BRIGIDINE COLLEGE RANDWICK

MATHEMATICS
EXTENSION 2

YEAR 12

HALF-YEARLY
2002

(TIME - 2 HOUR)

DIRECTIONS TO CANDIDATES

- *Put your **name** at the top of this paper and on **each of the 5 sections** to be collected.*
- *All **5 questions** may be attempted.*
- *All **5 questions** are to be answered on **separate pages** and will be collected in **separate bundles** at the end of this exam.*
- *All questions are of equal value.*
- *All necessary working should be shown in every question.*
- *Full marks may not be awarded for careless or badly arranged work.*

QUESTION 1 *(start a new page)*

a. Let $f(x) = \dfrac{(x - 2)(x + 1)}{5 - x}$ for $x \neq 5$.

0 = (x-2)(x+1)

i. Show that $f(x) = -x - 4 + \dfrac{18}{5 - x}$. *1 m*

0 -5x²

ii. Explain the behaviour of the graph as $x \to \pm\infty$. *1 m*

iii. Show that the graph of $y = f(x)$ has two stationary points.
(There is no need to find the y coordinates of these stationary values.) *2 m*

iv. Sketch the graph of $y = f(x)$.
Label all asymptotes and show the x intercepts. *2 m*

b. The function $f(x)$ is given by $f(x) = \dfrac{4(2x - 7)}{(x - 3)(x + 1)}$

i. By expressing $f(x)$ into partial fractions, show that there are turning points at $x = 2$ and $x = 5$. *3 m*

ii. Sketch the graph of $f(x)$ showing clearly: *3 m*

- the co-ordinates of any points of intersection with the x-axis and y-axis,
- the co-ordinates of any turning points,
- the equations of any asymptotes.

(there is no need to investigate points of inflection)

iii. Determine the area of the region bounded by this curve $f(x)$, the x-axis and the lines $x = 4$ and $x = 6$, expressing your answer as a single logarithm. *3 m*

QUESTION 2 *(start a new page)*

a. Evaluate $\displaystyle\int_0^{\pi/4} \frac{e^{\tan x}}{\cos^2 x}\,dx$ *2 m*

b. Evaluate $\displaystyle\int_0^{2} x\,e^x\,dx$ *2 m*

c. By considering the substitution $u = \sqrt{x}$, *3 m*

Evaluate $\displaystyle\int_4^{12} \frac{1}{(4 + x)\sqrt{x}}\,dx.$

d. Evaluate $\displaystyle\int_{-1}^{10} \frac{2x + 1}{x^2 + 2x + 2}\,dx$ *3 m*

e. i. Show that if $I_n = \displaystyle\int \tan^n \phi\, d\phi$ *3 m*

then $I_n = \dfrac{1}{n - 1}\tan^{n-1}\phi - I_{n-2}$

ii. Hence, use this to evaluate $\displaystyle\int_0^{\pi/4} \tan^4 \phi\, d\phi$ *2 m*

QUESTION 3 *(start a new page)*

a. Find $(1 - i)^3 (2 + 2i)^4$ in the form $x + yi$. *3 m*

b. Show that the expression $\dfrac{1 + 2i}{3 - 4i} + \dfrac{2 - i}{5i}$ is purely Real. *2 m*

c. Find the cube roots of $8 \text{ cis } (\pi/2)$. *3 m*

d. If $\left| \dfrac{z + 2}{z + 8} \right| = \dfrac{1}{2}$, show that $|z| = 4$. *3 m*

e. Express as complex equations the following loci *4 m*

i. The perpendicular bisector of AB, given that A and B are the points A(-1,2) and B (3,1).

ii. The Region outside the circle of centre (0,0) and radius 3 units.

iii. The circle $(x + 1)^2 + (y + 2)^2 = 4$.

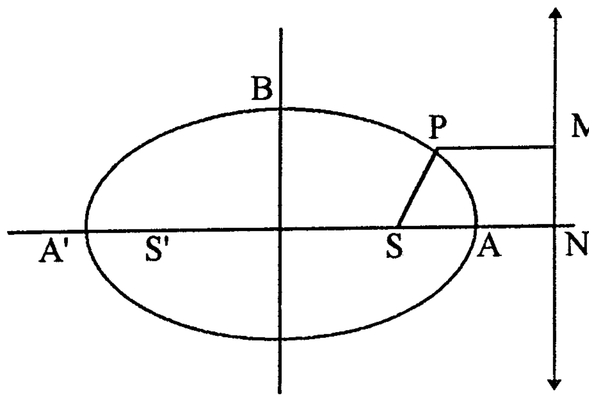
QUESTION 4 *(start a new page)*

a. i. Sketch the curve given by $|z - 2| + |z + 2| = 6$. *4 m*

ii. Express this curve in cartesian form.

5 m

c. The ellipse to the right has major axis 2a, minor axis 2b, eccentricity e and foci at S and S '.



By considering the definition that

$SP = e\,PM$

Show that

i. The equation of the directrix is $x = \frac{a}{e}$.

ii. The Focus at S has coordinates (ae,0).

iii. $b^2 = a^2(1 - e^2)$

6 m

d. i. Show that the Point $P\left(t, \frac{1}{t}\right)$ lies on the rectangular hyperbola $xy = 1$.

ii. Show that the tangent at P has equation $y = -x/t^2 + 2/t$.

iii. Show that the perpendicular from the origin to this tangent has equation $y = t^2 x$.

iv. Show that the foot of this perpendicular on the tangent has co-ordinates $\left(\frac{2t}{1 + t^4}, \frac{2t^3}{1 + t^4}\right)$.

QUESTION 5 *(start a new page)*

a. i. Show that the normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the point P $(5 \cos \phi, 3 \sin \phi)$ has equation $5x \sin \phi - 3y \cos \phi = 16 \sin \phi \cos \phi$.

3 m

ii. This normal cuts the major and minor axis of the ellipse at G and H respectively. Show that as P moves on the ellipse the midpoint GH describes another ellipse with the same eccentricity as the first. *4 m*

iii. On the same axes, sketch the two ellipses showing clearly the co-ordinates of the intercepts. *3 m*

b. i. Sketch the curves $y = \sin^2 x$ and $y = \cos 2x - 1$. *2 m*

ii. Determine the area enclosed by these two curves and the lines $x = \pi / 3$ and $x = \pi / 2$. *3 m*

- end of exam -

~~Ext~~ Ext 1 HY 02

Q1

a/ As a division

i) $-x+5 \overline{\,) x^2 - x - 2}$

or

$$\frac{(5-x)(-x-4)+18}{5-x}$$ 1

ii) $x \to \infty \mid y \to -x-4$ (below)

$x \to -\infty \mid y \to -x-4$ (above) 1

iii) $y = -x - 4 + 18(5-x)^{-1}$

$$\frac{dy}{dx} = -1 + 18(5-x)^{-2}$$

SV $\quad 0 = -1 + \dfrac{18}{(5-x)^2}$

$(5-x)^2 = 18$

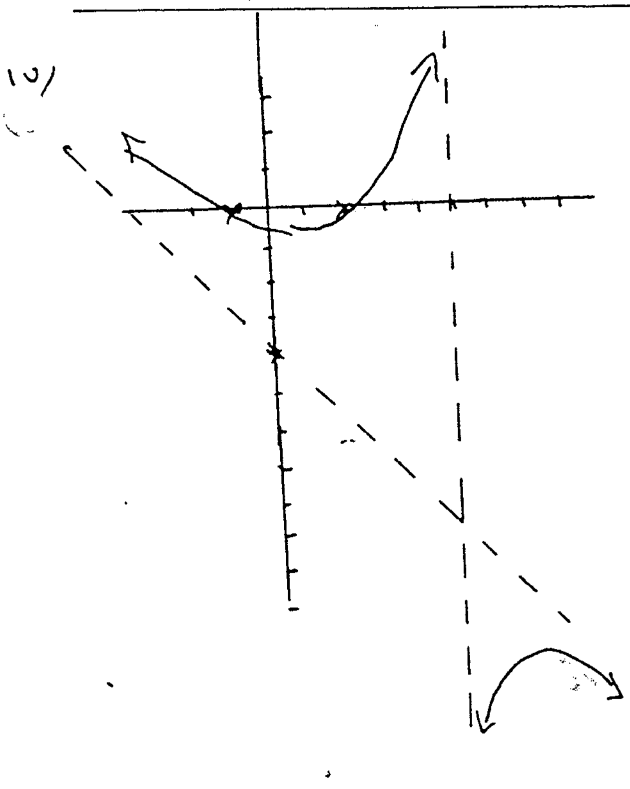
$5 - x = \pm 3\sqrt{2}$

$5 \pm 3\sqrt{2} = x$

$$\frac{d^2y}{dx^2} = -36(5-x)^{-3}$$ 2

test

iv)



2

b/ $f(x) = \dfrac{4(2x-7)}{(x-3)(x+1)}$

i) $4(2x-7) \equiv a(x+1) + b(x-3)$

$x = -1 \Rightarrow b = 9$

$x = 3 \Rightarrow a = -1$

$$f(x) = \frac{-1}{(x-3)} + \frac{9}{(x+1)}$$

$$f'(x) = \frac{1}{(x-3)^2} - \frac{9}{(x+1)^2}$$

$$= \frac{(x+1)^2 - 9(x-3)^2}{(x-3)^2(x+1)^2}$$ 3

$= x^2 + 2x + 1$
$\;\; -9x^2 + \cancel{54x} - 81 \quad$ /CD

$= -8x^2 + \cancel{56}x - 80 \quad$ /CD

$= -8[x^2 - \cancel{7}x + 10) \quad$ /CD

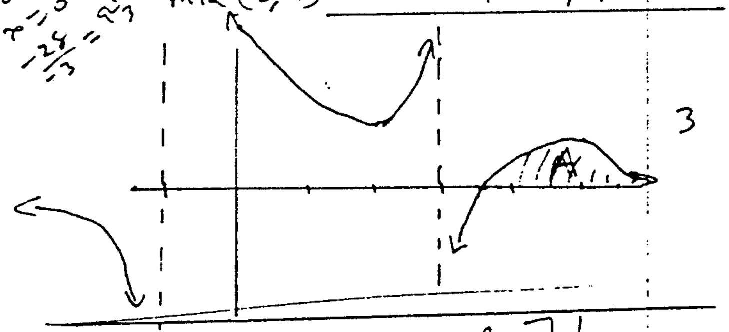
$= -8(x-2)(x-5) \quad$ /CD

SU. $0 \Rightarrow x = 2 \mid x = 5$

x	←	2	→	←	5	→
$f'(x)$	+	0	−	−	0	+

min (2, 4) $\qquad$ max (5, 1)

obs $x = 0$ $-\dfrac{28}{-3} = 9\frac{1}{3}$



3

$$A = \int_4^6 \left[\frac{-1}{x-3} + \frac{9}{x+1}\right] dx$$

$$= -\ln(x-3) + 9\ln(x+1)\Big|_4^6$$ 3

$= [-\ln 3 + 9\ln 7]$
$\quad - [-\ln 1 + 9\ln 5]$

$= \ln \dfrac{7^9}{3 \cdot 5^9}$ sq unit

Ext 2 ½ Yr 200?

Q_2

a) $\displaystyle\int_0^{\pi/4} \frac{e^{\tan x}}{\cos^2 x}\,dx$

$= \int e^{\tan x} \sec^2 x \, dx$

$= e^{\tan x} \Big|_0^{\frac{\pi}{4}}$

$= e^1 - e^0 = e - 1$

b) $\displaystyle\int_0^2 x e^x \, dx$

$u = x \qquad \frac{dv}{dx} = e^x$

$du = dx \qquad v = e^x$

$= x e^x - \int e^x \, dx$

$= x e^x - \left[e^x\right] \Big|_0^2$

$= (2e^2 - e^2) + (e^0)$

$= 2e^2 - e^2 + 1 = e^2 + 1$

c) (2) $\displaystyle\int_4^{12} \frac{dx}{(4+x)\sqrt{x}}$ (2)

$u = x^{1/2}$

$du = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}\,dx$

$\displaystyle\int_2^{2\sqrt{3}} \frac{2\,du}{4+u^2}$

$= \tan^{-1} \frac{x}{2} \Big|_2^{2\sqrt{3}}$

$= \tan^{-1}\sqrt{3} - \tan^{-1}\frac{2}{2}$

$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

d) $\displaystyle\int_{-1}^{10} \frac{2x+1}{x^2+2x+2}\,dx$

$\frac{2x+1}{x^2+2x+2} = \int \frac{2x+2}{x^2+2x+2} - \frac{1}{x^2+2x+2}$

$= \ln(x^2+2x+2) - \int \frac{1}{1+(x+1)^2}$

$\tan^{-1}(x+1)$

2

$= \ln 122 - \tan^{-1} 11 - \ln 1$
$+ \tan^{-1} 1$

$= \ln 122 - \tan^{-1} 11 + \frac{\pi}{4}$

e) $I_n = \int \tan^n \phi \, d\phi$

i) $= \int \tan^{n-2} \tan^2 \phi \, d\phi$

$= \int \tan^{n-2} (\sec^2\theta - 1)\, d\phi$

$= \int \tan^{n-2} \sec^2\phi \, d\phi - \int \tan^{n-2} \phi \, d\phi$

2

$I_n = \frac{1}{n-1} \tan^{n-1} - I_{n-2}$

ii) $\displaystyle\int_0^{\pi/4} \tan^4 \phi \, d\phi$

$= \frac{1}{3}\tan^3 \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \tan^2 \phi \, d\phi$

$= \frac{1}{3} - \left[\tan\frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \tan^0 \phi \, d\theta\right]$

$= \frac{1}{3} - 1 + \int_0^{\frac{\pi}{4}} d\theta$

$= \frac{1}{3} - 1 + \theta \Big|_0^{\frac{\pi}{4}}$

3

$= -\frac{2}{3} + \frac{\pi}{4}$

Ext 2 ½ Yr 2001

Q3

a/ $(1-i)^3 (2+2i)^4$

$= \left(\sqrt{2} \operatorname{cis} \frac{-\pi}{4}\right)^3 \left(2\sqrt{2} \operatorname{cis} \frac{\pi}{4}\right)^4$

$= (\sqrt{2})^3 (2\sqrt{2})^4 \operatorname{cis}\left(\frac{-3\pi}{4} + \frac{4\pi}{4}\right)$

$= 2\sqrt{2}\ 16(4) \operatorname{cis}\left(\frac{\pi}{4}\right)$

$= 128\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ 3

$= 128 + i\,128$

b/ $\frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} + \frac{2-i}{5i} \cdot \frac{i}{i}$

$\frac{3+10i-8}{9+16} + \frac{2i+1}{-5}$

$\frac{10i-5}{25} + \frac{-10i-5}{25}$

$= -\frac{10}{25} = -\frac{2}{5}$ ∴ purely Real 2

c/ $z^3 = 8 \operatorname{cis} \frac{\pi}{2}$

$(R\operatorname{cis}\theta)^3 = 8 \operatorname{cis} \frac{\pi}{2}$

$R^3 = 8$ $\quad \operatorname{cis} 3\theta = \operatorname{cis} \frac{\pi}{2}$

$R = 2$ $\quad 3\theta = \frac{\pi}{2} \pm 2n\pi$

$\theta = \frac{\pi}{6} \pm \frac{2n\pi}{3}$

$= \frac{\pi \pm 4n\pi}{6}$

$z_1 = 2\operatorname{cis}\left(\frac{\pi}{6}\right)$

$z_2 = \frac{\pi}{6} + \frac{4\pi}{6} = 2\operatorname{cis}\left(\frac{5\pi}{6}\right)$ 3

$z_3 = \frac{\pi}{6} + \frac{-4\pi}{6} = 2\operatorname{cis}\left(\frac{-\pi}{2}\right)$

d/ $\left|\frac{z+2}{z+8}\right| = \frac{1}{2}$

$2|z+2| = |z+8|$

$2|(x+2)+iy| = |(x+8)+iy|$

$\Rightarrow 4x^2 + 16x + 16 + 4y^2$

$= x^2 + 16x + 64 + y^2$

$3x^2 + 3y^2 = 48$

$x^2 + y^2 = 16$

$|z| = \sqrt{x^2+y^2}$

$= \sqrt{16}$

$\therefore |z| = 4$

e/ i/ $|z+1-2i| = |z-3-i|$

ii/ $|z| > 3$ 4

(ii) $|z+1+2i| = 2$

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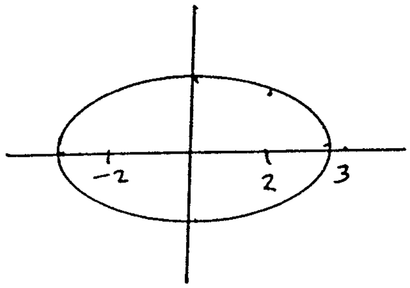
Q4/

a i/



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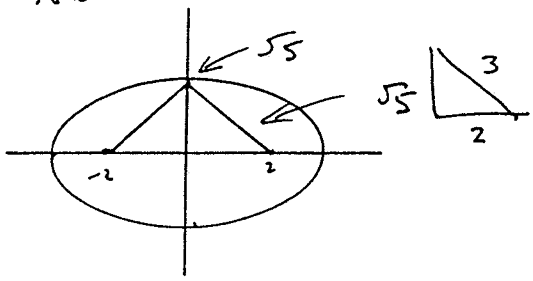


4

ii $\frac{x^2}{9} + \frac{y^2}{5} = 1$

b/ book work

5

c/ i/ $xy = 1$

$t \cdot \frac{1}{t} = 1$

1

ii/ $x = t \qquad y = t^{-1}$

$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = -t^{-2}$

$\frac{dy}{dx} = \frac{-t^{-2}}{1} = -\frac{1}{t^2}$

$y - \frac{1}{t} = -\frac{1}{t^2}(x - t)$

$y = \frac{-x}{t^2} + \frac{2}{t}$

2

iii/ $\perp \therefore m = t^2$

& $(0,0)$

$y - 0 = t^2(x - 0)$

$y = t^2 x$

iv/ $t^2 x = \frac{-x}{t^2} + \frac{2}{t}$

$t^4 x = -x + 2t$

$x(t^4 + 1) = 2t$

$x = \frac{2t}{t^4 + 1}$

$y = t^2 \left[\frac{2t}{t^4 + 1}\right]$

$y = \frac{2t^3}{t^4 + 1}$

4U Trial 2002

Q5 (a) i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

(Book work)

$\therefore 5x\sin\phi - 3y\cos\phi = 16\sin\phi\cos\phi$ 3

ii) G: $y = 0$

$\therefore \left(\frac{16}{5}\cos\phi, 0\right)$

H: $x = 0$

$\left(0, -\frac{16}{3}\sin\phi\right)$

midpoint M

$M = \left(\frac{8}{5}\cos\phi, -\frac{8}{3}\sin\phi\right)$

M on ellipse

$\frac{x^2}{\left(\frac{8}{5}\right)^2} + \frac{y^2}{\left(\frac{8}{3}\right)^2} = 1$

$\frac{x^2}{25} + \frac{y^2}{9} = 1$

$e^2 = (25-9)/25$

$= \frac{16}{25}$

$e = 4/5$ 4

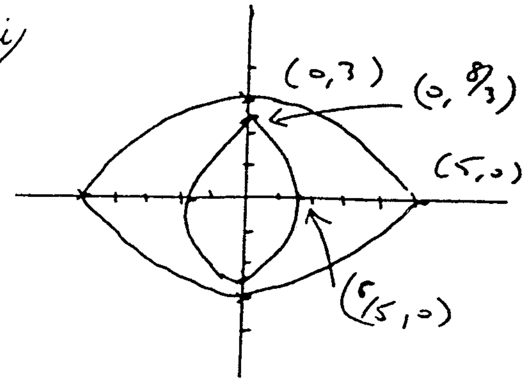
point M ellipse

$e^2 = \left[\left(\frac{8}{3}\right)^2 - \left(\frac{8}{5}\right)^2\right] / \left(\frac{8}{3}\right)^2$

$= \frac{16}{25}$

$e = 4/5$

iii)



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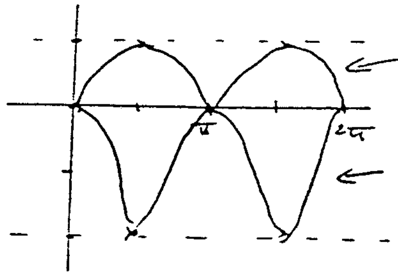
b) i)



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$A = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\sin^2 x - (\cos 2x - 1)\right] dx$

$= \int \left[\frac{1}{2}(1 - \cos 2x) - \cos 2x + 1\right] dx$

$= \int \left[\frac{3}{2} - \frac{3}{2}\cos 2x\right] dx$

$= \frac{3}{2}x - \frac{3}{4}\sin 2x \Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \left[\frac{3}{2}\frac{\pi}{2} - \frac{3}{4}(0)\right] - \left[\frac{3}{2}\frac{\pi}{3} - \frac{3}{4}\frac{\sqrt{3}}{2}\right]$

$= \frac{3\pi}{4} - \frac{\pi}{2} + \frac{3\sqrt{3}}{8}$ 3

$= \left(\frac{\pi}{4} - \frac{3\sqrt{3}}{8}\right)$ sq units