Student ______________________

BRIGIDINE COLLEGE RANDWICK

MATHEMATICS EXTENSION 2

HALF-YEARLY 2006

(Time: 2 hours + 5 minutes reading)

DIRECTIONS TO CANDIDATES

- *Put your **name** at the top of this paper and on **each of the 5 sections** to be collected.*
- *All **5 questions** may be attempted. and are to be answered on **separate pages** and will be collected in **separate bundles** at the end of this exam.*
- *All questions are of equal value.*
- *All necessary working should be shown in every question.*
- *Full marks may not be awarded for careless or badly arranged work.*

The following Outcomes may be Examined

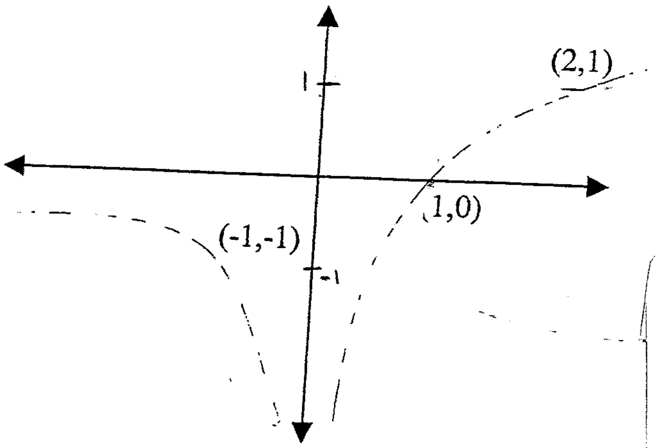
HE2	Uses inductive reasoning in the construction of proofs.
E6	Combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions.
HE6	Determines integrals by reduction to a standard from through a given substitution.
E8	Applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems.
E3	Uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.
E4	Uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials.

MATHEMATICS EXTENSION 2 HALF-YEARLY 2006

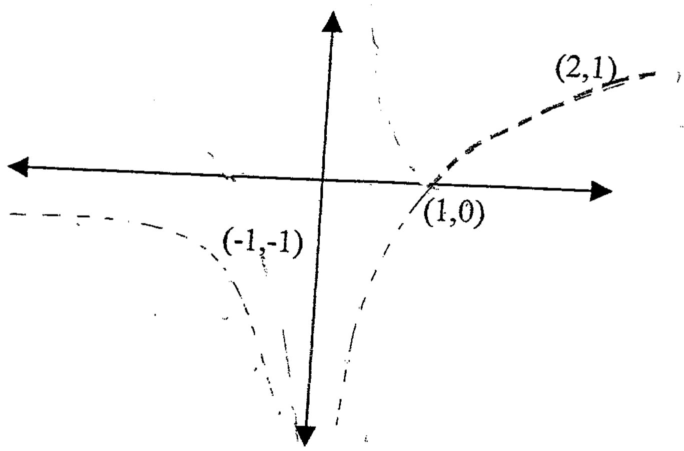
QUESTION 1 a Student ____________

Attach this page to your Answer Sheet

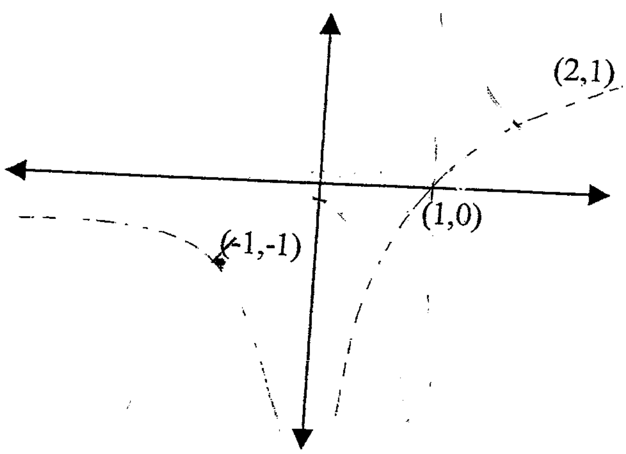
ii. $y = -\sqrt{f(x)}$



iii. $y = \left| f(|x|) \right|$

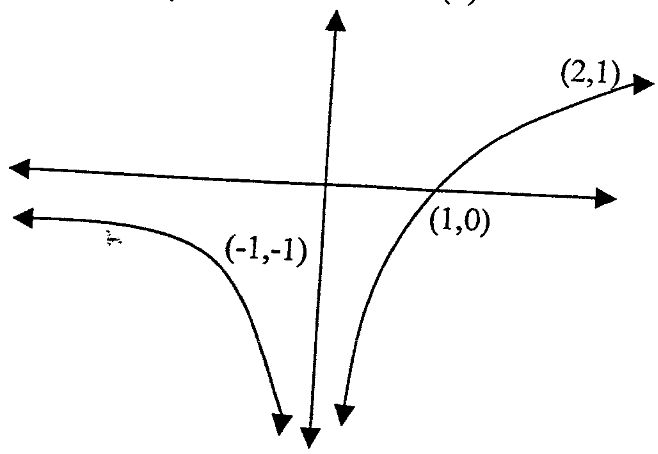


iv. $y = \dfrac{1}{f(x)}$



QUESTION 1 *(start a new page)*

a. The diagram below shows the graph of $y = f(x)$.



i. Comment on the first and second derivative of $f(x)$. *1 m*

A page has been provided (see back of Standard Integral Insert) to draw sketches of each of the following, showing all the features that assisted your sketch.

ii. $y = -\sqrt{f(x)}$ *1 m*

iii. $y = |f(|x|)|$ *2 m*

iv. $y = \dfrac{1}{f(x)}$ *2 m*

b. The function $f(x)$ is given by $f(x) = \dfrac{4(2x - 7)}{(x - 3)(x + 1)}$

i. By expressing $f(x)$ into partial fractions, show that there are turning points at $x = 2$ and $x = 5$. *4 m*

ii. Sketch the graph of $f(x)$ showing clearly *2 m*

- the co-ordinates of any points of intersection with the x-axis and y-axis,
- the co-ordinates of any turning points,
- the equations of any asymptotes.

(there is no need to investigate points of inflection)

iii. Determine the area of the region bounded by this curve $f(x)$, the x-axis and the lines $x = 4$ and $x = 6$, expressing your answer as a single logarithm. *3 m*

QUESTION 2 *(start a new page)*

a. $\int \frac{x^2}{x^6 + 9}\, dx$ (let $u = x^3$) *2 m*

b. Evaluate $\int_0^{\pi/4} \frac{e^{\tan x}}{\cos^2 x}\, dx$ *2 m*

c. Evaluate $\int_0^2 x^2 e^x \, dx$ *3 m*

d. $\int_0^2 \sqrt{4 - x^2}\, dx$ *3 m*

e. i. Show that $\int \cos^n x \, dx$ may be expressed as *3 m*

$$n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x \, dx$$

ii. Hence, or otherwise, evaluate $\int_0^{\pi/2} \cos^3 x \, dx$ *2 m*

QUESTION 3 *(start a new page)*

a. Given $z_1 = 1 - i$ and $z_2 = -1 + \sqrt{3}\,i$,

i. Determine the value of $\text{Im}(z_1)$. *1 m*

ii. Find $|z_1|$ and $|z_2|$ and write down the exact value of $|z_1 z_2|$ *2 m*

iii. Determine $\arg z_1$ and $\arg z_2$ and write down the value of $\arg z_1 z_2$ in terms of π. *2 m*

iv. By considering the product $z_1 z_2$ in the form $a + bi$, *3 m*

Show that $\cos \dfrac{5\pi}{12} = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$

b. Find the cube roots of $8 \text{ cis } (\dfrac{\pi}{2})$. *3 m*

c. Express as complex equations the following loci

i. The perpendicular bisector of AB, given that A and B are the points A(-1,2) and B (3,1). *2 m*

ii. The Region inside the circle $(x + 1)^2 + (y + 2)^2 = 4$. *2 m*

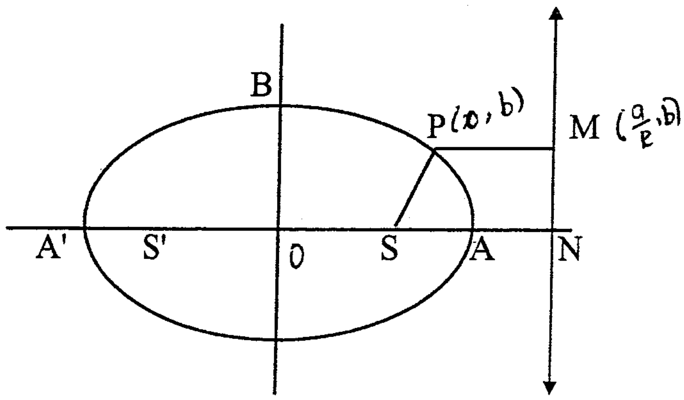
QUESTION 4 *(start a new page)*

a. Consider the ellipse given by the equation $\frac{x^2}{9} + \frac{y^2}{8} = 1$

i. Determine the length of the semi-major, semi-minor and state the eccentricity of this ellipse. *2 m*

ii. Express this ellipse in the form of a complex equation. *2 m*

b. The ellipse to the right has major axis 2a, minor axis 2b, eccentricity e and foci at S and S '.



By considering the definition that
SP = e PM

Show that

i. The equation of the directrix is $x = \frac{a}{e}$. *2 m*

ii. The Focus at S has coordinates (ae,0). *1 m*

iii. $b^2 = a^2(1 - e^2)$ *2 m*

c. i. Show that the Point P $(t, \frac{1}{t})$ lies on the rectangular hyperbola $xy = 1$. *1 m*

ii. Show that the tangent at P has equation $y = -x/t^2 + 2/t$. *1 m*

iii. Show that the perpendicular from the origin to this tangent has equation $y = t^2 x$. *1 m*

iv. Show that the foot of this perpendicular on the tangent has co-ordinates *3 m*

$$\left(\frac{2t}{1 + t^4}, \frac{2t^3}{1 + t^4}\right).$$

QUESTION 5 *(start a new page)*

a. i. Show that the normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the point P *3 m*

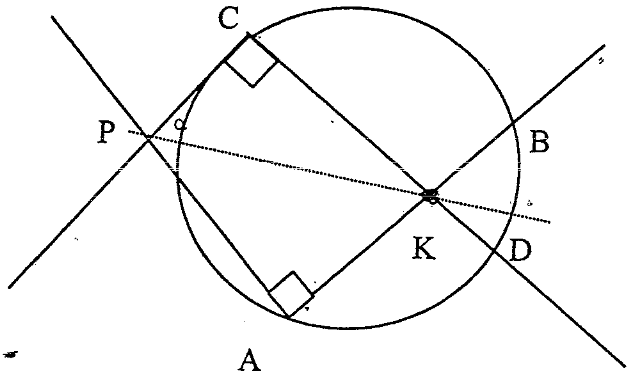
$(5 \cos \phi, 3 \sin \phi)$ has equation $5x \sin \phi - 3y \cos \phi = 16 \sin \phi \cos \phi$.

ii. This normal cuts the major and minor axis of the ellipse at G and H respectively. Show that as P moves on the ellipse the midpoint GH describes another ellipse with the same eccentricity as the first. *4 m*

iii. On the same axes, sketch the two ellipses showing clearly the co-ordinates of the intercepts. *3 m*

b. As shown below AB and CD are chords of a circle intersecting at K. *5 m*

P is a point such that $\angle DCP = \angle BAP = 90^\circ$ and $\angle CPK$ is α.



Redraw this figure on to your exam page and show that PK produced (intersecting BD at L) is perpendicular to BD.

- end of exam -

Ext 2 HY 06

Q1

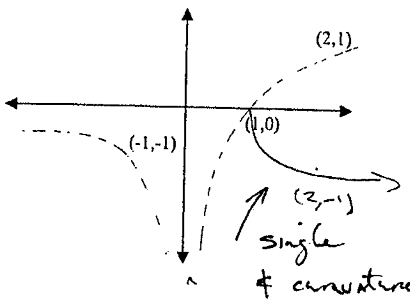
a)

i) $x<0 \quad f'(x)<0$

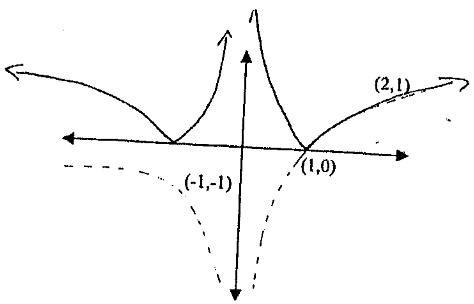
$x>0 \quad f'(x)>0$

$\forall x \quad f''(x)<0$

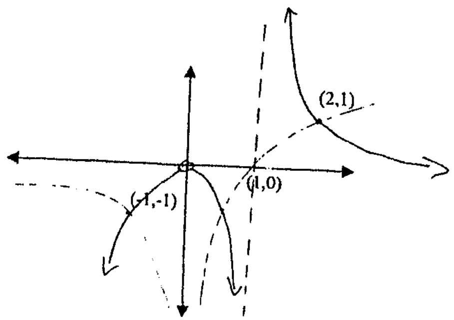
1 mark all 2/3 — 1



1



2



2

b) $f(x) = \dfrac{4(2x-7)}{(x-3)(x+1)}$

$4(2x-7) \equiv a(x+1) + b(x-3)$

$x=-1 \Rightarrow b=9$

$x=3 \Rightarrow a=-1$

$f(x) = \dfrac{-1}{x-3} + \dfrac{9}{x+1}$ ✓

$f'(x) = \dfrac{1}{(x-3)^2} - \dfrac{9}{(x+1)^2}$

$= \dfrac{(x+1)^2 - 9(x-3)^2}{(x-3)^2(x+1)^2}$

$\Rightarrow x^2+2x+1$ ✓

$-9x^2+54x-81$ /CD

S.P.

$0 = -8x^2+56x-80$ /

$0 = -8(x^2-7x+10)$

$0 = -8(x-2)(x-5)$

$x=2$ | $x=5$

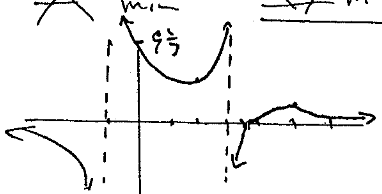
$(2,4)$ ✓ | $(5,1)$ ✓

x	←	2	→
y'	+	0	−

min

x	←	5	→
y'	−	0	+

max — 4



2

$A = \displaystyle\int_4^6 \left[\frac{-1}{x-3} + \frac{9}{x+1}\right] dx$ ✓

$= -\ln(x-3) + 9\ln(x+1) \Big|_4^6$

$= [-\ln 3 + 9\ln 7]$ ✓

$\quad - [-\ln 1 + 9\ln 5]$

$= \ln \dfrac{7^9}{3 \cdot 5^9}$ ✓ — 3

Ext 2 HY 06

Q2

a) $\left(\frac{1}{3}\right)\displaystyle\int \frac{3x^2\,dx}{(x^3)^2+9}$ $\quad u = x^3,\ du = 3x^2\,dx$

$= \frac{1}{3}\cdot\frac{1}{3}\displaystyle\int \frac{3\,du}{u^2+9}$ ✓

$= \frac{1}{9}\tan^{-1}\frac{u}{3} + c$

$= \frac{1}{9}\tan^{-1}\frac{x^3}{3} + c$ ✓ — 2

b) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{e^{\tan x}}{\cos^2 x}\,dx$

$\displaystyle\int_0^{\frac{\pi}{4}} e^{\tan x}\sec^2 x\,dx$

Let $u = \tan x$

$du = \sec^2 x\,dx$ ✓

$\displaystyle\int_0^1 e^u\,du$

$= e^u \Big|_0^1 = e - 1$ ✓ — 2

c) $\displaystyle\int_0^2 x^2 e^x\,dx$

$u = x^2 \quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 2x \quad v = e^x$ ✓

$= x^2 e^x \Big|_0^2 - \displaystyle\int_0^2 e^x\, 2x\,dx$

$= 4e^2 - 2\left[xe^x - e^x\right]\Big|_0^2$ ✓

$= 4e^2 - 2[2e^2 - e^2 - 1]$

$= 4e^2 - 4e^2 - 2e^2 + 2$

$= 2e^2 + 2$ ✓ — 3

d) $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$ ✓

$\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta}\;2\cos\theta\,d\theta$

$= \displaystyle\int_0^{\frac{\pi}{2}} 4|\cos\theta|\cos\theta\,d\theta$ ✓

$= \frac{4}{2}\displaystyle\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$

$= 2\displaystyle\int d\theta + \frac{1}{2}\int \cos 2\theta\,d\theta$

$= 2\frac{\theta}{2} + \frac{1}{2}\frac{\sin 2\theta}{2} \Big|_0^{\frac{\pi}{2}}$ ✓

$= \pi + 0 - 0 = \pi$

e) i) book work $\quad$ Appropriate

pg 132 eg 29 $\quad$ structure u_3..

ii) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$

$= \frac{1}{3}\cos^2 x \sin x \Big|_0^{\frac{\pi}{2}}$ ✓

$+ 2\displaystyle\int \cos x\,dx$

$= \left[\frac{1}{3}\left[(\cos\frac{\pi}{2})^2 \sin\frac{\pi}{2} - (\cos 0 \sin 0)\right]\right]$

$+ 2\sin x \Big|_0^{\frac{\pi}{2}}$

$= 0 + 2\sin\frac{\pi}{2} + 0$

$= 2$ ✓

Ext 2 HY 06

Q3 a)

$z_1 = 1 - i \quad z_2 = -1 + \sqrt{3}i$

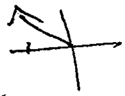
i) $\text{Im}(z_1) = -1$ — 1

ii) $|z_1| = \sqrt{2}, \quad |z_2| = 2$ ✓ either

$|z_1 z_2| = 2\sqrt{2}$ ✓ — 2

iii) $\arg z_1 = \frac{-\pi}{4}$ ok

$\arg z_2 = -\frac{\pi}{3}$ but



✓ for

$\therefore \arg z_2 = \frac{2\pi}{3}$ ✓ — 2

$\arg z_1 z_2 = \frac{-\pi}{4} + \frac{2\pi}{3} = \frac{5\pi}{12}$

iv) $z_1 z_2 = (1-i)(-1+\sqrt{3}i)$

$= (\sqrt{3}-1) + (\sqrt{3}+1)i$ ✓

Equate parts Above

parts ✓

$\sqrt{3} - 1 = 2\sqrt{2}\cos\frac{5\pi}{12}$ ✓ — 3

$\therefore \cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

b) $z^3 = 8\,\text{cis}\,\frac{\pi}{2}$

$r^3\,\text{cis}\,3\theta = 8\,\text{cis}\,\frac{\pi}{2}$

$r^3 = 8 \quad | \quad 3\theta = \frac{\pi}{2} + 2n\pi$

$\theta = \frac{\pi + \cancel{4}n\pi}{6}$ ✓

$n=0 \quad z_1 = 2\,\text{cis}\,\frac{\pi}{6}$

$n=-1 \quad z_2 = 2\,\text{cis}\,-\frac{\pi}{2}$ ✓ — 3

$n=1 \quad z_3 = 2\,\text{cis}\,\frac{5\pi}{6}$

method ✓

c) $A(-1,2)$ & $B(3,1)$

i) $\therefore |z - (-1+2i)| = |z - (3+i)|$

$|z + 1 - 2i| = |z - 3 - i|$ — 2

ii) $|z - (1+2i)| < \cancel{4}\ 2$

$|z - 1 - 2i| < \cancel{4}\ 2$ — 2

Q4

a) $\frac{x^2}{9} + \frac{y^2}{8} = 1 \qquad \therefore a = 3,\ b = 2\sqrt{2}$ ✓

i) $b^2 = a^2(1-e^2)$

$8 = 9 - 9e^2$

$9e^2 = 9 - 8$ ✓ — 2

$e^2 = \frac{1}{9} \quad e = \pm\frac{1}{3}$

ii) Since $ae = 3\left(\frac{1}{3}\right)$

$ae = \pm 1$

$a = 3$ method ✓

$|z-1| + |z+1| = 2a$ — 2

$|z-1| + |z+1| = 6$ ✓

b) i bookwork ✓✓

ii ✓

iii ✓✓ method $\cancel{2}$ — 5

page 63, 64

Q4 continued   Ext2 HY 06

c/ i) $xy = 1$

$t \cdot \frac{1}{t} = 1$ ✓   1

ii) $x = t \qquad y = t^{-1}$

$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = -t^{-2}$

$\frac{dy}{dx} = \frac{-t^{-2}}{1} = -\frac{1}{t^2}$

$y - \frac{1}{t} = -\frac{1}{t^2}(x - t)$

$y = \frac{-x}{t^2} + \frac{2}{t}$ 1

iii) $\perp \therefore m = t^2 \;\&\; (0,0)$

$y - 0 = t^2(x - 0)$

$y = t^2 x$ 1

iv) equate y's

$\left.\begin{aligned} t^2 x &= \frac{-x}{t^2} + \frac{2}{t} \\ t^4 x &= -x + 2t \end{aligned}\right]$ ✓

$x(t^4 + 1) = 2t$

$x = \frac{2t}{t^4+1}$ ✓

$y = t^2 \left[\frac{2t}{t^4+1}\right]$ 3

$y = \frac{2t^3}{t^4+1}$ ✓

Q5

a i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Book work

$5x\sin\phi - 3y\cos\phi$
$= 16\sin\phi\cos\phi$ 3

ii) G $y = 0$

$\therefore \left(\frac{16}{5}\cos\phi, 0\right)$ ✓

H $x = 0$

$\therefore \left(0, -\frac{16}{3}\sin\phi\right)$

midpt M ✓

$M \Rightarrow \left(\frac{8}{5}\cos\phi, -\frac{8}{3}\sin\phi\right)$

M on ellipse ✓

$\frac{x^2}{\left(\frac{8}{5}\right)^2} + \frac{y^2}{\left(\frac{8}{3}\right)^2} = 1$

for $\frac{x^2}{25} + \frac{y^2}{9} = 1$

$e^2 = \frac{(25-9)}{25} \Rightarrow \frac{16}{25}$

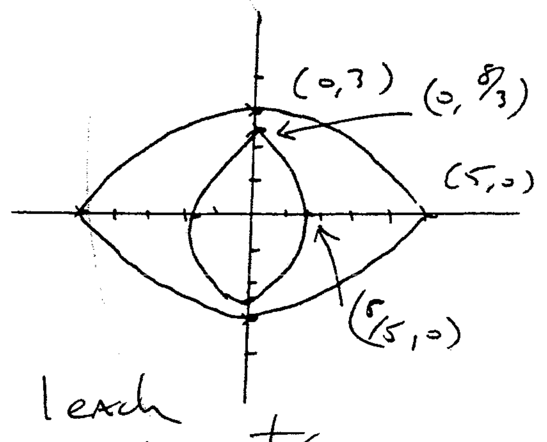
$e = 4/5$ compare ✓

for M

$e^2 = \frac{\left[\left(\frac{8}{3}\right)^2 - \left(\frac{8}{5}\right)^2\right]}{\left(\frac{8}{3}\right)^2}$ 4

$e^2 = \frac{16}{25}$

$e = \frac{4}{5}$

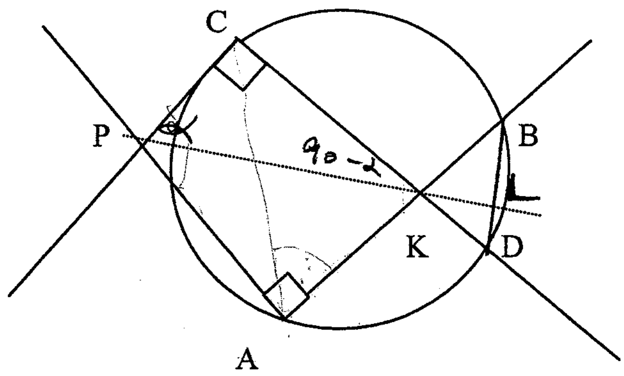


3

1 each
+ intercepts

Extn HY 06

Q5 b)



5 marks

$\angle CKP = 90 - \alpha$ ($\angle$ sum $\triangle$)

$\angle LKD = \angle CKP = 90 - \alpha$ (vert opp)

Quad PCKA

$\angle PCK + \angle KAP = 180^\circ$ ($\angle$ sum of quad)

$\therefore$ PCKA is cyclic (opp $\angle$ 180°)

$\angle CPK = \angle CAK$ ($\angle$'s in same segment of circle PCKA)

$\angle CAK = \alpha$

$\angle CDB = \angle CAB$ ($\angle$ in same segment)

$\angle KDL = (\angle CAK$ & $\angle KDL$ ie $\angle$'s $\angle CDB$ & $\angle CAB$)

$\angle KLD = 180^\circ - (\angle LKD + \angle KDL)$ $\triangle$ sum

$= 180 - (90 - \alpha + \alpha)$

$= 90^\circ$

$\therefore$ PK $\perp$ BD