Student _

BRIGIDINE COLLEGE
RANDWICK

PRELIMINARY
EXTENSION 1
MATHEMATICS

YEARLY

2007

(Time - 90 minutes)

Directions to candidates

* *Put your **name at the top** of this paper and on each of the **5 sections** that are to be collected.*

* *All 5 questions are to be attempted.*

* *All 5 questions are of equal value.*

* *All questions are to be answered on separate pages and will be collected in separate bundles at the end of this exam.*

* *All necessary working should be shown in every question.*

* *Full marks may not be awarded for careless or badly arranged work.*

Question 1 *(Start a new page)*

a. Differentiate the following with respect to x:

i. $\dfrac{6x}{1+3x}$ 2m

ii. $(5-7x)^4$ 2m

iii. $x\sqrt{x}$ 2m

b. The first term of a geometric series is 32 and the sixth term is 1. 3m

i. Find the common ratio.

ii. Find the limiting sum of this series.

c. The gradient function of the curve is given by $\dfrac{dy}{dx} = 1 + 4x - x^2$.
What is the equation of the curve if it passes through the point (3,0)? 2m

d. Expand and **simplify fully** $2(2^k - 1) + 2^{k+1}$ 2m

e. By considering arithmetic and/or geometric series evaluate

$$\sum_{n=1}^{30} \left(3n + \left(\frac{1}{2}\right)^n\right)$$ 3m

Question 2 *(Start a new page)*

a. If α and β are the roots to the equation $x^2 = 5x - 2$

i. State the value of $\alpha + \beta$ and $\alpha\beta$ 1m

hence find:

ii. $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ 1m

iii. $\alpha^2 + \beta^2$ 2m

b. Consider the parabola $(x + 2)^2 = 4y$

i. Show that if the line $y = mx$ intersects the parabola at two distinct points then $(4 - 4m)^2 - 16 > 0$ 2m

ii. Find the value(s) of m for which the line $y = mx$ is a tangent to the parabola. 2m

c. Solve $x^4 - 7x^2 - 18 = 0$ 3m

d. Consider the parabola $x^2 - 2x - 4y + 1 = 0$.

i. Express the equation in the form $(x - h)^2 = 4a(y - k)$ 2m

ii. Hence write down the coordinates of the vertex. 1m

iii. Find the coordinates of the focus 1m

iv. Write down the equation of the directrix 1m

Question 3 *(Start a new page)*

a. i. Factorize $u^2 - 6u - 16$ 1m

Hence or otherwise

ii. solve $(\log_2 x)^2 - 6\log_2 x - 16 = 0$ 2m

b. Given that $\log_a b = 2.75$ and $\log_a c = 0.25$ 2m
find the value of $\log_a (bc)^2$

c. i. Find the equation of the tangent and the normal to the curve $y=x^2 + 2x - 5$ at the point, P where $x = -2$. 4m

ii. The tangent meets the x-axis at A. Find the coordinates of point A. 1m

iii. The normal meets the y-axis at B. Find the coordinates of the point B. 1m

d. For the function $y = x + \dfrac{1}{x-2}$,

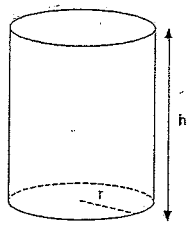
i. Show by using limits that $y = x$ is an asymptote to the curve. 2m

ii. State any other asymptotes. 1m

e. Find the primitive function of $\int \dfrac{3x^3 - 2x^2 + x^{-1}}{x^2}$ 2m

Question 4 *(Start a new page)*

a. A closed cylindrical can with height h cm and radius r cm is to be made from 216π cm^2 of metal sheeting , to package corn.



i. Show that $h = \dfrac{108}{r} - r$ 2m

ii. Find a formula for the volume of the cylinder, in terms of r . 2m

iii. Find the maximum volume that can be obtained from this amount of metal sheeting. 2m

b. A is the point (-3,-1) and B is the point (7,3). The point P(x,y) moves so that the angle APB is a right angle.

i. Show the locus of P is the circle, $x^2 - 4x + y^2 - 2y - 24 = 0$. 2m

ii. Find the centre and radius of this circle. 2m

c. Consider the function $f(x) = x^3 - x^2 - 8x - 3$.

i. Find the coordinates of the stationary points of the curve y=f(x) And determine their nature. 3m

ii. Sketch the curve, clearly labelling any stationary points and the y- intercept. 2m

iii. For what values of x is the function decreasing. 1m

Question 5 *(Start a new page)*

a. In a particular colony of birds, there were originally 2000 birds. From one autumn to the next, the population increases by 8%, but each autumn 200 birds migrate to a warmer climate and never return. If F_n represents the number of birds in the colony n autumns from when the original count was made, show that

i. $F_1 = 2000 \times 1.08 - 200$ 1m

ii. $F_n = 2500 - 500 \times 1.08^n$ 2m

iii. Hence find after how many autumns the colony will become extinct. 2m

b. A woman invests \$50 at the beginning of each month. Interest is compounded monthly at 1 % per month.

i. Determine the size of her first investment at the end of one month. 1m

ii. Show that at the end of 10 years her accumulated value of her investments may be given by $\$5050\,(1.01^{120} - 1)$ 2m

iii. Determine the accumulated value of her investment. 1m

c. A large company takes out a loan, to build a factory, the loan required is \$P with interest charged at an introductory rate of 6% p.a. for the first three months. The loan is to be initially repaid in equal monthly repayments of \$4000 over three years and interest is charged monthly before each repayment.

Let $\$A_n$ be the amount owing by the factory at the end of the nth repayment.

i. Find an expression for A_1. 2m

ii. Show that $A_3 = P(1.005)^3 - 4\,000(1 + 1.005 + 1.005^2)$ 2m

At the end of three months interest rates rise to 9% p.a. and the loan is to be repaid in total in equal monthly repayments of \$4800 for the next 2.75 years.

iii. If the loan's interest rate is fixed at 9% for the remainder of the loan, find the value of P. 3m

Year 11 Ext 1. Prelim 2007 Yearly.

1 i) $\frac{d}{dx}\left(\frac{6x}{1+3x}\right)=\frac{(1+3x)\times 6-6x\times 3}{(1+3x)^2}$

$=\frac{6+18x-18x}{(1+3x)^2}$

$=\frac{6}{(1+3x)^2}$

) $\frac{d}{dx}(5-7x)^4=4(5-7x)^3\times -7$

$=-28(5-7x)^3$

) $\frac{d}{dx}x\sqrt{x}=\frac{d}{dx}x^1\times x^{1/2}$

$=\frac{d}{dx}x^{3/2}$

$=\frac{3}{2}x^{1/2}$

$=\frac{3\sqrt{x}}{2}$

i) $a=32$ $T_6=1$ $r=?$

$T_n=ar^{n-1}$

$T_6=32r^5=1$

$r^5=\frac{1}{32}$

$r=\frac{1}{2}$

) $S_\infty=\frac{a}{1-r}$

$=\frac{32}{\frac{1}{2}}=64$

c $\frac{dy}{dx}=1+4x-x^2$

$y=x+\frac{4x^2}{2}-\frac{x^3}{3}+C$

given $x=3$, $y=0$

$0=3+2\times 9-\frac{27}{3}+C$

$\therefore C=-12$.

Eqn:

$y=x+2x^2-\frac{x^3}{3}-12$

$y=-\frac{1}{3}x^3+2x^2+x-12$.

d. $2\times 2^k-2\times 1+2^{k+1}$

$=2^{k+1}-2+2^{k+1}$

$=2\times 2^{k+1}-2$

$=2^{k+2}-2$

e. $\sum_{n=1}^{30}3n+\frac{1}{2}^n$

$=3+\frac{1}{2}+6+\frac{1}{4}+9+\frac{1}{8}+12+\frac{1}{16}$

$+\dots+90+\frac{1}{2^{30}}$

$=\underbrace{(3+6+\dots+90)}_{AP\{S_n=\frac{n}{2}(a+l)\}}+\underbrace{\left(\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^{30}}\right)}_{GP\{S_n=\frac{a(1-r^n)}{1-r}\}}$

$=\frac{30}{2}(3+90)+\frac{\frac{1}{2}(1-\frac{1}{2}^{30})}{\frac{1}{2}}$

$=1395+0.\dot{9}$

$=1396$

Marking Criteria

Question 1.

1 i) 2 marks use quotient rule correctly → correct answer

1 mark. 1 mistake in rule

ii) *Correct answer 2 marks

*with 1 mistake after using rule correctly 1 mark.

c.

1 mark for correct equation with ~~C~~

+1 mark for finding $C=-12$.

d. 1 mark expand

$2(2^k-1)\rightarrow 2^{k+1}-2$

+ 1 mark

$2^{k+1}-2+2^{k+1}\rightarrow 2.2^{k+1}-2$

$\rightarrow 2^{k+2}-2$

e. 1 mark writing writing out sum

$\rightarrow 3+\frac{1}{2}+6+\frac{1}{4}+\dots+90+\frac{1}{2^{30}}$

1 mark each for

Sum of Arithmetic series $3+6+\dots+90=1395$

& sum of Geometric series

$\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^{30}}=0.\dot{9}$

2a)

i) x^2-5x+2

$a=1,\ b=-5,\ c=2$

i. $\alpha+\beta = -\frac{b}{a} = 5$

$\alpha\beta = \frac{c}{a} = 2$

ii) $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$

$= \frac{5}{2}$

iii) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 5^2 - 2\times 2$

$= 21$

b)

i) $(x+2)^2 = 4y$ — ①

$y = mx$ ②

Pt of int. satisfies:

$(x+2)^2 = 4mx$

$x^2+4x+4 = 4mx$

$x^2+(4-4m)x+4=0$

2 solutions

$\Delta > 0$

$\Delta = (4-4m)^2 - 4\times 4$

$= (4-4m)^2 - 16$

$(4-4m)^2 - 16 > 0$

for 2 points of intersection.

ii tangent

$\Delta = 0$

$\Delta = (4-4m)^2 - 16 = 0$

$(4-4m)^2 = 16$

$4-4m = \pm\sqrt{16}$

$4-4m = 4$ or $4-4m = -4$

$-4m = 0$ | $m = 2$

$m = 0$ |

$y = mx$ is a tangent when $m=0$ & $m=2$.

c) $x^4 - 7x^2 - 18 = 0$

Let $x^2 = u$

$u^2 - 7u - 18 = 0$

$(u-9)(u+2) = 0$

$u = 9,\ u = -2$

$\therefore\ x^2 = 9 \qquad x^2 = -2$ No solution

$x = \pm 3$

iii $a = 1$

(focus) $(1,1)$

iv) directrix

$y = -1$

$x^2 - 2x - 4y + 1 = 0$

i) $x^2 - 2x + 1 = 4y - 1 + 1$

$(x-1)^2 = 4y$

ii) vertex $(1,0)$

Quest 2. Marking Criteria.

a i) 1 mark $\alpha+\beta = 5$, $\alpha\beta = 2$

ii) 1 mark $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{5}{2}$

1 mark if use wrong numbers from part (i) but correct method.

iii) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$ 1 mark

+ 1 mark correct answer

Only 1 mark if wrote $(\alpha+\beta)^2 + 2\alpha\beta$ wrong sign

b. 1 mark for point of intersection. $(x+2)^2 = 4mx$.

1 mark $\Delta = (4-4m)^2 - 16$ showing and stating $\Delta > 0$ for 2 solutions.

c. 3 marks correct solutions

lose 1 mark for not writing ± 3 only $+3$.

d. 2 marks for completing the square correctly, (lose mark if an error).

ii 1 mark correct vertex

iii 1 mark correct focus

iv. 1 mark correct directrix

{if use incorrect eqn. (i) to get ii) iii) iv) get marks.}

3·a(i) $u^2-6u-16=(u-8)(u+2)$

ii) $(\log_2 x)^2-6\log_2 x-16=0$

Let $u=\log_2 x$

$u^2-6u-16=0$

$(u-8)(u+2)=0$

$u=8 \quad u=-2$

$\therefore \log_2 x=8 \quad \log_2 x=-2$

$x=2^8 \quad , x=2^{-2}$

$x=256 \quad x=\frac{1}{2^2}$

$=\frac{1}{4}$

b) $\log_a(bc)^2 = 2\log_a bc$

$=2(\log_a b+\log_a c)$

$=2(2.75+0.25)$

$=2\times 3$

$=6$

c. i) $y=x^2+2x-5$

$\frac{dy}{dx}=2x+2$

at $x=-2$, $y=-5$

$\frac{dy}{dx}=2\times -2+2$

$=-2$

$\therefore$ gradient of tangent at $(-2,-5)$ $m=-2$.

Eqn. of tangent

$y--5=-2(x--2)$

$y+5=-2(x+2)$

$y=-2x-9$ or $2x+y+9=0$

gradient of normal $=\frac{1}{2}$

$y--5=\frac{1}{2}(x--2)$

$y+5=\frac{1}{2}x+1$

$y=\frac{1}{2}x-4$

or $x-2y-8=0$ $\Big\}$ Eqn of normal.

ii) x-intercept of tangent

when $y=0$

$2x+9=0$

$x=-4\frac{1}{2}$.

$A=(-4\frac{1}{2},0)$

iii) y-intercept normal

$x=0$

$y=-4$

$B(0,-4)$

d. $y=x+\frac{1}{x-2}$

$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(x+\frac{1}{x-2}\right)$

$\lim_{x\to\infty}\frac{1}{x-2}=0$

$\therefore$ $y=x$ asymptote

ii) $x=2$ asymptote. (as $x\neq 2$)

e. $\frac{3x^3}{x^2}-\frac{2x^2}{x^2}+\frac{x^{-1}}{x^2}=3x-2+x^{-3}$

Primitive

$\frac{3x^2}{2}-2x+\frac{x^{-2}}{-2}$

$=\frac{3x^2}{2}-2x-\frac{1}{2x^2}$

<u>Marking Criteria.</u>

3·a i 1 mark factorise correctly.

ii) 2 marks correct answers

1 mark $\log_2 x=8$, $\log_2 x=-2$

1 mark if either $x=256$ $x=\frac{1}{4}$

b. 1 mark

$\log_a(bc)^2=2(\log_a b+\log_a c)$.

1 mark substituting

c.

i) 2 marks eqn of tangent

2 marks eqn of normal

{1 mark for find gradient of tangent or normal}

ii) 1 mark for correct point using eqn in (i)

iii) 1 mark for correct answer. (using eqn in part (i))

d) 1 mark writing

i) $\lim_{x\to\infty}$

1 mark stating

$\lim_{x\to\infty}\frac{1}{x-2}\to 0$

ii) 1 mark $x=2$

e) 2 marks correct answer

(Needed to divide through by x^2 first).

Question 4.

a) $S.A = 216\pi$

i) $SA = 2\pi r^2 + 2\pi r h$ ✓

$\therefore 2\pi r^2 + 2\pi r h = 216\pi$

$2\pi(r^2 + rh) = 216\pi$ [1mark for equating to S.A. formula]

$r^2 + rh = 108$

$rh = 108 - r^2$

$h = \frac{108 - r^2}{r}$

$= \frac{108}{r} - \frac{r^2}{r}$ ✓

$\boxed{h = \frac{108}{r} - r}$ [1mark for rearraging]

ii) $V = \pi r^2 h$ ✓ (sub from (i))

$= \pi r^2 \left(\frac{108}{r} - r\right)$ [1mark for substituting]

$V = 108\pi r - \pi r^3$ ✓ [1mark answer]

iii) $\frac{dV}{dr} = 108\pi - 3\pi r^2$ ✓

$\frac{dV}{dr} = 0$ St.pt [1 mark finding $\frac{dV}{dr}$ and equating to 0 ($\frac{dV}{dr} = 0$)]

$108\pi - 3\pi r^2 = 0$

$3\pi r^2 = 108\pi$

$r^2 = \frac{108\pi}{3\pi}$

$r^2 = 36$ ✓

$r = 6$ cm

(note + as^{ll} radius)

Check if max: (1mark for testing to see if max).

$\frac{d^2V}{dr^2} = 6\pi r$

$r = 6$

$\frac{d^2V}{dr^2} = 36\pi > 0$

$\therefore$ maximum ✓

Maximum Volume

$V = 108 \times \pi \times 6 - \pi \times 6^3$ ✓

$= 432\pi$ ...3

b) $A(-3,-1)$ $B(7,3)$ $P(x,y)$

(i) $AP \perp BP$

$\therefore$ gradient AP $\times$ gradient PB $= -1$

$m_{AP} = \frac{y - -1}{x - -3}$ $m_{BP} = \frac{y-3}{x-7}$ ✓

$= \frac{y+1}{x+3}$

$m_{AP} \times m_{BP} = -1$ [2marks for finding locus using gradients or pythagoras theorem]

$\frac{y+1}{x+3} \times \frac{y-3}{x-7} = -1$

$\frac{y^2 - 3y + y - 3}{x^2 - 7x + 3x - 21} = -1$ [-1mark if an error]

$y^2 - 2y - 3 = -(x^2 - 4x - 21)$

$y^2 - 2y - 3 = -x^2 + 4x + 21$ ✓

$x^2 - 4x + y^2 - 2y - 24 = 0$

(ii) Complete the square

$x^2 - 4x + 4 + y^2 - 2y + 1 = 24 + 4 + 1$

$(x-2)^2 + (y-1)^2 = 29$ ✓ [1mark if made an error completing square]

Circle centre $(2,1)$ radius $\sqrt{29}$ ✓

[2 marks for completing square answer]

c) $f(x) = x^3 - x^2 - 8x - 3$

$f'(x) = 3x^2 - 2x - 8$

$f'(x) = 0$ $\boxed{\text{St pts}}$ ✓

$3x^2 - 2x - 8 = 0$

$(3x+4)(x-2) = 0$

$x = -\frac{4}{3}$, $x = 2$.

$f(-\frac{4}{3}) = -\frac{64}{27} - \frac{16}{9} + \frac{32}{3} - 3$

$= 3\frac{14}{27}$

$f(2) = 8 - 4 - 16 - 3$

$= -15$

Stationary points are:

$(-1\frac{1}{3}, 3\frac{14}{27})$ $(2,-15)$

[-1mark if error finding y coord]

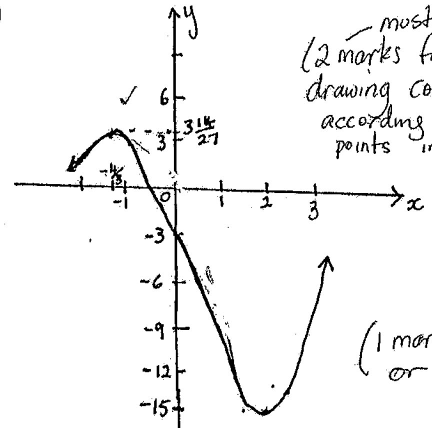
$f''(x) = 6x - 2$

$f''(-\frac{4}{3}) = -3\frac{1}{3} < 0$

$\therefore$ maximum turning point at $(-\frac{4}{3}, 3\frac{14}{27})$.

$f''(2) = 10 > 0 \therefore$ minimum turning point at $(2,-15)$

ii)



must show y-intercept (2 marks for drawing correctly according to points in(i))

(1 mark if did not show y-intercept or if stat-pt marked incorrectly)

y-intercept $x = 0$

$f(0) = -3$ $(0,-3)$.

iii) decreasing (-ve gradient)

$-\frac{4}{3} < x < 2$. ✓

1mark for correct answer.

Question 5

a. $P = 2000$, $r = 8\%$, $M = 200$

i) 1 mark correct answer.

$F_1 = 2000 \times \left(1 + \frac{8}{100}\right) - 200$

$= 2000 \times 1.08 - 200$

ii) 1 mark for deriving F_n

$F_2 = F_1 \times 1.08 - 200$

$= 2000 \times 1.08^2 - 200 \times 1.08 - 200$

$= 2000 \times 1.08^2 - 200(1 + 1.08)$

$F_3 = F_2 \times 1.08 - 200$

$= 2000 \times 1.08^3 - 200(1 + 1.08 + 1.08^2)$

$\vdots$

$F_n = 2000 \times 1.08^n - 200(1 + 1.08 + \dots + 1.08^{n-1})$

$= 2000 \times 1.08^n - 200\left(\frac{1(1.08^n - 1)}{1.08 - 1}\right)$

$= 2000 \times 1.08^n - \frac{200(1.08^n - 1)}{0.08}$

$= 2000 \times 1.08^n - \frac{200}{0.08}\underbrace{(1.08^n - 1)}_{\text{expand}}$

$= 2000 \times 1.08^n - 2500 \times 1.08^n + 2500$

$= (2000 - 2500)1.08^n + 2500$

$F_n = 2500 - 500 \times 1.08^n$ using sum of G.P

1 mark expanding and simplifying to get F_n

iii) Extinct means $F_n = 0$ ← No Birds

$2500 - 500 \times 1.08^n = 0$

1 mark for equating F_n to zero.

$2500 = 500 \times 1.08^n$

$1.08^n = \frac{2500}{500}$

$1.08^n = 5$

$\log 1.08^n = \log 5$

$n \log 1.08 = \log 5$

$n = \frac{\log 5}{\log 1.08}$

$= 20.9$

$n = 21$

After 21 autumns.

1 mark for solving & getting correct answer.

b. \$50 $r = 1\%$

i. $A_1 = 50 \times \left(1 + \frac{1}{100}\right)$

$= 50 \times 1.01$

$= \$50.50$

1 mark for correct answer.

ii) $n = 10$ year

$= 10 \times 12$ months

$= 120$ months

First investment at end of 120 months

50×1.01^{120}

2nd investment at end of 120 months (it is only there for 119 months.)

50×1.01^{119}

3rd investment there for 118 months.

50×1.01^{118}

$\vdots$

last \$50 invested for 1 month.

50×1.01

$\therefore A_n = 50 \times 1.01 + 50 \times 1.01^2 + \dots + 50 \times 1.01^{120}$

$A_n = 50(1.01 + 1.01^2 + \dots + 1.01^{120})$

$= 50\left(\frac{1.01(1.01^{120} - 1)}{1.01 - 1}\right)$

$= \frac{50 \times 1.01(1.01^{120} - 1)}{0.01}$

$= 5050(1.01^{120} - 1)$

1 mark for sum of geometric series.

1 mark for multiplying and simplifying to answer.

iii) 1 mark finding correct answer

$5050 \times (1.01^{120} - 1) =$

$\$11\,616.95$

c.

$P = \$P \quad r = 6\%$ p.a

$= \frac{6}{12} = 0.5\%$ p month

$M = 4000$

$n = 3$ years

$= 36$ months.

i) $A_1 = P \times \left(1 + \frac{0.5}{100}\right) - 4000$

$A_1 = P \times 1.005 - 4000$

ii) $A_2 = P \times 1.005^2 - 4000 \times 1.005 - 4000$

$A_3 = P \times 1.005^3 - 4000 \times 1.005^2 - 4000 \times 1.005 - 4000$

$= P \times 1.005^3 - 4000(1 + 1.005 + 1.005^2)$

sum of G.P $\frac{1(1.005^3 - 1)}{0.005}$

iii) $r = \frac{9}{12}\%$ p. month

$= 0.75\%$

next 33 months.

$M = 4800$

$A_4 = A_3 \times 1.0075 - 4800$

$A_5 = A_3 \times 1.0075^2 - 4800 \times 1.0075 - 4800$

⋮

$A_{36} = A_3 \times 1.0075^{33} - 4800 \times (1.0075^{32} + \cdots + 1.0075^2 + 1.0075)$

$= A_3 \times 1.0075^{33} - 4800\left(\frac{1.0075^{33} - 1}{0.0075}\right)$

$A_{36} = 0$ loan is repaid

$A_3 \times 1.0075^{33} - 4800\left(\frac{1.0075^{33} - 1}{0.0075}\right) = 0$

$\therefore A_3 = 139\,858.1858$

But $A_3 = P \times 1.005^3 - 4000\left(\frac{1.005^3 - 1}{0.005}\right)$ from ii)

$\therefore P \times 1.005^3 - 4000\left(\frac{1.005^3 - 1}{0.005}\right) = 139\,858.1858$

$P = \$149\,662.11$

c. Marking Criteria

i 2 marks for correct answer

1 mark if use 6% p.a instead of p month.

ii) Finds A_2 with working 1 mark

Finds A_3 from A_2.

iii) Finds A_4 correctly 1 mark.

Finds A_{36} and equates to zero 1 mark.

Finds Answer 1 mark.

(if found $A_{33} = 0$ and then answer \$138 492.70 1 mark.)

(If equated A_{last} to zero and found answer correctly 1 mark)