Student ___

BRIGIDINE COLLEGE RANDWICK

Extension 1 Mathematics

HSC

- 2010 -

HALF-YEARLY

TimeAllowed : 2 hours

DIRECTIONS TO CANDIDATES

- *Put your* ***name*** *at the top of this paper and on* ***each of the 6 sections*** *to be collected.*
- *All* ***6 questions*** *may be attempted.*
- *All* ***6 questions*** *are to be answered on* ***separate pages*** *and will be collected in* ***separate bundles*** *at the end of this exam.*
- *All questions are of equal value.*
- *All necessary working should be shown in every question IN PEN.*
- *Full marks may not be awarded for careless or badly arranged work.*

Question 1

a. Find the coordinates of the point that divides H(-3,4) and K(9,-6) externally in the ratio of 3:5. **2**

b. Prove that $\text{Tan}\left(\frac{\pi}{4}+x\right)=\frac{\text{Cos } x+\text{Sin } x}{\text{Cos } x-\text{Sin } x}$ **2**

c. Solve the inequality $\frac{2}{x} > x-1$. **4**

d. Use the Principle of Mathematical Induction to prove that

$5^n > 3^n + 2^n$ for all integers $n > 1$ **4**

Question 2 (Start a new page)

a. Show that $\lim_{x\to 0} \frac{\text{Tan } x}{3x} = \frac{1}{3}$ **2**

b. Find the exact value of $\text{Sin } 105^0$. **2**

c. i. If $\text{Tan } \frac{\theta}{2}=t$ show that $\text{Cos } \theta=\frac{1-t^2}{1+t^2}$ **2**

ii. Hence solve $5\text{Sin}\theta+12\text{Cos}\theta=5$ for $0^0 \le \theta \le 360^0$ **3**

d. Express Cosx - 2Sinx in the form $A\text{Cos}(x+\alpha)$ and hence or otherwise , solve $\text{Cos}x-2\text{Sin}x=\sqrt{5}$ for $0^0 \le x \le 360^0$ to the nearest minute. **3**

Question 3 (Start a New Page)

a. Find the obtuse angle between the lines $2x - 3y = 0$ and $4x + y - 2 = 0$ to the nearest degree. **3**

b. Find the Cartesian equation of the curve whose parametric equations are $x = \sin t$, $y = \cos 2t$. **2**

c. $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$. The tangent at P meets the x axis at T. The normal at P meets the y axis at N.

i. Find the coordinate of M, the midpoint of TN. **3**

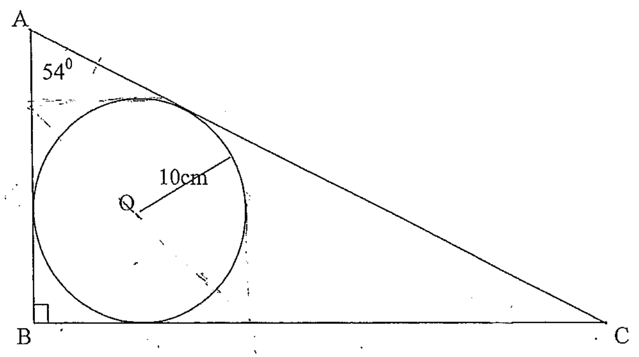
ii. Show that the locus of M is the parabola $x^2 = \frac{ay - a^2}{2}$ **2**

iii. Find the vertex and the equation of the directrix of the locus of M. **2**

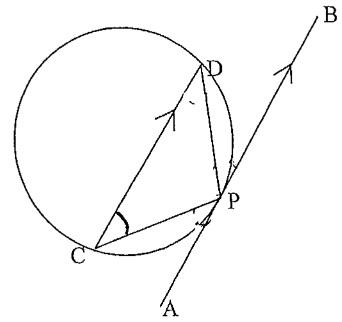
Question 4 (Start a new page)

a. Find the radius of the circle that circumscribes a triangle (verticies touch the circumference) whose sides measure 9, 40 and 41? **2**

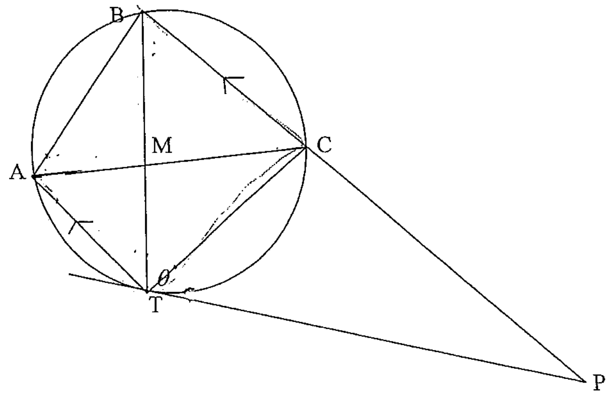
b. ABC is a right triangle. $\angle ACB = 36$ degrees, $\angle ABC = 90$ degrees. A circle centre O radius 10cm is placed inside the triangle so that it touches all 3 sides. Find the length of BC. **3**



c. A tangent AB is drawn at P. AB is parallel to chord CD. Prove that triangle PCD is isosceles. **2**



d. The diagram below shows points A,B,C and T lie on a circle. PT is a tangent to the circle at T. Line AT is parallel to the line BP and the point C lies on BP. The lines AC and BT intersect at M.



i. Copy the diagram onto your answer page.

ii. By letting $\angle BTP = \theta$, prove that ΔPTB is similar to ΔBAT **2**

iii. Show that AB = CT **3**

Question 5 (Start a New Page)

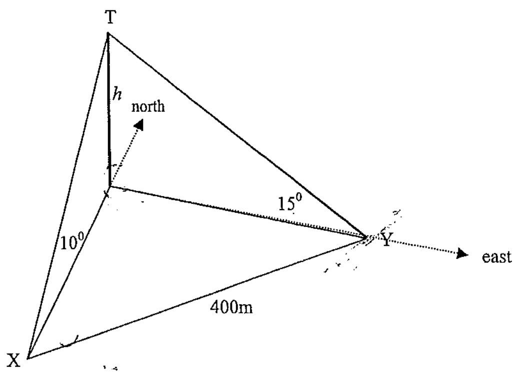
a. Use the substitution $u = x + 1$ to evaluate $\int_0^3 \frac{x-2}{\sqrt{x+1}}\,dx$. **4**

b. At any point on the curve $y = f(x)$, the gradient function is given by $\frac{dy}{dx} = \text{Cos}^2 x$.
Find the exact value of $f(\frac{3\pi}{4}) - f(\frac{\pi}{4})$. **4**

c. Evaluate $\int_1^{49} \frac{1}{4(x+\sqrt{x})}\,dx$ using the substitution $u^2 = x$, where $u > 0$. Give your answer in simplest exact form. **4**

Question 6 (Start a New Page)

a. A surveyor at X observes a tower due north.
The angle of elevation to the top of the tower is 10^0.
He then walks 400m to a position Y which is due east of the tower.
The angle of elevation from Y to the top of the tower is 15^0.



i) Write an expression for OY in terms of h. **1**

ii) Show that $h = \frac{400}{\sqrt{\text{Cot}^2 15 + \text{Cot}^2 10}}$ hence evaluate h. **3**

iii) Find the bearing of Y from X. **2**

b. Consider the curve $y = \frac{x-4}{x^2}$

i. Show that there is one turning point at $(8, \frac{1}{16})$ and determine its nature. **2**

ii. Investigate what happens to the curve as $x \to \pm\infty$. **1**

iii. Sketch the curve **showing** the turning point, any asymptotes, where/if it cuts the axes and any other necessary features. **1**

iv. On a separate number plane sketch $y = \left|\frac{x-4}{x^2}\right|$ **2**

END OF EXAM

yr12 EXT1 2010 ½ yrly

a) 3 : −5

(−3, 4) (9, −6)

$$\left(\frac{-5\times-3+3\times9}{3+-5}, \frac{-5\times4+3\times-6}{3+-5}\right)$$

$$(-21, 19)$$

b) $\tan\left(\frac{\pi}{4}+x\right)$

$$= \frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\times\tan x}$$

$$= \frac{1+\tan x}{1-\tan x}$$

$$= \frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}}$$

$$= \frac{\frac{\cos x+\sin x}{\cos x}}{\frac{\cos x-\sin x}{\cos x}}$$

$$= \frac{\cos x+\sin x}{\cos x-\sin x}$$

c) $x^2 \times \frac{2}{x} > x-1 \times x^2$ $\boxed{x\neq 0}$

$2x > x^3 - x^2$

$x^3 - x^2 - 2x < 0$

$x(x^2 - x - 2) < 0$

$x(x-2)(x+1) < 0$

1a) 2marks : $(-21, 19)$

1mark: $\left(1\frac{1}{2}, \frac{1}{4}\right)$

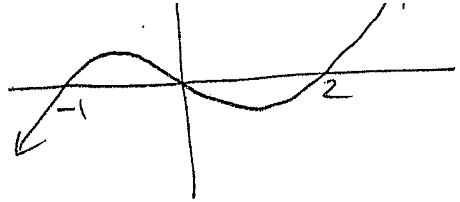
1b) 2 marks: correctly showing RHS with complete steps.

(1mark) $\frac{1+\tan x}{1-\tan x}$

c) (4marks): $x<-1$, $0<x<2$

(3marks): $x\leq-1$, $0\leq x\leq 2$

2marks: $x(x-2)(x+1)<0$



$x<-1$, $0<x<2$

d) $5^n > 3^n + 2^n$

show true for $n=2$

$5^2 > 3^2 + 2^2$

$25 > 9+4$ ∴ true for $n=2$

assume true for $n=k$

$5^k > 3^k + 2^k$

ie $5^k - 3^k - 2^k > 0$

show true for $n=k+1$

$5^{k+1} - 3^{k+1} - 2^{k+1} > 0$

$5\times5^k - 3\times3^k - 2\times2^k > 0$

$5\underbrace{[5^k-3^k-2^k]}_{>0 \text{ from above}} + \underbrace{2\times3^k}_{>0 \text{ for all } k} + \underbrace{3\times2^k}_{>0 \text{ for all } k} > 0$

∴ True for $n=k+1$

∴ since true for $n=2$, then true for $n=3, 4, \ldots$ etc.

Hence true for all $n>1$ (integers).

1mark: $x^3-x^2-2x<0$

d) (4marks) correct working (see solutions)

(3marks) showing all lines correctly but no conclusion or incorrect conclusion.

(2marks) showing $n=2$, $n=k$ and $n=k+1$ lines. and correct conclusion.

but not doing this correctly

(1mark) showing true for $n=2$

Q2a)

$$\lim_{x\to 0} \frac{\tan x}{3x}$$

$$\lim_{x\to 0} \frac{\frac{\sin x}{\cos x}}{3x}$$

$$\lim_{x\to 0} \frac{\sin x}{3x\cos x}$$

$$\lim_{x\to 0} \frac{\sin x}{x} \times \frac{1}{3} \times \frac{1}{\cos x}$$

$$= 1 \times \frac{1}{3} \times 1$$

$$= \frac{1}{3}$$

b) $\sin 105^\circ$

$$= \sin(60+45)$$

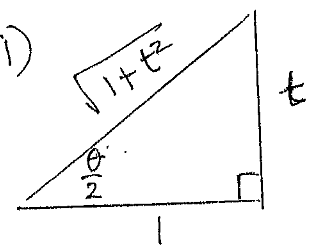
$$= \sin 60 \cos 45 + \cos 60 \sin 45$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

c) i)



$$\cos\theta = \cos\left(2 \times \frac{\theta}{2}\right)$$

$$= 2\cos^2\frac{\theta}{2} - 1$$

$$=$$

a) 2 marks: see working

1 mark: one mistake.

b) 2 marks: $\frac{\sqrt{3}+1}{2\sqrt{2}}$

1 mark: $\sin 60 \cos 45 + \cos 60 \sin 45$

c) i) 2 marks see working

1 mark: $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$$= 2 \times \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - 1$$

$$= \frac{2}{1+t^2} - \frac{1+t^2}{1+t^2}$$

$$= \frac{2-1-t^2}{1+t^2}$$

$$= \frac{1-t^2}{1+t^2}$$

ii) $5\sin\theta + 12\cos\theta = 5$

$$\sin\theta = \frac{2t}{1+t^2}$$

$$5 \times \frac{2t}{1+t^2} + 12 \times \frac{1-t^2}{1+t^2} = 5$$

$$10t + 12 - 12t^2 = 5 + 5t^2$$

$$\cancel{0 = 12t^2 - 10t}$$

$$0 = 17t^2 - 10t - 7$$

$$0 = (17t+7)(t-1)$$

$t = -\frac{7}{17}$ or $t = 1$

$\tan\frac{\theta}{2} = -\frac{7}{17}$ $\quad$ $\tan\frac{\theta}{2} = 1$

$\frac{\theta}{2} = 157^\circ 37'$ $\quad$ $\frac{\theta}{2} = 45^\circ$

$\theta = 315^\circ 14'$ $\quad$ $\theta = 90^\circ$

d) $\cos x - 2\sin x \equiv A\cos(x+\alpha)$

$$A = \sqrt{1^2+2^2} = \sqrt{5}$$

$$\tan\alpha = \frac{2}{1}$$

$$\alpha = 63^\circ 26'$$

$$\therefore \cos x - 2\sin x \equiv \sqrt{5}\cos(x + 63^\circ 26')$$

ii) 3 marks: $\theta = 315^\circ 14'$, 90

2 marks: $\tan\frac{\theta}{2} = -\frac{7}{17}$, $\tan\frac{\theta}{2} = 1$

1 mark: $0 = 17t^2 - 10t - 7$

d) 3 marks: $x = 296^\circ 34'$

2 marks: $\cos(x + 63^\circ 26') = 1$

1 mark: $A = \sqrt{5}$, $\alpha = 63^\circ 26'$

Q3c ii)

$x = \frac{ap}{2}$

$p = \frac{2x}{a}$

$y = \frac{2a + ap^2}{2}$

$\therefore y = \frac{2a + a\left(\frac{2x}{a}\right)^2}{2}$

$y = \frac{2a + \frac{4x^2}{a}}{2}$

$2y = 2a + \frac{4x^2}{a}$

$2ay = 2a^2 + 4x^2$

$ay = \cancel{2}a^2 + 2x^2$

$2x^2 = ay - \cancel{2}a^2$

$x^2 = \frac{ay - a^2}{2}$

2marks: see solution

1mark: 1mistake

iii) $x^2 = \frac{1}{2}a(y-a)$

vertex $= (0, a)$

focal length: $4a = \frac{1}{2}a$

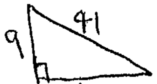
$a = \frac{1}{8}a$

$\therefore$ directrix: $y = a - \frac{1}{8}a = \frac{7}{8}a$

2marks: $V(0, a)$

directrix $= y = \frac{7}{8}a$

Q4a)

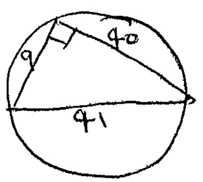


The triangle is a right∠Δ

ie $41^2 = 40^2 + 9^2$

$1681 = 1600 + 81$

$\therefore$

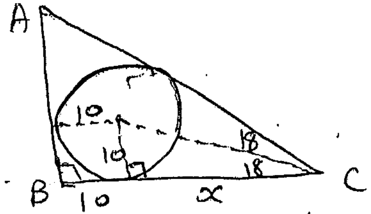


$\therefore$ radius $= \underline{20.5}$

2marks: 20.5

1mark: recognising a right ∠ Δ

b)



$\tan 18 = \frac{10}{x}$

$x = \frac{10}{\tan 18} = 30.7768$

$\therefore BC = 10 + 30.7768 = \underline{40.78}$

3marks: 40.78

2marks: $x = 30.7768$

1mark: $10 + \square$

c) let $\angle DPB = \theta$

$\angle DCP = \theta$ (∠'s in alternate segment)

$\angle CPA = \theta$ (alternate angles are =)

$\angle CDP = \theta$ (∠'s in alternate segment)

$\therefore$ PCD is isosceles since

$\angle PCD = \angle CDP$ (from above)

2marks: correct reasoning

1mark: one mistake.

d) ii) $\angle BTP = \theta = \angle BAT$ (∠'s in alt segment are =)

$\angle ATB = \angle TBP$ (alternate ∠'s are =)

$\angle ABT = \angle TPB$ (∠'s left after ∠ sum Δ = 180)

$\therefore \Delta PTB \parallel\!\mid \Delta BAT$ (equiangular)

2marks: correct reasoning

1mark: ~~one mistake~~

$\angle BTP = \angle BAT$ (∠'s in alt seg =)

or one mistake.

iii) $\angle CAT = \angle TBC$ (∠'s on circumf. standing on same arc =)

$\angle MAT = \angle MTA$

$\therefore \Delta AMT$ = isosceles (~~base ∠'s =~~)

$\therefore AM = MT$ (equal sides isos Δ)

$\angle BTA = \angle ACB$ (∠'s on circumference standing on same arc =)

$\therefore \Delta MBC$ = isosceles since $\angle MBC = \angle MCB$

$\therefore BM = CM$

(S) $BM = CM$ (proven above)

(A) $\angle BMA = \angle CMT$ (vert opp)

(S) $AM = MT$ (proven above)

$\therefore \Delta BMA \equiv \Delta CMT$ by SAS

$BA = CT$ (corresp sides of congruent Δ's)

3marks: correct reasoning

2marks: proving congruence

1mark: proving ΔAMT isosceles or ΔMBC is isosceles.

or correct conclusion from congr / attempt proof

Q2d cont

$\sqrt{5}\cos(x+63^\circ 26') = \sqrt{5}$

$\cos(x+63^\circ 26') = 1$

$x+63^\circ 26' = 0, 360^\circ$

$x = \underbrace{-63^\circ 26'}_{\text{out of domain}}, 296^\circ 34'$

So $\underline{\underline{x = 296^\circ 34'}}$

3a) $2x-3y=0 \rightarrow m = \frac{2}{3}$

$4x+y-2=0 \longrightarrow m = -4$

$\therefore \tan\alpha = \left| \dfrac{\frac{2}{3} - -4}{1 + \frac{2}{3} \times -4} \right|$

$= \left| -\frac{14}{5} \right|$

$\tan\alpha = \frac{14}{5}$

$\alpha = 70^\circ 21'$

$\alpha = 70^\circ$

$\therefore$ obtuse $\angle = 180 - 70$

$= \underline{110^\circ}$

b) $x = \sin t$ $\quad y = \cos 2t$

~~$y = 2\sin$~~

$y = 1 - 2\sin^2 t$

$y = 1 - 2x^2$

3marks: 110°

2marks: $\tan\alpha = \frac{14}{5}$

or finding correct gradients + using $\tan\alpha = \ldots$ then $180-\alpha$

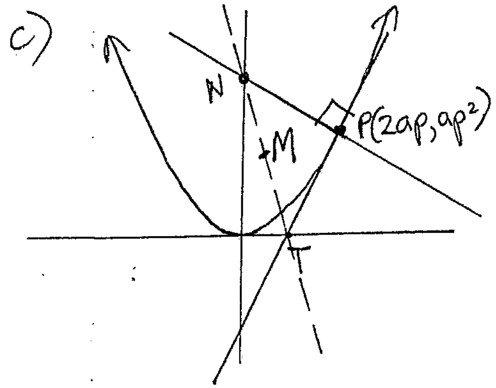
using incorrect α

1mark: finding both gradients.

2marks: $y = 1 - 2x^2$

1mark: $y = 1 - 2\sin^2 t$

c)



i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a}$

$y' = \frac{x}{2a}$

$\therefore m = \frac{2ap}{2a}$

$m = p \longrightarrow m_{normal} = -\frac{1}{p}$

$\therefore$ equ of tang

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$y - ap^2 = -\frac{x}{p} + 2a$

$y = -\frac{x}{p} + 2a + ap^2$

$\therefore$ T:

$0 = px - ap^2$

$px = ap^2$

$x = ap$

$\therefore T(ap, 0)$

N:

~~$0 = -\frac{x}{p} + 2a + ap^2$~~

~~$0 = -x + 2ap + ap^3$~~

$y = \frac{0}{p} + 2a + ap^2$

$y = 2a + ap^2$

$N(0, 2a+ap^2)$

$\therefore M = \left(\frac{ap}{2}, \frac{2a+ap^2}{2}\right)$

3marks: $M = \left(\frac{ap}{2}, \frac{2a+ap^2}{2}\right)$

2marks: $T(ap, 0)$ and $N(0, 2a+ap$

1mark:

$y = px - ap^2 \leftarrow$ Tangent or

$y = -\frac{x}{p} + 2a + ap^2 \leftarrow$ Norma

or finding T or N correctly from incorrect Tangents and Normals.

or using incorrect T and N to find midpt.

Q5

a) $\int_0^3 \frac{x-2}{\sqrt{x+1}}\,dx$

$u = x+1 \qquad x = u-1$

$\frac{du}{dx} = 1$

$du = dx$

$\int \frac{u-1-2}{\sqrt{u}}\,du$

$\int \frac{u-3}{\sqrt{u}}\,du$

$\int u^{\frac{1}{2}} - 3u^{-\frac{1}{2}}\,du$

$\frac{2u^{\frac{3}{2}}}{3} - \frac{3u^{\frac{1}{2}}}{\frac{1}{2}}$

$x=3$, $u=4$; $x=0$, $u=1$

$\left[\frac{2u^{\frac{3}{2}}}{3} - 6\sqrt{u}\right]_1^4$

$\left(\frac{16}{3} - 12\right) - \left(\frac{2}{3} - 6\right)$

$= -1\frac{1}{3}$

b) $f'(x) = y' = \cos^2 x \qquad \cos 2x = 2\cos^2 x - 1$

$\frac{\cos 2x + 1}{2} = \cos^2 x$

$\therefore \int \cos^2 x$

$f(x) = \int \frac{\cos 2x + 1}{2} = \frac{1}{2}\int \cos 2x + 1$

$= \frac{1}{2}\left[\frac{\sin 2x}{2} + x\right]$

$f(x) = \frac{1}{4}\sin 2x + \frac{x}{2}$

$f\left(\frac{3\pi}{4}\right) = \frac{1}{4}\sin\frac{3\pi}{2} + \frac{3\pi}{8}$

$= -\frac{1}{4} + \frac{3\pi}{8}$

4 marks: $-1\frac{1}{3}$

3 marks: $\left[\frac{2u^{\frac{3}{2}}}{3} - 6\sqrt{u}\right]_1^4$

2 marks: $\int_1^4 \frac{u-3}{\sqrt{u}}\,du$ or well into an integration

1 mark:

$\int \frac{u-3}{\sqrt{u}}\,du$

or limits as $\int_1^4$

4 marks: ~~$\frac{1}{4}\left[\frac{\pi}{2} - 2\right]$ or $\frac{\pi}{8} - \frac{1}{2}$~~ $\frac{\pi}{4} - \frac{1}{2}$

3 marks: $\left(\frac{3\pi}{8} - \frac{1}{4}\right)$ and $\left(\frac{1}{4} + \frac{\pi}{8}\right)$

or one mistake.

2 marks: $\frac{3\pi}{8} - \frac{1}{4}$ or $\frac{1}{4} + \frac{\pi}{8}$

$f\left(\frac{\pi}{4}\right) = \frac{1}{2}\left[\frac{\sin\frac{\pi}{2}}{2} + \frac{\pi}{4}\right]$

$= \frac{1}{2}\left[\frac{1}{2} + \frac{\pi}{4}\right] = \frac{1}{4} + \frac{\pi}{8}$

$\therefore f\left(\frac{3\pi}{4}\right) - f\left(\frac{\pi}{4}\right)$

$-\frac{1}{4} + \frac{3\pi}{8} - \frac{1}{4} - \frac{\pi}{8}$

$\underline{\underline{\frac{\pi}{4} - \frac{1}{2}}}$

c) $\int_1^{49} \frac{1}{4(x+\sqrt{x})}\,dx \qquad u^2 = x$

$u = \sqrt{x}$ since $u > 0$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

$x = 49 \quad u = 7$

$x = 1 \quad u = 1$

$\therefore \int_1^7 \frac{1}{4(u^2+u)} \times 2u\,du$

$\frac{1}{2}\int_1^7 \frac{1}{u+1}\,du$

$\frac{1}{2}\left[\ln(u+1)\right]_1^7$

$\frac{1}{2}\left[\ln 8 - \ln 2\right]$

$\frac{1}{2}\ln 4 = \underline{\underline{\ln 2}}$

1 mark:

$\frac{1}{2}\int \cos 2x + 1$

4 marks: $\ln 2$

3 marks:

$\frac{1}{2}\left[\ln 8 - \ln 2\right]$

2 marks:

$\frac{1}{2}\int \frac{1}{u+1}\,du$

or $\int_1^7 \frac{1}{4(u^2+u)} \times 2u\,du$

1 mark:

limits as $\int_1^7$

or $dx = 2u\,du$

Q6a) i) $\tan 15 = \dfrac{h}{OY}$

$OY = \dfrac{h}{\tan 15}$

ii) $OX = \dfrac{h}{\tan 10}$

$\cancel{400}\,XY^2 = OX^2 + OY^2$

$400^2 = \left(\dfrac{h}{\tan 10}\right)^2 + \left(\dfrac{h}{\tan 15}\right)^2$

$400^2 = (h\cot 10)^2 + (h\cot 15)^2$

$400^2 = h^2\cot^2 10 + h^2\cot^2 15$

$h^2(\cot^2 10 + \cot^2 15) = 400^2$

$h^2 = \dfrac{400^2}{\cot^2 10 + \cot^2 15}$

$h = \dfrac{400}{\sqrt{\cot^2 10 + \cot^2 15}}$

$h = 58\cdot 9\,m$

iii) $OX = \dfrac{58\cdot 9}{\tan 10} = 334\,m$

$\therefore \cos\theta = \dfrac{334}{400}$ (using ΔXOY)

$\theta = 33^\circ 23'$

Ans = $\underline{\underline{033^\circ}}$

1mark: $OY = \dfrac{h}{\tan 15}$ or $h\cot 15$

3marks: correctly showing h and h = 58.9

2marks: correctly showing h

1mark: h = 58.9 or using pythagoras on ΔXOY

2marks: 033° or N33°E

1mark: OX = 334 or OY = 219.9

b) i) $y = \dfrac{x-4}{x^2}$ $\quad x \neq 0$

$y' = \dfrac{1\times x^2 - 2x(x-4)}{x^4}$

$= \dfrac{x^2 - 2x^2 + 8x}{x^4}$

$y' = \dfrac{8-x}{x^3} = 0$

$x = 8$

$y = \dfrac{1}{16}$

x	←	8	→
y'	+	0	−
	/	—	\

$\therefore (8, \frac{1}{16})$ is a max turning pt.

ii) $x \to \infty \quad y \to 0^+$

$x \to -\infty \quad y \to 0^-$

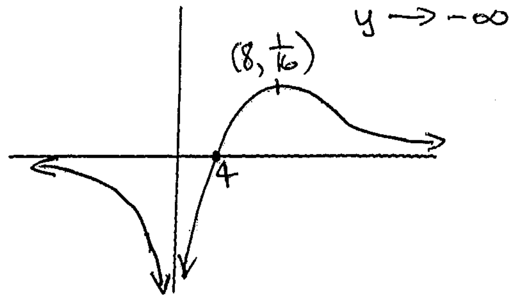
iii) cuts x-axis: $0 = \dfrac{x-4}{x^2}$

$x = 4$

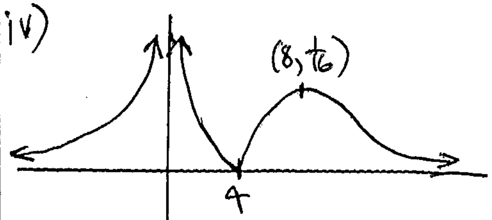
~~cuts y-axis~~ $x \neq 0$

$x \to 0^+$
$y \to -\infty$

$x \to 0^-$
$y \to -\infty$



iv)



2marks: correctly showing $(8, \frac{1}{16})$ is a max

1mark: finding $(8, \frac{1}{16})$ but no test.

1mark: see solution

1mark: see solution

2marks: see solution or correct from their sketch (NOTE sketch must have 2 parts below x-axis for 2marks)

1mark: correctly flipping vertically across x-axis