x = -5t^2 + 110$

$$x = -5(3.7)^2 + 110$$
$$= 41.55$$

Distance $= (110 - 41.55)$ m

$= 68.45$ m.

(b) (i) $P(\text{Sat Dry}) = P(WD)$ or $P(DD)$

$$= \frac{1}{2}\times\frac{5}{6} + \frac{1}{2}\times\frac{1}{2}$$
$$= \frac{2}{3}.$$

(ii) $P(\text{Sat Sun Wet}) = P(WWW)$ or $P(DWW)$

$$= \frac{1}{2}\times\frac{1}{6}\times\frac{1}{6} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{6}$$
$$= \frac{1}{18}.$$

(iii) $P(\text{at least 1 of Sat/Sun being dry})$

$$= 1 - P(\text{both wet})$$
$$= 1 - \frac{1}{18}$$
$$= \frac{17}{18}.$$

(c) (i) $A_1 = 100\,000(1.006) - M$

$$A_2 = A_1(1.006) - M$$
$$= [100\,000(1.006) - M]1.006 - M$$
$$= 100\,000(1.006)^2 - M(1.006) - M$$
$$= 100\,000(1.006)^2 - M(1 + 1.006).$$

(ii) Continuing the pattern for n months:

$$A_n = 100\,000(1.006)^n - M\left(1 + 1.006 + \ldots + 1.006^{n-1}\right)$$

$\left(1 + 1.006 + \ldots + 1.006^{n-1}\right)$ is a geometric series with $a = 1, r = 1.006$, and n terms.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{1\left((1.006)^n - 1\right)}{1.006 - 1}$$

$$\therefore A_n = 100\,000(1.006)^n - M\left(\frac{(1.006)^n - 1}{0.006}\right).$$

(iii) Using $n = 120$ and $M = 780$:

$$A_{120} = 100\,000(1.006)^{120} - 780\left(\frac{(1.006)^{120} - 1}{0.006}\right)$$
$$= 68499.458\ldots$$
$$= 68500 \quad \text{(nearest 100)}$$

$\therefore$ he still owes \$68 500.

(iv) Let B_n be the amount owing after the one off payment. Using the pattern from part (ii):

$$B_n = 48\,500(1.006)^n - 780\left(\frac{(1.006)^n - 1}{0.006}\right)$$

To be repaid in full $B_n = 0$:

$$0 = 48\,500(1.006)^n - 780\left(\frac{(1.006)^n - 1}{0.006}\right)$$

$$48\,500(1.006)^n = 780\left(\frac{(1.006)^n - 1}{0.006}\right)$$

$$48\,500(1.006)^n = 130\,000(1.006^n - 1)$$

$$48\,500(1.006)^n = 130\,000(1.006^n) - 130\,000$$

$$81\,500(1.006)^n = 130\,000$$

$$1.006^n = \frac{130\,000}{81\,500} = \frac{260}{163}$$

$$n\ln(1.006) = \ln\left(\frac{260}{163}\right)$$

$$n = \ln\left(\frac{260}{163}\right) \div \ln(1.006) = 78.055\ldots$$

Loan is repaid after a further 79 months.

Question 15

(a) (i) $C = Ae^{-0.14t}$

$$\frac{dC}{dt} = -0.14 \times Ae^{-0.14t} = -0.14C$$

(ii) When $t = 0$, $C = 130$:

$$130 = Ae^{-0.14(0)} = Ae^0$$

$$A = 130.$$

(iii) When $t = 7$:

$$C = 130e^{-0.14t} = 130e^{-0.14(7)} = 48.7904\ldots = 48.8 \quad (1\text{ d.p.})$$

$\therefore$ about 48.8 mg remains.

(iv) Let $C = \frac{1}{2}A$:

$$\frac{1}{2}A = Ae^{-0.14t}$$

$$\frac{1}{2} = e^{-0.14t}$$

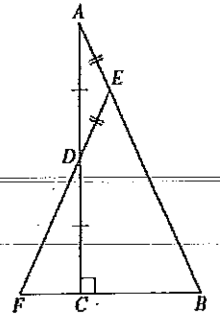
$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{-0.14t}\right)$$

$$\ln\left(\frac{1}{2}\right) = -0.14t$$

$$t = \ln\left(\frac{1}{2}\right) \div -0.14 = 4.95 \quad (2\text{ d.p.})$$

$\therefore$ after about 5 hours.

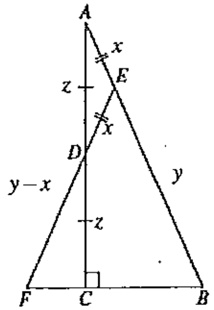
(b)



(i) In $\triangle ACB$ and $\triangle DCF$:
$\angle ACB = \angle DCF$ (given $AC \perp FB$)
$\angle BAC = \angle ADE$ (base angles of isosceles $\triangle ADE$)
$\angle ADE = \angle FDC$ (vertically opposite angles)
$\therefore\ \angle BAC = \angle FDC$
$\angle ABC = \angle DFC$ (angle sum of triangle)
$\therefore \triangle ACB\ |||\ \triangle DCF$ (equiangular).

(ii) $\angle DFC = \angle ABC$ (matching angles of similar triangles)
These are equal angles in $\triangle EFB$.
$\therefore \triangle EFB$ is isosceles.

(iii) Let $AE = x$, $EB = y$ and $DC = z$



$ED = x$ (given $AE = ED$)
$EF = y$ ($\triangle EFB$ is isosceles, part(ii))
$\therefore DF = y - x$
Ratio of the sides of similar triangles from part (i):

$$\frac{DF}{AB} = \frac{DC}{AC}$$

$$\frac{y-x}{x+y} = \frac{z}{2z}$$

$$\frac{y-x}{x+y} = \frac{1}{2}$$

$$2(y-x) = x+y$$

$$2y - 2x = x + y$$

$$y = 3x$$

$\therefore EB = 3AE$

(c) (i) When $\frac{dV}{dt} = 0$:

$$80\sin(0.5t) = 0$$

$$\sin(0.5t) = 0$$

$$0.5t = 0, \pi, 2\pi, \ldots$$

$$t = 0, 2\pi, 4\pi, \ldots$$

The volume is increasing at $t = 0$, then the volume must be decreasing just after $t = 2\pi$ hours.

(ii) $V = \int 80\sin(0.5t)\,dt$

$$= -160\cos(0.5t) + C$$

When $t = 0$, $V = 1200$:

$$1200 = -160\cos(0.5(0)) + C$$

$$1200 = -160 + C$$

$$C = 1360$$

$t = 0, 2\pi, 4\pi, \ldots$

When $t = 3$:

$$V = 1360 - 160\cos(0.5(3)) = 1348.682\ldots = 1348.7 \quad (1\text{ d.p.})$$

$\therefore$ about 1349 litres.

(iii) *Method 1:*
Maximum or minimum values when $\frac{dV}{dt} = 0$. From part (i), this occurs for $t = 0, 2\pi, 4\pi, \ldots$.

When $t = 0$, it is given that the volume is increasing $\therefore$ the maximum should be when $t = 2\pi$:

$$V = 1360 - 160\cos(0.5(2\pi)) = 1360 - 160 \times -1 = 1520$$

$\therefore$ Greatest volume is 1520 litres.

OR

Method 2:
Since $-1 \le \cos(0.5t) \le 1$:

$$-160 \le 160\cos(0.5t) \le 160$$

$$160 \ge -160\cos(0.5t) \ge -160$$

$$1520 \ge 1360 - 160\cos(0.5t) \ge 1200$$

$$1520 \ge V \ge 1200$$

$\therefore$ Greatest volume is 1520 litres.

Question 16

(a) (i) $y = x^2 - 7x + 10$

$$= (x-2)(x-5)$$

For x-intercepts, $y = 0$:

$$(x-2)(x-5) = 0$$

$$x - 2 = 0 \quad \text{or} \quad x - 5 = 0$$

$$x = 2, 5$$

For A, $x = 2$ and for B, $x = 5$.

(ii) For the y-intercept, $x=0$:

$$y=x^2-7x+10$$
$$=(0)^2-7(0)+10$$
$$=10$$

For the x coordinates:

$$y=x^2-7x+10$$
$$10=x^2-7x+10$$
$$0=x^2-7x$$
$$=x(x-7)$$
$$x=0,\ 7$$

$\therefore\ C$ is $(7,10)$.

(iii) $$\int_0^2 x^2-7x+10\,dx=\left[\frac{1}{3}x^3-\frac{7}{2}x^2+10x\right]_0^2$$
$$=\left[\frac{1}{3}(2)^3-\frac{7}{2}(2)^2+10(2)\right]-[0]$$
$$=\frac{26}{3}$$
$$=8\frac{2}{3}.$$

(iv) The area found in part (iii) is the area under the curve between O and A.

The required area is the area under the line AC (a triangle) less the area under the curve BC (by symmetry, this is the same as for part (iii)).

$$A=\Delta ACD-\int_0^2 x^2-7x+10\,dx$$
$$=\left(\frac{1}{2}\times(7-2)\times10\right)-\frac{26}{3}$$
$$=25-8\frac{2}{3}$$
$$=16\frac{1}{3}$$

$\therefore$ the area is $16\frac{1}{3}$ units ^{2}.

(b) For a volume around the y-axis:

$$V=\pi\int_0^6 x^2\,dy \text{ and } y=8\log_e(x-1)$$

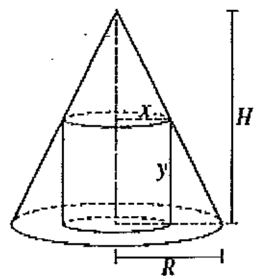
Make x the subject:

$$y=8\log_e(x-1)$$
$$\frac{y}{8}=\log_e(x-1)$$
$$x-1=e^{\frac{y}{8}}$$
$$x=e^{\frac{y}{8}}+1$$
$$x^2=\left(e^{\frac{y}{8}}+1\right)^2$$
$$=\left(e^{\frac{y}{8}}\right)^2+2\left(e^{\frac{y}{8}}\right)+(1)^2$$
$$=e^{\frac{y}{4}}+2e^{\frac{y}{8}}+1$$

$$V=\pi\int_0^6 x^2\,dy$$
$$=\pi\int_0^6\left(e^{\frac{y}{4}}+2e^{\frac{y}{8}}+1\right)dy$$
$$=\pi\left[4e^{\frac{y}{4}}+16e^{\frac{y}{8}}+y\right]_0^6$$
$$=\pi\left[4e^{\frac{1}{4}y}+16e^{\frac{1}{8}y}+y\right]_0^6$$
$$=\pi\left\{4e^{\frac{1}{4}(6)}+16e^{\frac{1}{8}(6)}+6-\left[4e^{\frac{1}{4}(0)}+16e^{\frac{1}{8}(0)}+0\right]\right\}$$
$$=\pi\left\{4e^{\frac{3}{2}}+16e^{\frac{3}{4}}+6-[4+16]\right\}$$
$$=118.748\ldots$$
$$=118.7\ \ (1\text{ d.p.}).$$

$\therefore$ the volume is about 118.7 units ^{3}.

(c) (i)



Method 1:
Using ratio of sides in similar triangles:

$$\frac{x}{R}=\frac{H-y}{H}$$
$$\frac{xH}{R}=H-y$$
$$y=H-\frac{xH}{R}$$
$$=H\left(1-\frac{x}{R}\right)$$
$$=H\left(\frac{R-x}{R}\right)$$

OR

Method 2:
Using ratio of sides in similar triangles:

$$\frac{y}{H}=\frac{R-x}{R}$$
$$y=H\left(\frac{R-x}{R}\right)$$

From either method, find the volume:
Volume of a cylinder, $r=x$ and $h=y$:

$$V=\pi r^2h$$
$$=\pi x^2y$$
$$=\pi x^2\left(\frac{R-x}{R}\right)H$$
$$=\frac{H}{R}\pi x^2(R-x).$$

(ii) $$V=\frac{H}{R}\pi x^2(R-x)$$
$$=\frac{H}{R}\pi\left(Rx^2-x^3\right)$$

$$\frac{dV}{dx}=\frac{H}{R}\pi\left(2Rx-3x^2\right)$$

For $\frac{dV}{dx}=0$:

$$\frac{H}{R}\pi\left(2Rx-3x^2\right)=0$$
$$x(2R-3x)=0$$
$$2R-3x=0 \text{ or } x=0$$
$$x=\frac{2}{3}R \quad (\text{since } x\neq0)$$

Test for a max or min:

$$\frac{d^2V}{dx^2}=\frac{H}{R}\pi(2R-6x)$$
$$=\frac{H}{R}\pi\left(2R-6\left(\frac{2}{3}R\right)\right)$$
$$=\frac{H}{R}\pi(2R-4R)$$
$$=\frac{H}{R}\pi(-2R)$$
$$=-2H\pi<0$$

$\therefore$ Maximum value when $x=\frac{2}{3}R$.

$$V_{CYLINDER}=\frac{H}{R}\pi\left(\frac{2}{3}R\right)^2\left(R-\frac{2}{3}R\right)$$
$$=\frac{H}{R}\pi\frac{4}{9}R^2\left(\frac{1}{3}R\right)$$
$$=\frac{4\pi HR^2}{27}$$

$$V_{CONE}=\frac{1}{3}\pi R^2H$$

Consider the ratio of volumes:

$$\frac{V_{CYLINDER}}{V_{CONE}}=\frac{4\pi HR^2}{27}\div\frac{1}{3}\pi R^2H$$
$$=\frac{4\pi HR^2}{27}\times\frac{3}{\pi R^2H}$$
$$=\frac{4}{9}$$

$\therefore$ The maximum volume of the inscribed cylinder does not exceed $\frac{4}{9}$ of the volume of the cone.

End of Mathematics solutions