BRIGIDINE COLLEGE RANDWICK

Year 12 Mathematics **Student** ______________________

14 June 2006 **Teacher** ______________________
Time 45 minutes

Show all necessary working.
Neatness may be taken into consideration in the awarding of marks.

1. Determine the following

a. $\int \frac{1}{e^{2x}}\,dx$ **(1)**

b. $\int \frac{3}{x+1}\,dx$ **(1)**

c. $\int \text{Sin } 2x \; dx$ **(2)**

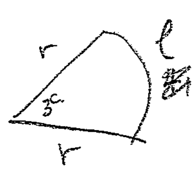
2. Evaluate $\int_0^{\frac{\pi}{4}} 2\,\text{Sec}^2x\,dx$ **(2)**

3. Find $\frac{d}{dx}\ln\sqrt{x}$ **(2)**

4. State the domain of the function $y = \ln(5\text{-}2x)$. **(2)**

5. The perimeter of a sector is 40 cm. If the angle at the centre is 3 radians, find the radius of the circle. **(3)**

$\frac{5\pi}{6} \times \frac{180}{\pi} = 150^\circ$



6. Prove that the area under the curve $y = e^x$ from $x = 0$ to $x = \ln 3$ is equal to the area under the curve $y = \frac{4}{x}$ from $x = 1$ to $x = e^{0.5}$ (Clearly show your working) **(4)**

7. Find $\frac{d}{dx}\ln(x^4 + 1)$, hence or otherwise evaluate $\int_0^{\sqrt{2}} \frac{2x^3}{x^4+1}dx$ (correct to 3 sig figs). **(3)**

8. a. Sketch the graph of $y = 3\text{Cos } 2x$ over the domain $0 \leq x \leq 2\pi$ **indicating important features of the curve.** **(3)**

b. Find the area bounded by the curve and the axes from $x = 0$ to $x = \frac{\pi}{3}$ **(4)**

(leave your answer in fully simplified exact form).

9. a. If $f(x) = e^{-\frac{1}{2}x^2}$ find $f'(x)$ and show that $f''(x) = (x^2 - 1)e^{-\frac{1}{2}x^2}$ **(2)**

b. Show that there is only one stationary point at (0,1) and determine its nature. **(2)**

c. Determine the coordinates of the point(s) of inflection to the curve. **(2)**

d. Show that $y = f(x)$ is an even function. **(1)**

e. Use the above information to sketch the curve, showing clearly what happens as $x \to \pm\infty$ **(2)**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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YEAR 12 MATHEMATICS ASSESSMENT TASK – 14 June, 2006

MARKING SCHEME

QUESTION 1: Omission of "C" not penalised

(a) Correct answer $-\frac{1}{2}e^{-2x} + C$: 1 mark

(b) $3\ln(x+1) + C$ or $3\log(x+1) + C$: 1 mark

(c)

$-\frac{1}{2}\cos 2x + C$:	2 marks
$\frac{1}{2}\cos 2x + C$:	1 mark
$-\cos 2x + C$:	1 mark
$-2\cos 2x + C$:	1 mark
$\cos 2x + C$:	0 mark

QUESTION 2

Correct definite integral, $\left[2\tan x\right]_0^{\frac{\pi}{4}}$: 1 mark

2 or correct evaluation of definite integral: 1 mark

QUESTION 3

$\ln\sqrt{x} = \ln\left(x^{\frac{1}{2}}\right)$ or $\frac{1}{2}\ln x$: 1 mark

Correct answer $\frac{1}{2x}$ or appropriate answer from incorrect working: 1 mark

Note: Mathematically incorrect statements were penalised.

QUESTION 4

$5 - 2x > 0$: 1 mark

Correct answer, $x < \frac{5}{2}$, or appropriate answer following incorrect working: 1 mark

QUESTION 5

Perimeter	$= 2r + r\theta$	1 mark
	$= 5r$:	1 mark
r	$= 8$cm; $13\frac{1}{3}$	1 mark

QUESTION 6

$$A_1 = \int_0^{\ln 3} e^x\,dx$$

Correct integration 1mark

Correct evaluation 1 mark

$$A_2 = \int_1^{e^{0.5}} \frac{4}{x}\,dx$$

Correct integration 1mark

Correct evaluation 1 mark

QUESTION 7

Correct differentiation of $\ln(x^4 + 1)$ 1 mark

Correct integration of $\int_0^{\sqrt{2}} \frac{2x^3}{x^4 + 1}\,dx$ 1 mark

Evaluation of definite integral 1 mark

QUESTION 8

(a) Graph of $y = 3\cos 2x$

Correct shape – cosine curve; includes correct x-intercepts, appropriate scales on axes.	1 mark
Correct amplitude	1 mark
Correct period	1 mark

(b)

Correct statement of area	1 mark
Correct integration	1 mark
Correct exact values	1 mark
Correct simplification	1 mark

QUESTION 9

(a) Correct $f'(x)$ 1 mark

Correct $f''(x)$ 1 mark

(b) Stationary point at (0, 1) 1 mark

Note: it is not sufficient to substitute $x = 0$ into $f'(x)$. This only demonstrates that a stationary point exists for $x = 0$, not that there is only one stationary point. To show only one stationary point exists it is necessary to solve the equation

$$-xe^{-\frac{1}{2}x^2} = 0$$

Nature of stationary point 1 mark

(c) Coordinate(s) of point(s) of inflection 1 mark

Showing concavity changes 1 mark

(d) Showing $f(-x) = f(x)$ 1mark

(e) Sketching the curve

Information to be included on graph:

- Maximum stationary point at (0, 1) – (b)
- Points of inflection at $(-1, e^{-\frac{1}{2}})$ and $(1, e^{-\frac{1}{2}})$ – (c)
- The graph is symmetrical about the y-axis – (d)
- The x-axis is an asymptote (above)

Any two of the of the above	1 mark
All four of the above	2 marks

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YEAR 12 MATHEMATICS ASSESSMENT TASK – 14 June, 2006

SOLUTIONS AND ANSWERS

QUESTION 1

(a)

$$\int \frac{1}{e^{2x}}dx = \int e^{-2x}dx$$
$$= -\tfrac{1}{2}e^{-2x} + C$$

(b)

$$\int \frac{3}{x+1}dx = 3\ln(x+1) + C$$

(c)

$$\int \sin 2x\, dx = -\tfrac{1}{2}\cos 2x + C$$

QUESTION 2

$$\int_0^{\frac{\pi}{4}} 2\sec^2 x\, dx = [2\tan x]_0^{\frac{\pi}{4}}$$
$$= 2\left[\tan\frac{\pi}{4} - \tan 0\right]$$
$$= 2$$

QUESTION 3

$$\frac{d}{dx}(\ln\sqrt{x}) = \frac{d}{dx}\ln\left(x^{\frac{1}{2}}\right)$$
$$= \frac{d}{dx}\left(\tfrac{1}{2}\ln x\right)$$
$$= \frac{1}{2x}$$

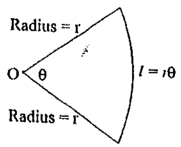
QUESTION 4: *State the domain of $y = \ln(5-2x)$*

The function $y = \ln(5-2x)$ is only defined for

$$5 - 2x > 0$$
$$-2x > -5$$
$$x < \frac{5}{2} \text{ or } x < 2\frac{1}{2} \text{ etc}$$

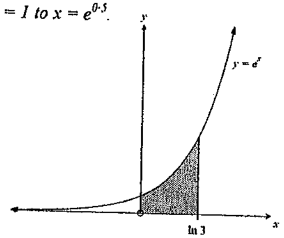
QUESTION 5

Perimeter	=	$2r + r\theta$
	=	$2r + r\times 3$
	=	$5r$
Thus $5r$	=	40cm
r	=	8cm

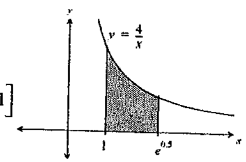


QUESTION 6: *Prove that the area under the curve $y = e^x$ from $x = 0$ to $x = \ln 3$ is equal to the area under the curve $y = \frac{4}{x}$ from $x = 1$ to $x = e^{0.5}$.*

$$A_1 = \int_0^{\ln 3} e^x dx$$
$$= \left[e^x\right]_0^{\ln 3}$$
$$= e^{\ln 3} - e^0$$
$$= 3-1$$
$$= 2$$



$$A_2 = \int_1^{e^{0.5}} \frac{4}{x}dx$$
$$= 4[\ln x]_1^{e^{0.5}}$$
$$= 4\left[\ln e^{0.5} - \ln 1\right]$$
$$= 4[0.5 - 0]$$
$$= 2$$
$$= A_1$$



QUESTION 7: *Find $\frac{d}{dx}\ln(x^4+1)$, hence evaluate $\int_0^{\sqrt{2}} \frac{2x^3}{x^4+1}dx$ (correct to 3 sig figs.*

$$\frac{d}{dx}\ln\left(x^4+1\right) = \frac{4x^3}{x^4+1}$$

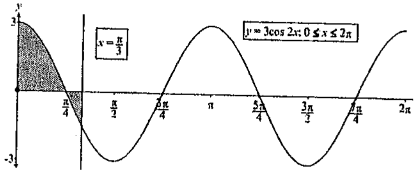
Now

$$\frac{2x^3}{x^4+1} = \tfrac{1}{2}\times\frac{4x^3}{x^4+1}$$

$$\therefore \int_0^{\sqrt{2}} \frac{2x^3}{x^4+1}dx = \tfrac{1}{2}\int_0^{\sqrt{2}}\frac{4x^3}{x^4+1}dx$$
$$= \tfrac{1}{2}\left[\ln\left(x^4+1\right)\right]_0^{\sqrt{2}}$$
$$= \tfrac{1}{2}\left[\ln\left(\sqrt{2}^{\,4}+1\right) - \ln\left(0^4+1\right)\right]$$
$$= \tfrac{1}{2}[\ln 5 - \ln 1]$$
$$= \tfrac{1}{2}[\ln 5 - 0]$$
$$= \tfrac{1}{2}\ln 5$$
$$= 0.805 \text{(3 sig. figs)}$$

QUESTION 8

(a) *Sketch the curve $y = 3\cos 2x$ over the domain $0 \le x \le 2\pi$ **indicating the important features of the curve***



(b) *Find the area bounded by the curve and the axes from $x = 0$ to $x = \frac{\pi}{3}$ **(leave your answer in fully simplified exact form).***

$$\text{Area} = \int_0^{\frac{\pi}{4}} 3\cos 2x\,dx + \left|\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 3\cos 2x\,dx\right|$$
$$= \frac{3}{2}[\sin 2x]_0^{\frac{\pi}{4}} + \left|\frac{3}{2}[\sin 2x]_{\frac{\pi}{4}}^{\frac{\pi}{3}}\right|$$
$$= \frac{3}{2}\left[\sin\frac{\pi}{2} - \sin 0 + \left|\sin\frac{2\pi}{3} - \sin\frac{\pi}{2}\right|\right]$$
$$= \frac{3}{2}\left[1 - 0 + \left|\frac{\sqrt{3}}{2} - 1\right|\right]$$
$$= \frac{3}{2}\left[1 - \left(\frac{\sqrt{3}}{2} - 1\right)\right]$$
$$= \frac{3}{2}\left[2 - \frac{\sqrt{3}}{2}\right]$$
$$= \frac{3}{2}\left[\frac{4}{2} - \frac{\sqrt{3}}{2}\right]$$
$$= \frac{3}{4}\left[4 - \sqrt{3}\right] \text{ unit}^2$$

QUESTION 9

(a) *If $f(x) = e^{-\frac{1}{2}x^2}$, find $f'(x)$ and show that $f''(x) = (x^2-1)e^{-\frac{1}{2}x^2}$*

$$f(x) = e^{-\frac{1}{2}x^2}$$
$$f'(x) = -xe^{-\frac{1}{2}x^2}$$
$$f''(x) = (-x)\frac{de^{-\frac{1}{2}x^2}}{dx} + e^{-\frac{1}{2}x^2}\frac{d(-x)}{dx}$$
$$= (-x)(-xe^{-\frac{1}{2}x^2}) - e^{-\frac{1}{2}x^2}$$
$$= x^2e^{-\frac{1}{2}x^2} - e^{-\frac{1}{2}x^2}$$
$$= e^{-\frac{1}{2}x^2}(x^2-1)$$

(b) *Show that there is only one stationary point at (0, 1) and determine its nature.*

For a stationary point $f'(x) = 0$.

i.e. $-xe^{-\frac{1}{2}x^2} = 0$

As $e^{-\frac{1}{2}x^2} > 0$ for all x

then $x = 0$

Now $f(0) = e^0$
$= 1$

Thus there is only one stationary point with coordinates (0, 1).

For $x = 0^-$ (i.e. slightly before 0)

$$f'(x) = -(0^-)e^{-\frac{1}{2}(0^-)^2}$$
$$\to (-)(-)(+) \to (+)$$

i.e. immediately before (0, 1) $f'(x)$ is positive.

For $x = 0^+$ (i.e. slightly after 0)

$$f'(x) = -(0^+)e^{-\frac{1}{2}(0^+)^2}$$
$$\to (-)(+)(+) \to (-)$$

i.e. immediately after (0, 1) $f'(x)$ is negative.

∴ (0, 1) is a maximum turning point.

(c) *Determine the coordinates of the point(s) of inflection to the curve.*

For a point of inflection $f''(x) = 0$ and the concavity changes.

$$e^{-\frac{1}{2}x^2}(x^2-1) = 0$$
$$(x^2-1) = 0$$
$$x = \pm 1$$

$x = 1$: $f''(1^-) = e^{-\frac{1}{2}(1^-)^2}((1^-)^2-1) < 0$
$\to (+)(-)$ (1^- is a fraction)

$f''(1^+) = e^{-\frac{1}{2}(1^+)^2}((1^+)^2-1) < 0$
$\to (+)(+)$ (1^+ is greater than 1)

Thus, as the concavity changes, there is a point of inflection at $x = 1$.

$x = -1$: $f''(-1^-) = e^{-\frac{1}{2}(1^-)^2}((-1^-)^2-1) > 0$

$f''(-1^+) = e^{-\frac{1}{2}(1^+)^2}((-1^+)^2-1) < 0$

Thus, as the concavity changes, there is a point of inflection at $x = -1$.

The points of inflection are $(-1, e^{-\frac{1}{2}})$ and $(1, e^{-\frac{1}{2}})$

(d) *Show that $y = f(x)$ is an even function.*

For an even function $f(\text{-a}) = f(\text{a})$

$$f(-a) = e^{-\frac{1}{2}(-a)^2}$$
$$= e^{-\frac{1}{2}a^2}$$
$$f(a) = e^{-\frac{1}{2}(a)^2}$$
$$= e^{-\frac{1}{2}a^2}$$
$$= f(-a)$$

(e) *Use the above information to sketch the curve, showing clearly what happens as $x \to \pm\infty$*

As $x \to -\infty$, $e^{-\frac{1}{2}x^2} \to 0$, and as $x \to \infty$, $e^{-\frac{1}{2}x^2} \to 0$.
Thus the x-axis is an asymptote to the curve.

Information to be included on graph:

- Maximum stationary point at (0, 1) – (b)
- Points of inflection at $(-1, e^{-\frac{1}{2}})$ and $(1, e^{-\frac{1}{2}})$ – (c)
- The graph is symmetrical about the y-axis – (d)
- The x-axis is an asymptote (above)