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FORTITER ET SUAVITER

Mathematics
HSC Assessment
June 10
Task 3 2015

General Instructions

- Time Allowed – 50 minutes
- Write using blue or black pen
- Draw any relevant diagrams using pencil
- Board-approved calculators may be used
- All necessary working should be shown in every question

Total marks (41)
- Attempt Questions 1 –7

Place Multiple Choice answers at the top of Question 5.

Q1. If $\sin x = -\frac{1}{5}$ and $\pi \le x \le \frac{3\pi}{2}$, then $\cot x$ equals:

A. $-\frac{1}{2\sqrt{6}}$ B. $-2\sqrt{6}$ C. $\frac{1}{2\sqrt{6}}$ D. $2\sqrt{6}$

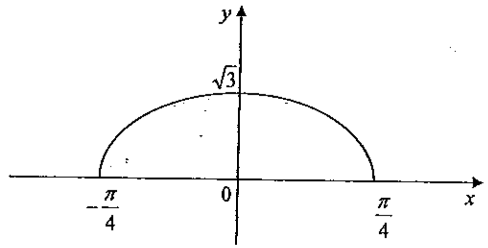
Q2. The acceleration of a particle is given by $\ddot{x} = 4\cos 2t$ where x is the displacement in metres and t is in seconds. Which of the following is a possible expression for its displacement?

A. $-2\sin 2t$ B. $2\sin 2t$ C. $\cos 2t$ D. $-\cos 2t$

Q3. A particle moves so that its dispalcement in metres from the origin at time t seconds is given by $x = 20t - 5t^2$. At what time is it stationary?

A. 0 seconds B. 2 seconds C. 4 seconds D. 6 seconds

Q4. The diagram shows the region bounded by the curve $y = \sqrt{3\cos 2x}$ and the x-axis for $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$.
The region is rotated about the $x-$axis to form a solid.



Which of the following gives the volume of the solid?

(A) $V = 3\pi\int_0^{\frac{\pi}{4}} \cos 2x \, dx$ (B) $V = 9\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x \, dx$

(C) $V = 6\pi\int_0^{\frac{\pi}{4}} \cos 4x \, dx$ (D) $V = 6\pi\int_0^{\frac{\pi}{4}} \cos 2x \, dx$

Question 5(Trigonometric Functions 19 marks)

a. Differentiate $y = 2x\tan\frac{x}{2}$ 2

b. Find the exact slope of the tangent to the curve $y = \cos\left(x + \frac{\pi}{3}\right)$ at the point $\left(0, \frac{1}{2}\right)$. 2

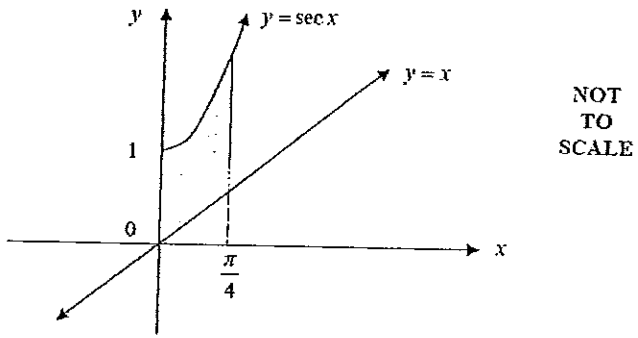
c. i. Sketch the curve $y = 3\cos\frac{x}{2}$ for $-\pi \le x \le \pi$. 2

ii. Use your graph to determine the number of solutions to the equation $\cos\frac{x}{2} = \frac{x+4}{6}$ that exist in the domain $-\pi \le x \le \pi$. 2

d. Using the Trapezoidal rule with three function values, find an approximation to the value for

$$\int_1^2 \operatorname{cosec}\frac{\pi x}{6}. \quad \text{(correct to 2 decimal places)}$$ 3

e.



The shaded region is boounded by the y axis, $y = x$ and the curve $y = \sec x$ from $x = 0$ to $x = \frac{\pi}{4}$.

Find the volume formed when this region is rotated about the x axis.
Leave answer in exact from. 3

f. i. Differentiate $\cos^2 3x$ with respect to x. 2

ii. Hence evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 3x \sin 3x\, dx$. 3

Question 6 (Start a New Page - Applications of Calculus - 18 marks)

a. A particle P moves such that the displacement $x\,cm$ from the origin after t seconds is given by $x = t^3 - 3t$, $t \geq 0$.

i. Find the initial velocity. 1

ii. Determine when the particle is at rest. 2

iii. Is the particle speeding up or slowing down at $t = 2$? Give reasons. 2

b. A tank is emptied by a tap from which the water flows so that, until the water ceases, the rate of flow after t minutes is R litres/minute where $R = -(t-6)^2$.

i. What is the initial rate of flow? 1

ii. How long does it take to empty the tank? 1

iii. How long will it take (to the nearest second) for the flow to drop to 20 litres/minute? 2

iv. How much water was in the tank initially? 2

c. Two particles A and B start from the origin at the same time and move along a straight line so that their velocities in m/s at any time t seconds are given by:

$$v_A = t^2 + 2 \quad \text{and} \quad v_B = 8 - 2t$$

Clearly show that the two particles never move with the same acceleration. 2

d. A property agent asumes that for her property, the rate of increase in the value is directly proportional to the value V.

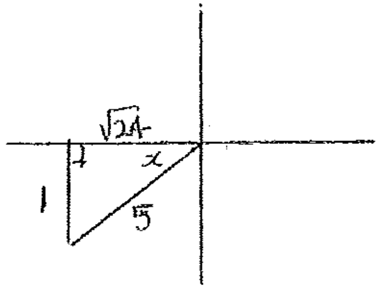
i. Show that the equation $V = V_0 e^{kt}$ satisfies this assumtion, where V_0 and k are constants and t is the time in years. 1

ii. She bought an apartment 10 years ago for \$150 000, and calculated that it would be worth \$600 000 after 20 years.
Show that $k = 0.1\ln 2$ 2

iii. If the apartment is currently valued at \$350 000, calculate whether this is more or less than she had anticipated and by how much. 2

Yr12 Maths Task 3
June 2015

Q1/ $\sin x = -\frac{1}{5}$



$\therefore \cot x = \frac{\sqrt{24}}{1} = \sqrt{24} = 2\sqrt{6}$

$\therefore D$

Q2/ $\ddot{x} = 4\cos 2t$

$\dot{x} = -2\sin 2t$

$\dot{x} = -\cos 2t \quad \therefore D$

Q3/ $x = 20t - 5t^2$

$V = 20 - 10t = 0$

$t = 2s \quad \therefore B$

Q4/ $y = \sqrt{3\cos 2x}$

$y^2 = 3\cos 2x$

$\therefore V = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 3\cos 2x\,dx$

$V = 2\pi\int_0^{\frac{\pi}{4}} 3\cos 2x\,dx$

$= 6\pi\int_0^{\frac{\pi}{4}} \cos 2x\,dx$

$\therefore D$

Q5 left out.

Q6a) $y = 2x\tan\frac{x}{2}$

$y' = 2\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2} \times 2x$

$y' = 2\tan\frac{x}{2} + x\sec^2\frac{x}{2}$

b) $y = \cos\left(x + \frac{\pi}{3}\right)$

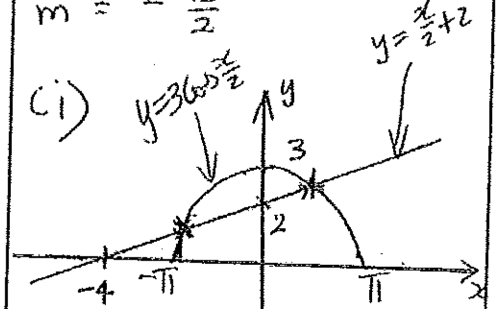
$y' = -\sin\left(x + \frac{\pi}{3}\right)$

at $x = 0$

$m = -\sin\frac{\pi}{3}$

$m = -\frac{\sqrt{3}}{2}$

c i)



ii) $\cos\frac{x}{2} = \frac{x+4}{6}$

$3\cos\frac{x}{2} = \frac{x+4}{2}$

$= \frac{x}{2} + 2$

2 marks : see soln

1 mark :

showing $\frac{1}{2}\sec^2\frac{x}{2}$ somewhere in a product rule attempt.

or a product rule used.

b) 2 marks : $m = \frac{-\sqrt{3}}{2}$

1 mark :

$m = -\sin\frac{\pi}{3}$

c i) 2 marks see soln

1 mark:

ii) 2 marks: 2 solns <u>with</u> full marks.

MUST have $y = \frac{x}{2} + 2$ for

must show GRAPHICALLY correct reasoning

ii) 1 mark: incorrect sketches but answered correctly from them

or sketching $y = \frac{x}{2} + 2$

Q6d)

x	1	1.5	2
y	$\operatorname{cosec}\frac{\pi}{6}$	$\operatorname{cosec}\frac{1.5\pi}{6}$	$\operatorname{cosec}\frac{2\pi}{6}$

$h = 0.5$

$$\therefore A \approx \frac{0.5}{2}\left[\operatorname{cosec}\frac{\pi}{6} + 2\operatorname{cosec}\frac{1.5\pi}{6} + \operatorname{cosec}\frac{2\pi}{6}\right]$$

$A \approx 1.50$ units^2

e) $y = \sec x$

$y^2 = \sec^2 x$

$y = x$

$y^2 = x^2$

$$\therefore V = \pi\int_0^{\frac{\pi}{4}} \sec^2 x - x^2$$

3marks: 1.50 units^2

2marks: $A \approx \frac{0.5}{2}\left(\operatorname{cosec}\frac{\pi}{6} + 2\operatorname{cosec}\frac{1.5\pi}{6} + \operatorname{cosec}\frac{2\pi}{6}\right)$

1mark: $h = 0.5$

3marks: $\pi\left[1 - \frac{\pi^3}{192}\right]$

2marks: $\pi\left[\tan x - \frac{x^3}{3}\right]_0^{\frac{\pi}{4}}$

1mark: $V = \int_0^{\frac{\pi}{4}} \sec^2 x - x^2$

$$V = \pi\left[\tan x - \frac{x^3}{3}\right]_0^{\frac{\pi}{4}}$$

$$V = \pi\left[\left(\tan\frac{\pi}{4} - \frac{\left(\frac{\pi}{4}\right)^3}{3}\right) - (\tan 0 - 0)\right]$$

$$V = \pi\left[1 - \frac{\pi^3}{192}\right]$$

f) i) $y = (\cos 3x)^2$

$y' = 2(\cos 3x) \times -3\sin 3x$

$y' = -6\sin 3x\cos 3x$

ii) $(\cos 3x)^2 \underset{\text{Int}}{\overset{\text{Diff}}{\rightleftarrows}} -6\sin 3x\cos 3x$

$\therefore \int -6\sin 3x\cos 3x = (\cos 3x)^2$

$$\int_{\frac{\pi}{6}}^{\pi} \sin 3x\cos 3x = \left[\frac{(\cos 3x)^2}{-6}\right]_{\frac{\pi}{6}}^{\pi}$$

$$\frac{(\cos 3\pi)^2}{-6} - \frac{\left(\cos\frac{\pi}{2}\right)^2}{-6}$$

$= -\frac{1}{6} + 0$

$= -\frac{1}{6}$

i) 2marks: $-6\sin 3x\cos 3x$

1mark: Some correct working eg $-6\sin 3x$

ii) 3marks: $-\frac{1}{6}$

2marks: $\left[\frac{(\cos 3x)^2}{-6}\right]_{\frac{\pi}{6}}^{\pi}$

1mark: Some correct working

Q7a) i) $x = t^3 - 3t$

$\dot{x} = 3t^2 - 3$

$t=0 \quad V = 3\times 0^2 - 3$

$V = -3$

ii) $\dot{x} = 0$ rest

$3t^2 - 3 = 0$

$t^2 = 1$

$t = \pm 1$

but $t \geq 0 \therefore t = 1$

because time can't be negative

iii) $\dot{x} = 3\times 2^2 - 3$

$\dot{x} = 9\text{cm/s}$

$\ddot{x} = 6t$ when $t=2$

$\ddot{x} = 12\text{cm/s}^2$

$\therefore$ since $\dot{x} > 0, \ddot{x} > 0$ then particle is speeding up.

1mark: $V = -3$

2marks: $t=1$ with acknowledgement of a negative t in the solution.

1mark: $t^2 = 1$

2marks: see solution

1mark: $\dot{x} = 9$ and $\ddot{x} = 12$ or incorrect $\dot{x}, \ddot{x}$ but correctly interpreted.

b) i) $R = -(t-6)^2$

is already a rate (L/min) so no need to differentiate

$\therefore R = -(0-6)^2 = -36\text{L/min}$

ii) empty tank = zero flow rate.

$\therefore R = 0$

$0 = -(t-6)^2$

$(t-6)^2 = 0$

$t - 6 = 0$

$t = 6$

iii) $R = -20, \quad t = ?$

$-20 = -(t-6)^2$

$t - 6 = \pm\sqrt{20}$

$t = 6 \pm \sqrt{20}$

$t = 6 + \sqrt{20}$ — this is more than the time to empty the tank. so reject this time

$t = 6 - \sqrt{20}$

$t = 1.52786$ mins

$\underline{t = 1\text{min } 32\text{sec}}$

i) 1mark = -36L/min

ii) 1mark: $t = 6$

Note: If you integrate then you MUST find C

iii) 2marks: 1min 32 sec

1mark: $t = 6 \pm \sqrt{20}$

or $\dfrac{12 \pm \sqrt{80}}{2}$

Note: your solution MUST have 2 answers to be considered for a mark.

Q7b iv) Litres/min

$R = -(t-6)^2$

$\int R = \int -(t-6)^2$

$V = -\dfrac{(t-6)^3}{3} + c$

Volume of water in litres

$t = 6$, volume $(V) = 0$

took 6 seconds to empty

$0 = -\dfrac{(6-6)^3}{3} + c$

$\therefore c = 0$

$V = -\dfrac{(t-6)^3}{3}$

$\therefore$ when $t = 0$, $V = ?$

$V = -\dfrac{(0-6)^3}{3} = 72$ litres.

2 marks: 72 litres with correct working (includes finding $c = 0$)

1 mark: $V = -\dfrac{(t-6)^3}{3} + c$

or $V = 72$ without finding a constant.

c) $V_A = t^2 + 2$ $\quad V_B = 8 - 2t$

$\dot{V}_A = 2t$ $\quad \dot{V}_B = -2$

since $t \geq 0$, $\dot{V}_A$ can never be negative which is what $\dot{V}_B$ is.

2 marks: see soln

1 mark: $\dot{V}_A = 2t$, $\dot{V}_B = -2$

d i) $\dfrac{dv}{dt} = kV$ ← show

$\therefore V = V_0 e^{kt}$

$\dfrac{dv}{dt} = V_0 \times k e^{kt}$

but $V = V_0 e^{kt}$

$\therefore \dfrac{dv}{dt} = kV$

1 mark: see soln

ii) $t = 0$, $V = 150000$

$\therefore V_0 = 150000$

$t = 20$, $V = 600000$, $k = ?$

$600000 = 150000 e^{20k}$

$4 = e^{20k}$

$\log_e 4 = 20k$

$k = \frac{1}{20} \ln 4 = \frac{1}{20} \ln 2^2$

$= 2 \times \frac{1}{20} \ln 2$

$= 0.1 \ln 2$

2 marks: see solution

1 mark: $\ln 4 = 20k$

Q7 d iii) current value : $t=10$

$V=?$

$$V = 150000\, e^{0.1 \ln 2 \times 10}$$

$V = \$300000$ (expected value)

but appartment is worth \$350000 now

It is worth more than she anticipated

by $\$350000 - \$300000$

$= \$50000$

OR $\quad 350\,000 = 150000\, e^{0.1 \ln 2 \times t}$

$2\frac{1}{3} = e^{0.1 \times \ln 2 \times t}$

$\log_e(2\frac{1}{3}) = 0.1 \ln 2 t$

$t = \dfrac{\ln(2\frac{1}{3})}{0.1 \ln 2} = 12.2$ years

So since it only took 10 years to reach \$350 000 then it was more than anticipated (look at the top for how much)

2marks: see solutions

1mark: $V = \$300\,000$

or wrong V but correctly answered from it.

1mark for $t = 12.2$ years.