Student : Melissa Ng
Teacher : Milanov

BRIGIDINE COLLEGE RANDWICK

Ave $\frac{48}{75}$ = 64%

Year 12 Extension 1

2004 Half - Yearly Examination

Time Allowed : 2 hours

Instructions :

* There are 5 Questions worth 15 marks each.
* Start each question on a new page.
* Show all necessary working.
* Marks may not be awarded for careless or badly arranged work.
* Attempt all questions.

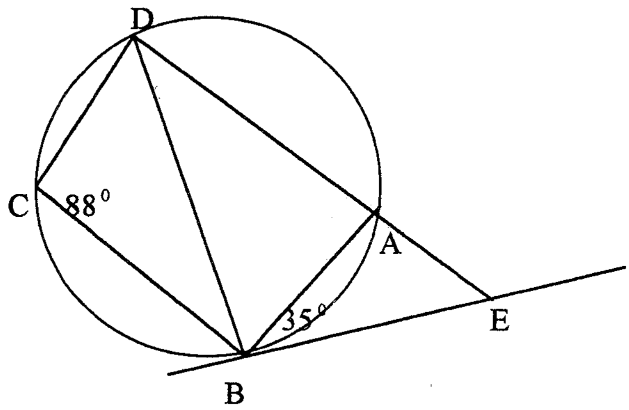
Question 1. (START A NEW PAGE)

a. Solve the following inequalities :

i. $\frac{5}{3-x} \leq 1$ (3)

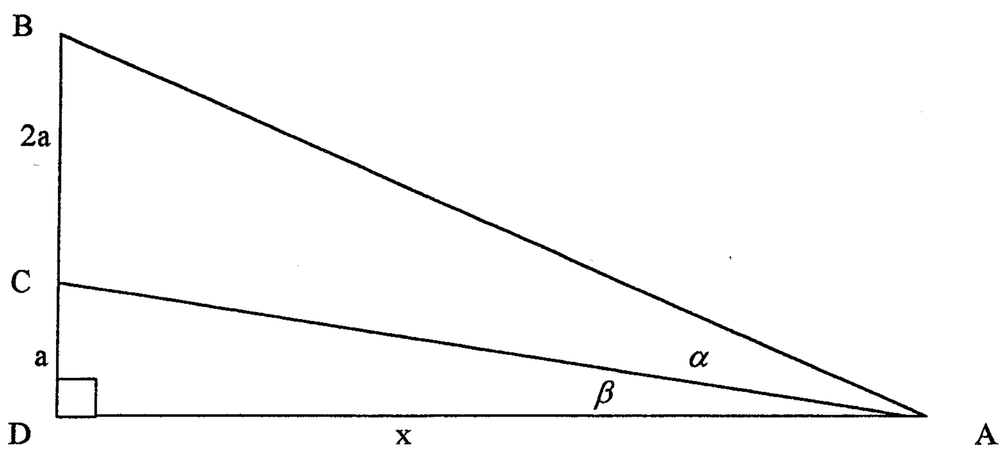
ii. $\frac{5}{|3-x|} \leq 1$ (3)

b. If $\angle BCD = 88^0$, $\angle EBA = 35^0$ find $\angle BAE$ and $\angle BDE$, giving reasons for your answer. (BE is a tangent). (2)



c. Solve for x if $0 \leq x \leq 2\pi$ for $2\,\text{Cos}\,2x < -\sqrt{3}$ (4)

d. By considering Tan $(\alpha + \beta)$, prove $\text{Tan}\,\alpha = \frac{2ax}{x^2 + 3a^2}$ if (3)



Question 2. (START A NEW PAGE)

a. Find the acute angle between the tangents formed by $y = \ln x$ at $x = 1$ and $x = 2$. (2)

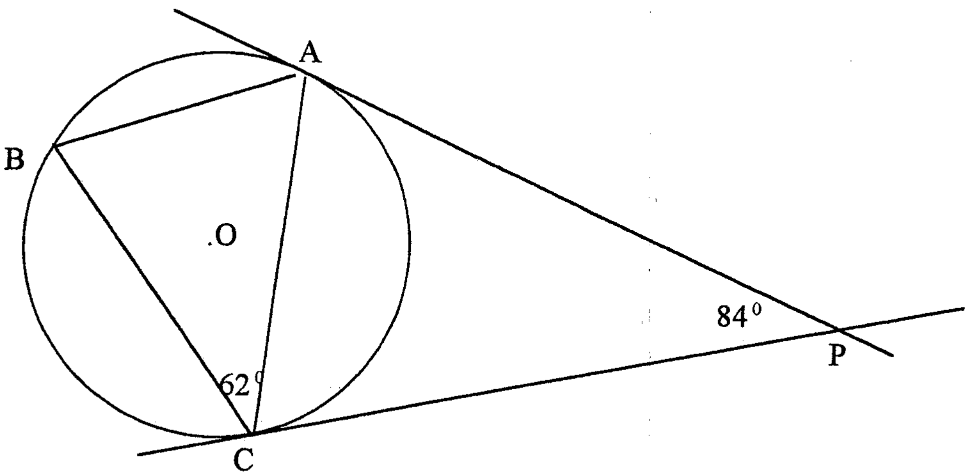
b. Find $\int_{-1}^{0} x\,(2x+1)^3\,dx$ by using the substitution $u = 2x + 1$. (3)

c. Find the ratio in which the line $2x + y - 4 = 0$ divides the interval joining the points A(-3,4) and B(3,1) internally. (3)

d. ABC is an acute angled triangle inscribed in a circle centre O and $\angle ACB = 62^0$. The tangents to the circle at A and C meet at P. If $\angle APC = 84^0$

i. find the size of $\angle BAC$ (giving reasons). (3)

ii. determine $\angle OAC$ (giving reasons). (1)



e. Find the exact value of $\int_{0}^{\frac{\pi}{4}} \text{Cos}^2 \frac{1}{2}x\,dx$ (3)

Question 3. (START A NEW PAGE)

a. Find θ if $0 \le \theta \le 360^0$ for which $3\text{Cos}\,\theta + \sqrt{3}\,\text{Sin}\,\theta = \sqrt{3}$ (3)

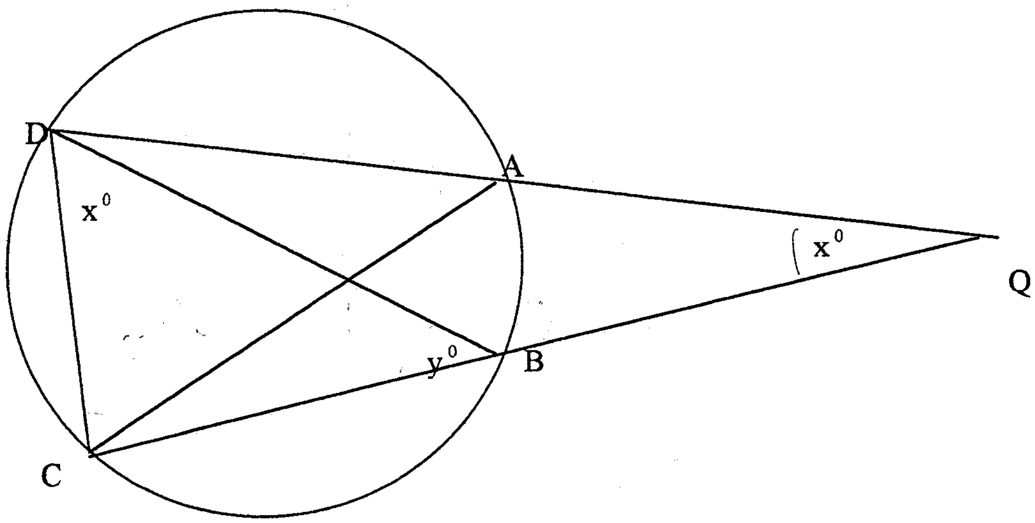
b. Find the limit of $\dfrac{\text{Sin } 4h}{\text{Tan } 5h}$ as h approaches 0 (2)

c. i. Express Cos A -Cos B as a product. (1)

ii. Hence or otherwise find , in simplest surd form, the value of $\text{Sin}\,52^0 30' \times \text{Sin}7^0 30'$. (3)

d. Find the size of all angles θ for which $\text{Sin}\ 2\theta = \frac{1}{2}\text{Cos}\theta$ (3)

e. In the diagram below, DA produced and CB produced meet at Q. (3)
If $\angle CDB = \angle DQC = x^0$ and $\angle DBC = y^0$, prove that CD = CA.



Question 4. (START A NEW PAGE)

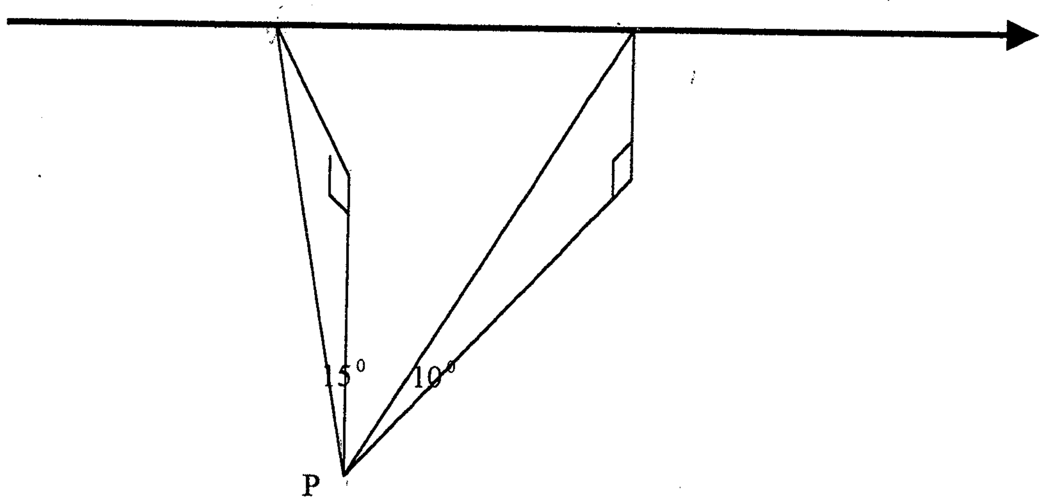
a. Assuming that Cos x $\neq 0$, make Tan x the subject of Sin $(x + \theta) = a$ Cos x. (2)

b. i. Sketch $\frac{x-2}{x^2}$ showing where/if it cuts the coordinate axes, any turning points, asymptotes and any other necessary features (4)

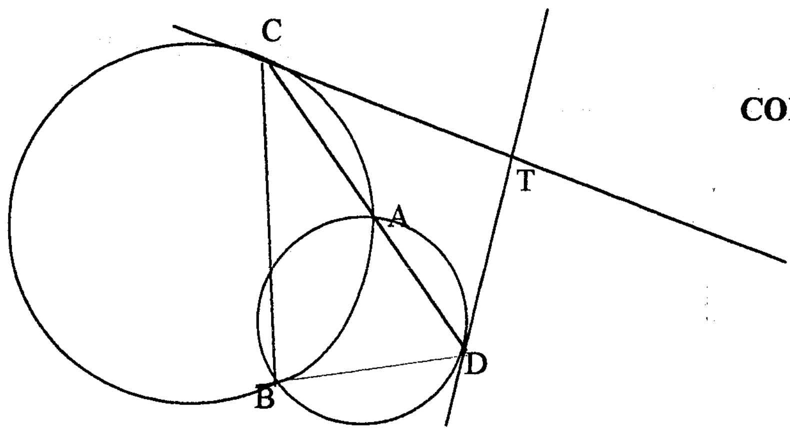
ii. With the aid of your graph above you sketch $y = \left|\frac{x-2}{x^2}\right|$ **on a separate number plane.** (2)

c. An aeroplane is flying due east at a constant altitude and speed of 450km/h. It is observed due north of a certain point ,P, on ground level with an angle of elevation of 15^0. One minute later , the angle of elevation of the plane from P is 10^0. Find the altitude of the plane . (4)

COPY THE DIAGRAM



d. Two circles intersect at A and B. CAD is a straight line. Tangents at C and D intersect at T. Prove that CBDT is a cyclic quadrilateral. (3)

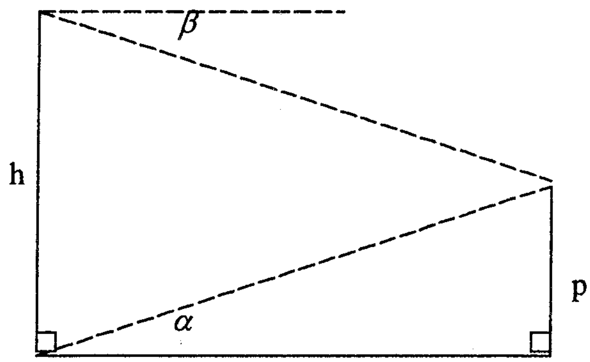


COPY THE DIAGRAM

(CONTINUED OVER)

Question 5. (START A NEW PAGE)

a. From the foot of a mast, the angle of elevation of a building p metres high is α . From the top of a mast, the angle of depression of the building is β. Show that the height h of the mast is given by $h = \dfrac{p \operatorname{Sin}(\alpha + \beta)}{\operatorname{Sin}\alpha \operatorname{Cos}\beta}$ (3)



COPY THE DIAGRAM

b. A secant PQ passes through point P($2ap, ap^2$) and Q($2aq, aq^2$) on the parabola $x^2 - 4ay = 0$ (7)

i. Find the coordinate of the focus S.

ii. Show that the gradient of PQ is $\dfrac{p+q}{2}$, hence show that $pq = -1$.

iii. Show that the coordinates of T, the point where the tangents at P and Q intersect are $\left(\dfrac{a(p^2 - 1)}{p}, -a\right)$.

iv. Find the locus of T.

c. If n = positive integers , prove by induction that:

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots\ldots\ldots + (-1)^{n-1} n^2 = (-1)^{n-1} \times \frac{1}{2} n(n+1)$$ (5)

(end of exam)

yr 12 ½ yrly EXT 1 2004

Q1 a) i) $\frac{5}{3-x} \leq 1 \quad x \neq 3$

$5(3-x) \leq (3-x)^2$

$15-5x \leq 9-6x+x^2$

$0 \leq -6-x+x^2$

$x^2-x-6 \geq 0$

$(x-3)(x+2) \geq 0$

$x \leq -2 \quad x \geq 3$ but $x \neq 3$

$\therefore x \leq -2 \quad x > 3$

ii) $\frac{5}{|3-x|} \leq 1 \qquad x \neq 3$

$|3-x|$ is always + so inequality doesn't change.

$5 \leq |3-x|$

$|3-x| \geq 5$

$3-x \geq 5 \qquad 3-x \leq -5$

$-x \geq 2 \qquad -x \leq -8$

$x \leq -2 \qquad x \geq 8$

b) $\angle BAE = 88°$ (ext $\angle$ of cyclic quad = int opp $\angle$)

$\angle BDE = 35°$ ($\angle$ in alt segment)

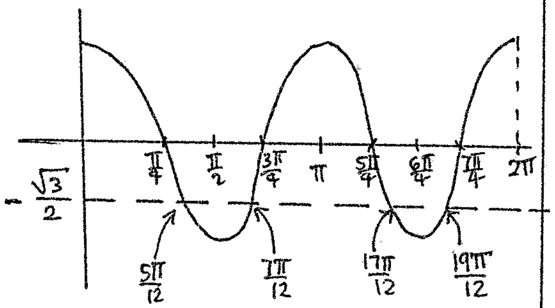
c)

$\cos 2x < -\frac{\sqrt{3}}{2}$

$2x = 150°, 210°, 510°, 570°$

$2x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}$

$x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$



$\therefore \frac{5\pi}{12} < x < \frac{7\pi}{12}, \quad \frac{17\pi}{12} < x < \frac{19\pi}{12}$

d) $\tan\beta = \frac{a}{x} \qquad \tan(\alpha+\beta) = \frac{3a}{x}$

$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{3a}{x}$

$\frac{\tan\alpha + \frac{a}{x}}{1 - \tan\alpha \times \frac{a}{x}} = \frac{3a}{x}$

$\frac{\frac{x\tan\alpha + a}{x}}{\frac{x - a\tan\alpha}{x}} = \frac{3a}{x}$

$\frac{x\tan\alpha + a}{x - a\tan\alpha} = \frac{3a}{x}$

$x^2\tan\alpha + ax = 3ax - 3a^2\tan\alpha$

$x^2\tan\alpha + 3a^2\tan\alpha = 2ax$

$\tan\alpha = \frac{2ax}{x^2+3a^2}$

Q2 a) $y = \ln x$

$y' = \frac{1}{x}$

at $x=1$, $m=1$ — at $x=2$, $x = \frac{1}{2}$

$\therefore \tan\alpha = \frac{1-\frac{1}{2}}{1+1\times\frac{1}{2}}$

$\alpha = 18°26'$

b) $\int_{-1}^{0} x(2x+1)^3\,dx$

$u = 2x+1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$x=0,\ u=1$

$x=-1,\ u=-1$

also $x = \frac{u-1}{2}$

$\int_{-1}^{1} \frac{u-1}{2} \times u^3 \times \frac{1}{2}\,du$

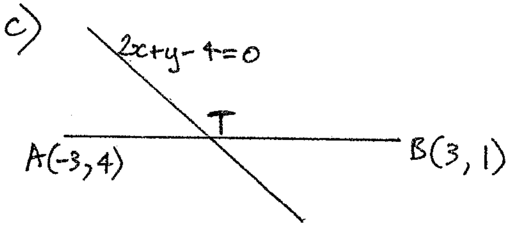
$\frac{1}{4}\int_{-1}^{1} u^4 - u^3\,du$

$\frac{1}{4}\left[\frac{u^5}{5} - \frac{u^4}{4}\right]_{-1}^{1}$

$\frac{1}{4}\left[\left(\frac{1}{5}-\frac{1}{4}\right) - \left(-\frac{1}{5} - \frac{1}{4}\right)\right]$

$= \frac{1}{10}$

c)



Find equ of AB first

$m = \frac{4-1}{-3-3} = -\frac{1}{2}$

$y-1 = -\frac{1}{2}(x-3)$

$y-1 = -\frac{1}{2}x + \frac{3}{2}$

$y = -\frac{1}{2}x + \frac{5}{2}$

Point of intersection:

$2x + \left(-\frac{1}{2}x + \frac{5}{2}\right) - 4 = 0$

$\frac{3}{2}x = \frac{3}{2}$

$x = 1$

$y = 2 \qquad \therefore T = (1,2)$

AT : TB

$\sqrt{(-3-1)^2+(4-2)^2} : \sqrt{(3-1)^2+(1-2)^2}$

$\sqrt{20} : \sqrt{5}$

$2 : 1$ (or $1 : 2$)

d) i) PA = PC (equal tangents from same external pt)

$\therefore \triangle PAC$ = isos

$\therefore \angle PAC = \frac{180-84}{2} = 48$ (base $\angle$ isos $\triangle$)

$\angle ACP = 48°$

$\angle CBA = 48°$ ($\angle$ in alt segment)

$\angle BAC = 180-48-62$ ($\angle$ sum $\triangle$)

$\angle BAC = 70°$

ii) $\angle OAC + \angle CAP = 90°$ (tangents meet radii at 90° at pt of contact)

$\angle OAC + 48 = 90°$

$\angle OAC = 42°$

e) $\int_0^{\frac{\pi}{4}} \cos^2 \frac{1}{2}x\,dx$

$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$\cos^2\frac{\theta}{2} = \frac{1}{2}[\cos\theta + 1]$

$\frac{1}{2}\int_0^{\frac{\pi}{4}} \cos x + 1\,dx$

Q2e) continued

$\frac{1}{2}\left[\sin x + x\right]_0^{\frac{\pi}{4}}$

$\frac{1}{2}\left[\left(\frac{1}{\sqrt{2}}+\frac{\pi}{4}\right)-(0+0)\right]$

$\frac{1}{2}\left[\frac{1}{\sqrt{2}}+\frac{\pi}{4}\right]$

Q3a) $3\cos\theta + \sqrt{3}\sin\theta = \sqrt{3}$

$a\cos x + b\sin x = r\cos(x-\alpha)$

$r = \sqrt{3^2+(\sqrt{3})^2} = \sqrt{12}$

$\tan\alpha = \frac{\sqrt{3}}{3}$

$\alpha = 30° = \frac{\pi}{6}$

$\therefore \sqrt{12}\cos\left(x-\frac{\pi}{6}\right) = \sqrt{3}$

$\sqrt{12}\cos(x-30°) = \sqrt{3}$

$\cos(x-30°) = \frac{1}{2}$

$x - 30° = 60°, 300°$

$x = 90°, 330°$

b) $\lim_{h\to 0} \frac{\sin 4h}{\tan 5h}$

$\lim_{h\to 0} \frac{\sin 4h}{4h} \times \frac{4h}{1} \times \frac{5h}{\tan 5h} \times \frac{1}{5h}$

$\downarrow\ 1 \times 4h \times 1 \times \frac{1}{5h}$

$= \frac{4}{5}$

(if desperate, use your calc in RADS)

c) i) $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

ii) $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) = -\frac{1}{2}\left[\cos A - \cos B\right]$

$\sin 52°30' \sin 7°30'$

$\frac{A+B}{2} = 52°30' \quad \frac{A-B}{2} = 7°30'$

$A+B = 105° \cdots ① \quad A-B = 15° \cdots ②$

①+②

$2A = 120°$

$A = 60°$

$B = 45°$

$\therefore \sin 52°30' \sin 7°30' = -\frac{1}{2}\left[\cos 60 - \cos 45\right]$

$= -\frac{1}{2}\left[\frac{1}{2} - \frac{1}{\sqrt{2}}\right]$

$= -\frac{1}{2}\left[\frac{\sqrt{2}-2}{2\sqrt{2}}\right]$

$= -\frac{1}{2}\left[\frac{\sqrt{2}-2}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right]$

$= -\frac{1}{2}\left[\frac{2-2\sqrt{2}}{4}\right]$

$= \frac{\sqrt{2}-1}{4}$

d) $\sin 2\theta = \frac{1}{2}\cos\theta$

$2\sin\theta\cos\theta = \frac{1}{2}\cos\theta$

$4\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta\left[4\sin\theta - 1\right] = 0$

$\cos\theta = 0 \qquad \sin\theta = \frac{1}{4}$

$\cos\theta = \cos\frac{\pi}{2} \qquad \sin\theta = \sin 14°29'$

$\cos\theta = \cos 90° \qquad \sin\theta = \sin 14°29'$

$\theta = n.360 \pm 90°$ or $\theta = n.180 + (-1)^n 14°29'$

e) $\angle DAC = y°$ ($\angle$'s in the same segment)

$\angle BDQ + x = y$ (ext $\angle$ Δ)

$\angle BDQ = y - x$

$\therefore \angle CDA = x + y - x = y$

$\therefore \angle CDA = y = \angle DAC$

so ΔDAC is isosceles

$\therefore DC = CA$ (2 equal sides)

Q4 a) $\sin(x+\theta) = a\cos x$

$\frac{\sin x\cos\theta}{\cos x} + \frac{\cos x\sin\theta}{\cos x} = \frac{a\cos x}{\cos x}$

$\tan x \cos\theta + \sin\theta = a$

$\tan x = \frac{a - \sin\theta}{\cos\theta}$

b) i) $y = \frac{x-2}{x^2}$

$x \neq 0$

when $y=0$, $x=2$ $\quad (2,0)$

$\lim_{x\to 0^+} y \to -\infty$

$\lim_{x\to 0^-} y \to -\infty$

$\lim_{x\to\infty} \frac{x-2}{x^2} = \frac{\frac{x}{x^2} - \frac{2}{x^2}}{\frac{x^2}{x^2}} = \frac{\frac{1}{x} - \frac{2}{x^2}}{1}$

$\therefore \lim_{x\to\infty} y \to 0$ (from above)

$\lim_{x\to -\infty} y \to 0$ (from below)

$y' = \frac{1.x^2 - 2x(x-2)}{x^4}$

$y' = \frac{x^2 - 2x^2 + 4x}{x^4}$

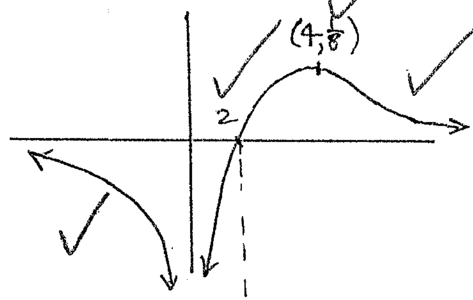
$y' = \frac{4x - x^2}{x^4} = \frac{x(4-x)}{x^4} = \frac{4-x}{x^3}$

st pts $\frac{4-x}{x^3} = 0$

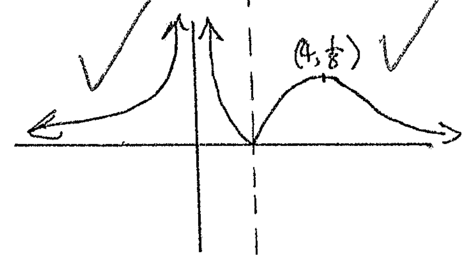
$x = 4,\ y = \frac{1}{8}$

$(4, \frac{1}{8})$

x	←	4	→
y'	+	0	−

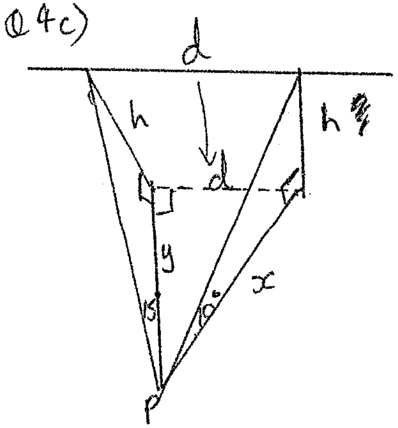


ii) $y = \left|\frac{x-2}{x^2}\right|$



c) over

Q4c)



$d = ?$ $\quad S = \frac{D}{T}$

$D = S \times T$

$= 450\,\text{km/hr} \times \frac{1}{60}$

$= 7\frac{1}{2}\,\text{km}$

$\tan 10 = \frac{h}{x} \rightarrow x = \frac{h}{\tan 10}$

$\tan 15 = \frac{h}{y} \rightarrow y = \frac{h}{\tan 15}$

$\therefore \left(\frac{h}{\tan 10}\right)^2 = \left(7\frac{1}{2}\right)^2 + \left(\frac{h}{\tan 15}\right)^2$

$\frac{h^2}{\tan^2 10} = \frac{225}{4} + \frac{h^2}{\tan^2 15}$

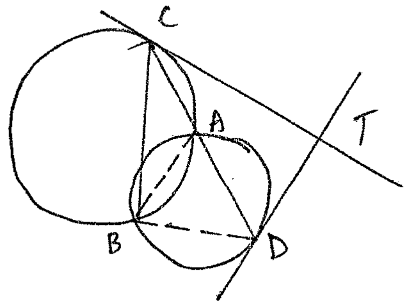
$h^2\left[\frac{1}{\tan^2 10} - \frac{1}{\tan^2 15}\right] = 56.25$

$h^2 = \frac{56.25}{\left(\frac{1}{\tan^2 10} - \frac{1}{\tan^2 15}\right)}$

$h^2 = 3.08\ldots$

$h = 1.756\,\text{km}$

d)



join BA and BD

let $\angle CDT = \theta$

$\therefore \angle ABD = \theta$ ($\angle$ in alt segment)

let $\angle DCT = \phi$

$\therefore \angle CBA = \phi$ ($\angle$ in alt segment)

$\angle CTD = 180 - \theta - \phi$ ($\angle$ sum Δ)

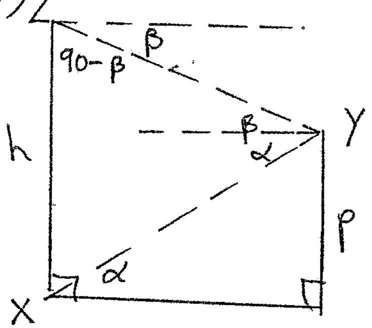
$\angle CBD = \theta + \phi$ ($\angle$ sum)

$\therefore \angle CTD + \angle CBD = 180 - \theta - \phi + \theta + \phi$

$= 180°$

$\therefore$ true CBDT is cyclic quad

Q5a)



$\sin\alpha = \frac{p}{XY} \rightarrow XY = \frac{p}{\sin\alpha}$

$\angle XZY = 90 - \beta$

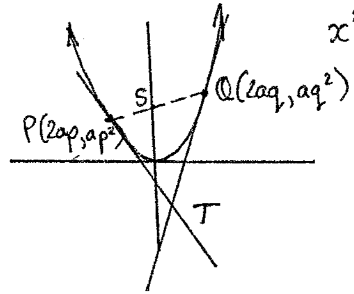
$\therefore \frac{XY}{\sin(90-\beta)} = \frac{h}{\sin(\alpha+\beta)}$

$\sin(90-\beta) = \cos\beta$

$\frac{\frac{p}{\sin\alpha}}{\cos\beta} = \frac{h}{\sin(\alpha+\beta)}$

$\frac{p\sin(\alpha+\beta)}{\sin\alpha\cos\beta} = h$

b)



i) $x^2 = 4ay \quad \therefore S = (0, a)$

ii) $m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq} = \frac{a(p-q)(p+q)}{2a(p-q)}$

$m_{PQ} = \frac{p+q}{2}$

(forgot to write focal chord on paper (no marks))

$m_{SQ} = m_{PQ}$

$m_{SQ} = \frac{aq^2 - a}{2aq - 0} = \frac{a(q-1)(q+1)}{2aq}$

$\frac{p+q}{2} = \frac{(q-1)(q+1)}{2q}$

$pq + q^2 = q^2 - 1$

$pq = -1$

iii) $PT \Rightarrow y = px - ap^2 \ldots (1)$

$QT \rightarrow y = qx - aq^2 \ldots (2)$

but $q = -\frac{1}{p}$ sub in (2)

$y = \frac{-x}{p} - \frac{a}{p^2} \ldots (3)$

(3) = (1)

$px - ap^2 = -\frac{x}{p} - \frac{a}{p^2}$

$p^3x - ap^4 = -xp - a$

$x(p^3 + p) = ap^4 - a$

$x(p^3 + p) = a(p^2-1)(p^2+1)$

$xp(p^2+1) = a(p^2-1)(p^2+1)$

$x = \frac{a(p^2-1)}{p}$ sub in (1)

$y = p \times \frac{a(p^2-1)}{p} - ap^2$

$y = ap^2 - a - ap^2$

$y = -a$

$\therefore T = \left(\frac{a(p^2-1)}{p}, -a\right)$

iv) $y = -a$

c. $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1}n^2 = (-1)^{n-1} \times \frac{1}{2}n(n+1)$

(1) Show true for $n=1$

LHS = 1, RHS = $\frac{1}{2} \times 2 = 1$ $\therefore$ True for $n=1$

(2) Assume true for $n=k$

$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1}k^2 = (-1)^{k-1} \times \frac{1}{2}k(k+1)$

Show true for $n = k+1$

$1^2 - 2^2 + 3^2 + \ldots + (-1)^{k-1}k^2 + (-1)^k(k+1)^2 = (-1)^k \times \frac{1}{2}(k+1)(k+2)$

$(-1)^{k-1} \times \frac{1}{2}k(k+1) + (-1)^k(k+1)^2$

$(-1)^k \cdot -1 \times \frac{1}{2}k(k+1) + (-1)^k(k+1)^2$

$(-1)^k \cdot -1 \times \frac{1}{2}k(k+1) + \frac{2(-1)^k(k+1)^2}{2}$

$(-1)^k \times \frac{1}{2}\left[-1 \times k(k+1) + 2(k+1)^2\right]$

$(-1)^k \times \frac{1}{2}(k+1)\left[-k + 2(k+1)\right]$

$(-1)^k \times \frac{1}{2}(k+1)(k+2)$

= RHS

$\therefore$ if true for $n=k$ then true for $n=k+1$

(3) Since true for $n=1$, then true for $n = 2, 3, 4, \ldots$ hence true for all positive integral n.