BRIGIDINE COLLEGE RANDWICK

Extension 1
MATHEMATICS

YEAR 12

HALF-YEARLY
2003

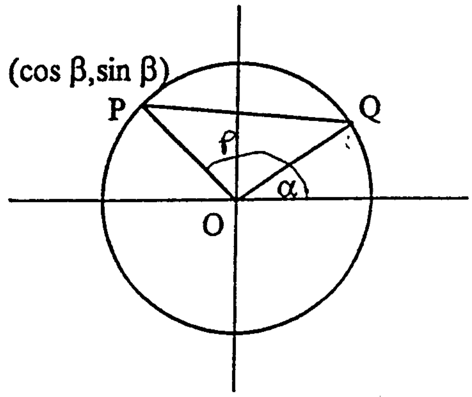
(TIME - 2 HOUR)

DIRECTIONS TO CANDIDATES

- *Put your **name** at the top of this paper and on **each of the 5 sections** to be collected.*
- *All 5 **questions** may be attempted.*
- *All 5 **questions** are to be answered on **separate pages** and will be collected in **separate bundles** at the end of this exam.*
- *All questions are of equal value.*
- *All necessary working should be shown in every question.*
- *Full marks may not be awarded for careless or badly arranged work.*

QUESTION 1 ***(start a new page)***

a. i. The figure to the right represents a unit circle with points P & Q as shown. Show that the point Q may be represented by $(\cos\alpha, \sin\alpha)$ — 5 m



ii. By application of the distance formula and cosine rule, show that

$$\cos(\beta - \alpha) = \cos\beta \cos\alpha + \sin\beta \sin\alpha$$

iii. Hence, or otherwise, determine the exact value of $\cos 15^\circ$ in exact form.

b. Prove that — 3 m

$$\tan(\pi/4 - x) = \frac{\cos x - \sin x}{\cos x + \sin x}$$

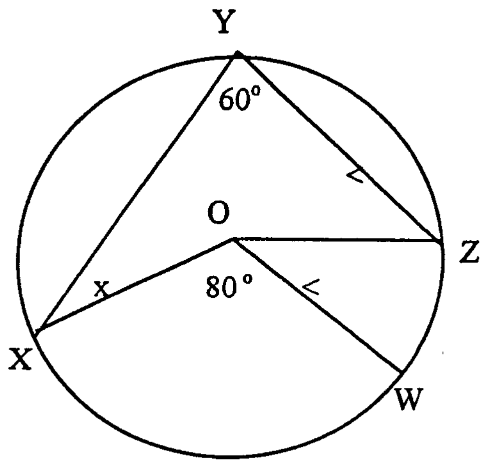
c. Solve the inequality $\frac{2x - 3}{4x - 5} \leq -2$ — 4 m

d. Sketch the circle showing chords AB and CD intersecting at X. — 3 m

Prove that $AX.XB = CX.XD$.

QUESTION 2 *(start a new page)*

a.



This figure represents a circle centre 0

(sides and angle as marked)

YZ is parallel to OW

Copy this figure onto your exam page and find the value of x, giving reasons. *3 m*

b. Find the coordinates of the point internally dividing the join of (-1,5) and (6,-4) in the ratio of 3 : 2. *2 m*

c. Consider the curve given by $y = 4x^2 + \dfrac{1}{x}$. *5 m*

i. State any asymptotes for this curve.

ii. Show that this curve has a minimum value when $x = \frac{1}{2}$.

iii. Hence, or otherwise, sketch this curve $y = 4x^2 + \dfrac{1}{x}$.

d. Let $\tan \dfrac{\theta}{2} = t$. *5 m*

i. Show that $\sin \theta = \dfrac{2t}{1 + t^2}$ and $\cos \theta = \dfrac{1 - t^2}{1 + t^2}$.

ii. Hence, or otherwise, prove $\dfrac{1 - \cos\theta}{\sin \theta} = \tan\dfrac{\theta}{2}$.

QUESTION 3 *(start a new page)*

a. Find the acute angle between the lines with equations $2y = x + 1$ and $2x - 3y = 4$. (answer to the nearest minute) *3 m*

b. Prove by the method of mathematical induction that *3 m*

$$2 + 5 + 8 + \dots + (3n - 1) = \frac{n(3n + 1)}{2}$$

c. i. Show that the cartesian equation of the parabola whose parametric equations are *6 m*

$x = 2ap$ and $y = ap^2$ is given by $x^2 = 4ay$.

ii. A point P lies on this parabola ($x^2 = 4ay$), show that the normal at this point P may be given by $x + py = ap^3 + 2ap$.

iii. If this normal meets the axis of this parabola at R, draw a neat sketch to represent this information.

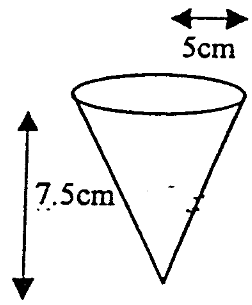
iv. Show that the locus of M, the midpoint of PR, is another parabola and use this information to write down its vertex and focal length.

d. Solve for x, in the interval $0 \leq x \leq 2\pi$, $\sin 2x = \cos x$. *3 m*

QUESTION 4 *(start a new page)*

a. Prove by the method of *mathematical induction* that $3^{2n} + 7$ has 8 as a factor. *4 m*

b. Filter paper is in the shape of an inverted cone, base radius 5 cm and altitude 7.5 cm. If water is flowing out from the bottom at a constant rate of 1.5 cm^3/s find the rate at which the level of the liquid is falling when the depth is 5 cm.



4 m

c. Find the general solution to $\sqrt{2}\sin\alpha + 1 = 0$ *2 m*

d. Find $\int 3x^2(x^3 - 1)$ by using the substitution $u = x^3 - 1$ *2 m*

e. Find $\int x\sqrt{x+1}\,dx$ by using the substitution $u = x + 1$ *3 m*

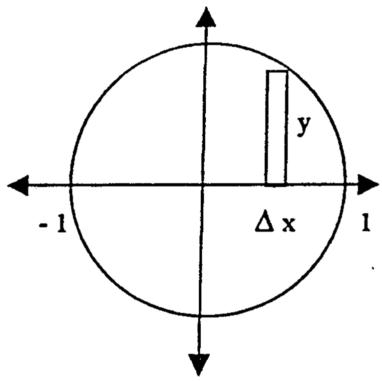
QUESTION 5 *(start a new page)*

a. i. Find $d(\tan^5 x)/dx$ *4 m*

ii. Hence, or otherwise, Evaluate $\int_0^{\pi/3} \tan^4 x \sec^2 x \, dx$

b. The circle to the right has the Cartesian equation *6 m*

$x^2 + y^2 = 1$



i. Explain why the area of this circle in the first quadrant may be given by

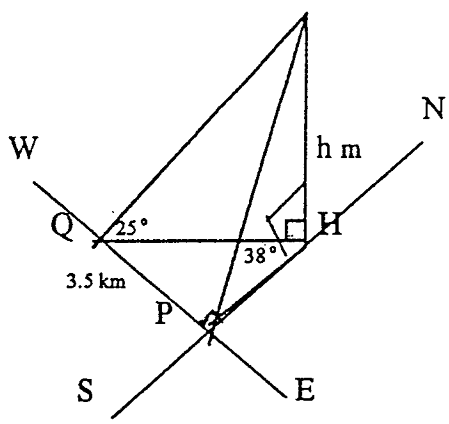
$$A \approx \lim_{\Delta x \to 0} \sum_{n=0}^{a} \sqrt{1 - x^2}\, \Delta x$$

ii. By considering the substitution $x = \sin\alpha$, show that the area of this circle is π square units.

c. The angular elevation of a hill at a place P due south of it is 38° and at a place Q due west of P the elevation is 25°. *5 m*

The distance from P to Q is 3.5 km.

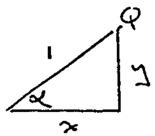
(P and Q are both at sea level.)



i. Show that $PH = \dfrac{h}{\tan 38°}$

ii. Show that $h^2 = \dfrac{(3.5\,\text{km})^2 (\tan^2 38)(\tan^2 25)}{\tan^2 38 - \tan^2 25}$

iii. Find the height "h" of the hill, to the nearest 10m.

i) 

$\cos\alpha = \frac{x}{1}$

$\sin\alpha = y$ 1

ii) book work 2

iii) $\cos(45-30)$

$= \cos 45 \cos 30 + \sin 45 \sin 30$

$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$ 2

LHS

$= \frac{\sin(\frac{\pi}{4}-x)}{\cos(\frac{\pi}{4}-x)}$

$= \frac{\sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x}{\cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x}$

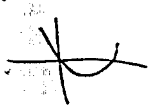
$= \frac{\frac{1}{\sqrt{2}}(\cos x - \sin x)}{\frac{1}{\sqrt{2}}(\cos x + \sin x)}$

$= \frac{\cos x - \sin x}{\cos x + \sin x} = RHS$ 3

$\frac{2x-3}{4x-5} + 2 \leq 0$

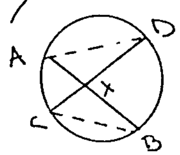
$(4x-5)^2 \left[\frac{2x-3+8x-10}{4x-5}\right] \leq 0$

$(4x-5)(10x-13) \leq 0$ 4



$\frac{5}{4} < x \leq \frac{13}{10}$

NB $x \neq \frac{5}{4}$

d) 

$\angle XAD = \angle XCB$

($\angle$'s inscribed arc)

$\angle AXD = \angle CXB$

(vertically opposite)

$\therefore \triangle XAD \parallel\!\mid \triangle XCB$

$\frac{AX}{XC} = \frac{XD}{XB}$ sides ratio 3

$\therefore AX.XB = CX.XD$

Q2

a) diagram

$\angle XOZ = \angle 120°$ insc $= \frac{1}{2}$ ctr

$\angle WOZ = 40°$

$\angle OZY = 40°$ Alt //

$x = 20°$ sum of [illegible] 3

b) formula

$\frac{3(6)+2(-1)}{3+2}, \frac{3(-4)+2(5)}{3+2}$

$= (3\frac{1}{5}, -\frac{2}{5})$ 2

c) i) $x = 0$

$x \to \infty \therefore y = 4x^2$ above

$x \to -\infty$ $y = 4x^2$ below

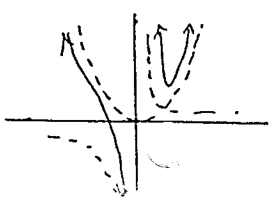
ii) $y = 4x^2 + x^{-1}$ 5

S.P. $y' = 8x - x^{-2}$

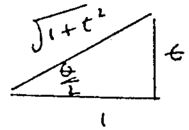
$0 = 8x^3 - 1$

$\frac{1}{8} = x^3$ | $y'' = 8 + x^{-3}$

$\frac{1}{2} = x$ | $y'' > 0$ ↺

iii 

Ext 1 4U 03

d) $\tan\frac{\theta}{2} = t$ 

* $\sin 2\left[\frac{\theta}{2}\right] = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= 2 \frac{t}{\sqrt{1+t^2}} \frac{1}{\sqrt{1+t^2}}$

$= \frac{2t}{1+t^2}$ 3

* $\cos 2\left[\frac{\theta}{2}\right] =$

$\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$= \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$

$= \frac{1-t^2}{1+t^2}$

ii) $\left[\frac{1-t^2}{1+t^2}\right] \div \left[\frac{2t}{1+t^2}\right]$ 2

$\left[\frac{1+t^2-(1-t^2)}{1+t^2}\right] \cdot \frac{1+t^2}{2t}$

$\frac{2t^2}{(1+t^2)} \cdot \frac{1+t^2}{2t} = t$

$= \tan\frac{\theta}{2}$

Q3

a) $y = \frac{1}{2}x + \frac{1}{2}$ $m_1 = \frac{1}{2}$

$3y = 2x - 4 \Rightarrow m_2 = \frac{2}{3}$

$\tan\theta = \frac{\frac{2}{3} - \frac{1}{2}}{1 + \frac{2}{3}(\frac{1}{2})}$ 3

$\tan\theta = \frac{1}{4}$ $\theta = 7°8'$ 1

b) Add both sides $3(n+1)-1$

RHS $\frac{n(3n+1)}{2} + (3n+2)$

$\frac{1}{2}[3n^2 + n + 6n + 4]$

$= \frac{1}{2}[3n^2 + 7n + 4]$

$= \frac{1}{2}(3n+4)(n+1)$

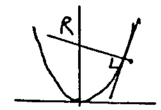
$= \frac{(n+1)[3(n+1)+1]}{2}$ 3

& statements

4 i) $y = a\left[\frac{x}{2a}\right]^2$ 1

$4ay = x^2$

ii) book work 1

iii) 

$P(2ap, ap^2)$

$R(0, ap^2+2a)$

iv) midpt

$M = \left[\frac{2ap+0}{2}, \frac{ap^2+2p^2+2a}{2}\right]$

$= (ap, ap^2+a)$

$x = ap$ | $y = ap^2 + a$

$\therefore y = a\left[\frac{x^2}{a^2}+1\right]$

$ay = x^2 + a^2$

$x^2 = a(y-a)$

Vertex $(0,a)$ Focal $= \frac{1}{4}a$

d) $2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ | $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}$ | $x = \frac{\pi}{6}$

$x = \frac{3\pi}{2}$ | $x = \frac{5\pi}{6}$

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Q4

a) Let $3^{2n}+7 = 8m$

$3^{2(n+1)} + 7$

$3^2\, 3^{2n} + 7$

$9\left[3^{2n}\right] + 7$

$9\left[8m-7\right]+7$

$9(8m) - 56$

$8\left[9m-7\right]$

$\therefore$ divisible by 8 + statement

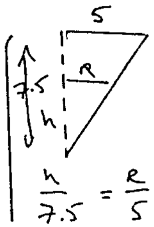
4

b) $\frac{dU}{dt} = -1.5$

$U = \frac{1}{3}\pi r^2 h$

$U = \frac{1}{3}\pi\left[\frac{2h}{3}\right]^2 h$

$U = \frac{4}{27}\pi h^3$



$\frac{h}{7.5} = \frac{R}{5}$

$\frac{2h}{3} = R$

$\frac{dU}{dh} = \frac{4}{9}\pi h^2$

4

$\frac{dh}{dt} = \frac{dh}{dU}\,\frac{dU}{dt}$ $\quad h=5$

$= \frac{9}{4\pi(25)} \cdot -\frac{3}{2}$

$= \frac{-27}{200\pi}$ cm/s²

$(\approx -.043)$

c) $\sin\alpha = \frac{-1}{\sqrt{2}}$

$\alpha = n\pi + (-1)^n\left(-\frac{\pi}{4}\right)$

2

or

d) $u = x^3 - 1$

$du = 3x^2\, dx$

$\int u\, du$

$= \frac{u^2}{2} + c$

$= \frac{1}{2}(x^3-1) + c$

2

e) $u = x+1$

$du = dx$

$u - 1 = x$

$\int (u-1)\,u^{1/2}\, du$

$\int \left[u^{3/2} - u^{1/2}\right] du$

$= \frac{2}{5}u^{5/2} - \frac{3}{2}u^{3/2} + c$

$= \frac{2}{5}(x+1)^{5/2} - \frac{3}{2}(x+1)^{3/2} + c$

3

Ext 1 HY Yr12 03

Q5

a) i) $\frac{d\,(\tan^5 x)}{dx} = 5\tan^4 x \sec^2 x$

ii) $\tan^5 x = 5\int \tan^4 x \sec^2 x\, dx$

$\int_0^{\frac{\pi}{3}} \tan^4 x \sec^2 x\, dx$

$= \frac{1}{5}\tan^5 x \Big|_0^{\frac{\pi}{3}}$

$= \frac{1}{5}\left(\tan\frac{\pi}{3}\right)^5$

$= \frac{1}{5}(\sqrt{3})^5 = \frac{9}{5}\sqrt{3}$

4

b) i) Area is built up as series of rectangles width 0

1

ii) $x^2 + y^2 = 1$

$y = \pm\sqrt{1-x^2}$

$\therefore A = 2\int_{-1}^{1} \sqrt{1-x^2}\, dx$

$A = 4\int_0^1 \sqrt{1-x^2}\, dx$

$x = \sin\alpha$ $\quad x=1$, $\alpha = \frac{\pi}{2}$

$dx = \cos\alpha\, d\alpha$ $\quad \alpha = 0$

$A = 4\int_0^{\frac{\pi}{2}} (1-\sin^2\alpha)^{1/2}\cos\alpha\, d\alpha$

$= 4\int |\cos\alpha|\cos\alpha\, d\alpha$

$= 4\int \frac{1}{2}(1+\cos 2\alpha)\, d\alpha$

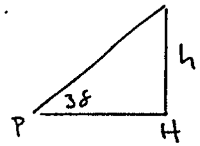
$= \frac{4}{2}\left[\alpha - \frac{\sin 2\alpha}{2}\right]\Big|_0^{\frac{\pi}{2}}$

5

$= 2\left[\left(\frac{\pi}{2} - 0\right) - (0-0)\right]$

$= \pi$ sq units

c) i)



$\tan 38 = \frac{h}{PH}$

$\therefore PH = \frac{h}{\tan 38}$

ii) NB $QH = \frac{h}{\tan 25}$

Pythagoras

$(3.5)^2 + \frac{h^2}{\tan^2 38} = \frac{h^2}{\tan^2 25}$

$(3.5)^2 \tan^2 38 \tan^2 25 + h^2\tan^2 25 = h^2 \tan^2 38$

$(3.5)^2 \tan^2 38 \tan^2 25 = h^2\left[\tan^2 38 - \tan^2 25\right]$

$h^2 = \frac{(3.5\text{ km})^2 \tan^2 38 \tan^2 25}{\tan^2 38 - \tan^2 25}$

iii) $h^2 = 4.13759\ldots$

$h = 2.034$ km