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BRIGIDINE COLLEGE RANDWICK

MATHEMATICS EXTENSION 1

YEAR 12

HALF-YEARLY 2006

(Time: 2 hours + 5 minutes reading)

DIRECTIONS TO CANDIDATES

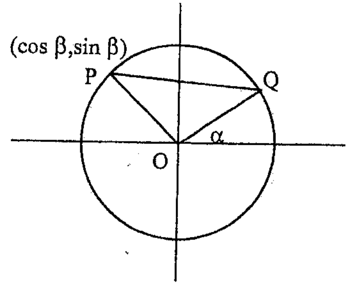
- *Put your **name** at the top of this paper and on **each of the 6 sections** to be collected.*
- *All **6 questions** may be attempted. and are to be answered on **separate pages** and will be collected in **separate bundles** at the end of this exam.*
- *All questions are of equal value.*
- *All necessary working should be shown in every question.*
- *Full marks may not be awarded for careless or badly arranged work.*

The following Outcomes may be addressed

PE2	Uses multi-step deductive reasoning in a variety of contexts
PE3	Solves problems involving circle geometry and parametric representation.
PE4	Uses the parametric representation together with differentiation to identify geometric properties of parabolas.
PE5	Determines derivatives which require the application of more than one rule of differentiation.
PE6	Makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations.
HE2	Uses inductive reasoning in the construction of proofs.
HE4	Uses the relationship between functions, inverse functions and their derivatives.
HE6	Determines integrals by reduction to a standard form though a given substitution.
HE7	Evaluates mathematical solutions to problems and communicates them in an appropriate form.

QUESTION 1 *(start a new page)*

a. i. The figure to the right represents a unit circle with points P & Q as shown. Show that the point Q may be represented by $(\cos\alpha, \sin\alpha)$ *1 m*



ii. By application of the distance formula and cosine rule, show that *3 m*

$$\cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$$

iii. Hence, or otherwise, determine the exact value of $\cos 15^\circ$ in exact form. *2 m*

b. Solve the inequality $\dfrac{2x - 3}{4x - 5} \le -2$ *4 m*

c. Find the general solution to $\sqrt{2}\cos\alpha + 1 = 0$ *2 m*

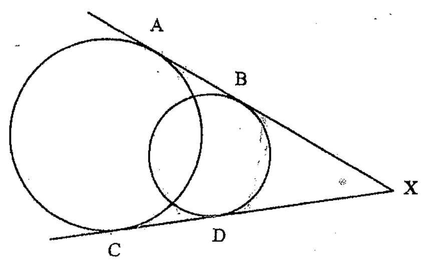
QUESTION 2 *(start a new page)*

a. i. Show that $\dfrac{1 + \cos 2A}{\sin 2A} = \cot A$. **2 m**

ii. Hence find the exact value of $\cot 15°$. **2 m**

b. Find the coordinates of the point internally dividing the join of $(-1,5)$ and $(6,-4)$ in the ratio of $3 : 2$. **2 m**

c. In the diagram below AB is common tangent to the two circles. **3 m**
Likewise CD is also a common tangent.
The two tangents meet externally at X.
Prove that $AC \parallel BD$.



d. Find the size of the acute angle between the tangents drawn to $y = \ln x$ at the points where $x = 1$ and $x = 2$ to the nearest minute. **3 m**

QUESTION 3 *(start a new page)*

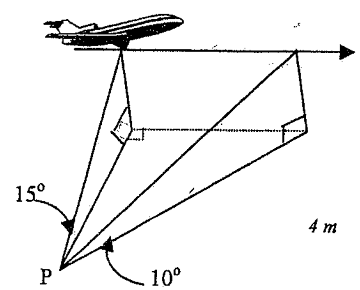
a. Prove by the method of Mathematical Induction, that $(n \geq 1)$ **3 m**

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{2(3n+2)}$$

b. An plane is flying due east at a speed of 450 km/hr and maintains a constant altitude. **4 m**

From a point P on ground level an observer noted the angle of elevation to be 15 °.

One minute late, this same observer (at point P) notes the angle of elevation to be 10 °.



Determine the altitude of this plane (nearest metre).

c. Find the equation of the tangent to the curve represented parametrically by $x = 2t + 5$ and $y = 3t^2 - 2$ at the point where $t = 1$. **3 m**

d. By considering the method of addition of ordinates, sketch the curve $y = x^2 - \dfrac{1}{x}$. **2 m**

QUESTION 4 *(start a new page)*

a. i. Show that $1 - \cos x = 2\sin^2 \frac{x}{2}$ *1 m*

ii. Hence, or otherwise, find $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$ *2 m*

b. Evaluate $\int_0^1 \frac{x}{\sqrt{1 + x}}\,dx$ by using the substitution $u = 1 + x$. *3 m*

c. Consider the function $f(x) = \frac{x}{4 - x^2}$

i. Show that $f(x)$ is odd. *1 m*

ii. Show that $f(x)$ is increasing throughout its domain. *2 m*

iii. Explain the behaviour of this function as x approaches $\pm\infty$. *1 m*

iv. Sketch the graph of $f(x)$ showing clearly the coordinates of any points of intersection with the x and y axis and the equations of any asymptotes. *2 m*

QUESTION 5 *(start a new page)*

a. $\frac{dx}{dt} = \frac{t - 1}{\sqrt{t^2 - 2t + 4}}$ and $x = 10$ when $t = 0$, *3 m*

find x in terms of t by using the substitution $u = t^2 - 2t + 4$.

b. Two points are defined by $x = 2\cos t$ and $y = \cos 2t$. *2 m*

Show that these points lie on a parabolic arc.

c. i. Show that the cartesian equation of the parabola whose parametric equations are $x = 2ap$ and $y = ap^2$ is given by $x^2 = 4ay$. *1 m*

ii. A point P lies on this parabola ($x^2 = 4ay$), show that the normal at this point P may be given by $x + py = ap^3 + 2ap$. *2 m*

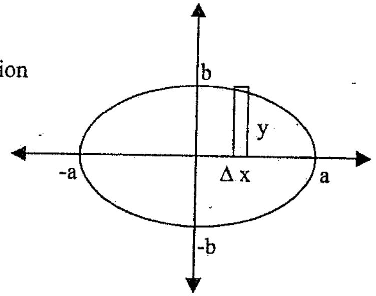
iii. If this normal meets the y-axis of this parabola at R, draw a neat sketch to represent this information. *1 m*

iv. Show that the locus of M, the midpoint of PR, is another parabola and use this information to write down its vertex and focal length *3 m*

QUESTION 6 *(start a new page)*

a. The ellipse to the right has the cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



The area of this ellipse may be given by

$$A \approx \lim_{\Delta x \to 0} \sum_{n=0}^{a} \sqrt{\frac{a^2 - x^2}{a^2} b^2} \; \Delta x$$

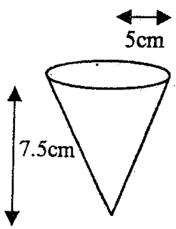
By considering the substitution $x = a \sin\alpha$,

show that the area of this ellipse is πab. *4 m*

b. Prove by the method of Mathematical Induction, that

$7^{2n-1} + 5$ is divisible by 12 for $n \geq 1$. *4 m*

c. Filter paper is in the shape of an inverted cone, base radius 5 cm and altitude 7.5 cm. If water is flowing out from the bottom at a constant rate of 1.5 cm^3/s find the rate at which the level of the liquid is falling when the depth is 5 cm.



4 m

- end of exam -

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0,\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Q5

a) $\int \frac{dx}{dt}\,dt = \frac{1}{2}\int \frac{2(t-1)}{\sqrt{t^2-2t+4}}\,dt$ ✓

$x = \frac{1}{2}\int u^{-1/2}\,du \qquad u = t^2-2t+4, \quad \frac{du}{dt} = 2t-2$

$x = \frac{1}{2}u^{1/2}\cdot 2 + c$ ✓

$x = \sqrt{t^2-2t+4} + c$ ✓

$10 = \sqrt{4} + c$

$\therefore x = \sqrt{t^2-2t+4} + 8$ | 3

b) $x = 2\cos t \quad | \quad y = \cos 2t$

$\cos 2t = 2\cos^2 t - 1$ ✓

$\therefore y = \cos 2t$

$y = 2\left[\frac{x^2}{4}\right] - 1$

$y = \frac{x^2}{2} - 1$ ✓

$\therefore$ they lie on parabola | 2

c)

i) $x = 2ap \quad | \quad y = ap^2$

$y = a\left[\frac{x}{2a}\right]^2$

$y = \frac{ax^2}{4a^2}$ ✓

$4ay = x^2 \Rightarrow x^2 = 4ay$ | 1

ii) $x = 2ap \quad | \quad y = ap^2$

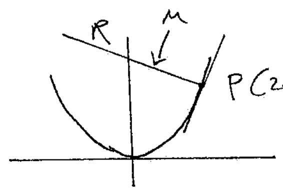
$\frac{dx}{dp} = 2a \quad | \quad \frac{dy}{dp} = 2ap$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$ ✓

$py - ap^3 = -x + 2ap$

$\therefore x + py = ap^3 + 2ap$ ✓ | 2

iii)



| 1

iv) $0 + py = ap^3 + 2ap$

$y = ap^2 + 2a$

$R\ [0,\ ap^2 + 2a]$

$\underline{M}$ $\quad x = \frac{1}{2}(0 + 2ap)$ ✓

$x = ap$

$y = \frac{1}{2}\left[ap^2 + 2a + ap^2\right]$

$y = \frac{1}{2}\left[2ap^2 + 2a\right]$

$y = ap^2 + a$

$y = a\frac{x^2}{a^2} + a$

$y = \frac{x^2}{a} + a$

$y = \frac{x^2 + a^2}{a}$

$ay = x^2 + a^2$

$x^2 = ay - a^2$ ✓

$x^2 = a(y - a)$

$x^2 = 4\left(\frac{a}{4}\right)(y - a)$

$\therefore$ parabola vertex $(0, a)$ ✓

$\frac{1}{4}a$ | 3

Q6

a) $A_{First} = \int_0^a y\,dx$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$y^2 = b^2\left[\frac{a^2 - x^2}{a^2}\right]$

$y = \frac{b}{a}\sqrt{a^2 - x^2}$ ✓

$+$ since 1st quad

$A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$ ✓

$x = a\sin\alpha \quad \Big| \quad dx = a\cos\alpha\,d\alpha$

$= \frac{4b}{a}\int \sqrt{a^2 - a^2\sin^2\alpha}\;a\cos\alpha\,d\alpha$

$= \frac{4b}{a}\int a\sqrt{1 - \sin^2\alpha}\,\cos\alpha\,d\alpha$

$= 4ab\int \cos^2\alpha \cdot d\alpha$ ✓

$= \frac{4ab}{2}\int_0^{\frac{\pi}{2}} \left[1 + \cos 2\alpha\right] dx$

$= 2ab\left[\alpha + \frac{\sin 2\alpha}{2}\right]_0^{\frac{\pi}{2}}$

$= 2ab\left[\left(\frac{\pi}{2} + 0\right) - 0 + 0\right]$

$= ab\pi \quad u^2$ ✓ | 4

b) statements ✓

$7^{2n-1} + 5 = 12m$

$7^{2n+2-1} + 5$

$7^{2n+1} + 5$

$7^{2n-1+2} + 5$

$49(12m - 5) + 5$

$= 49(12m) - 240$

$= 12\left[49m - 20\right]$

divisible by 12 $\rightarrow$ statement

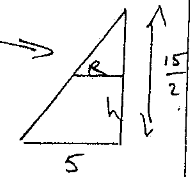
c) $\frac{dV}{dt} = -1.5 = -\frac{3}{2}$

$\frac{dV}{dt} = \frac{dV}{dh}\,\frac{dh}{dt}$ ✓

$V = \frac{1}{3}\pi R^2 h$

$= \frac{1}{3}\pi\,\frac{4h^2}{9}\,h$

$= \frac{4\pi h^3}{27}$ ✓



$\frac{R}{5} = \frac{2h}{15}$

$R = \frac{10h}{15}$

$R = \frac{2h}{3}$

$\frac{dV}{dh} = \frac{4}{9}\pi h^2$

$h = 5$

$= \frac{4}{9}\pi\,25$

$\frac{dV}{dt} = \frac{dV}{dh}\,\frac{dh}{dt}$ ✓

$-\frac{3}{2} = \frac{100\pi}{9}\,\frac{dh}{dt}$

$\therefore \frac{dh}{dt} = -\frac{3}{2}\cdot\frac{9}{100\pi}$

$= \frac{-27}{200\pi}$ cm³/s ✓ | 4

$[\approx 0.04297]$

Ext 1 HY 06 Yr12

Book work

∴ marks

i) show Trig ✓

ii) formulae ✓

equate ✓

iii) $\cos(45-30)$

$= \cos 45 \cos 30 - \sin 45 \sin 30$

$= \frac{1}{\sqrt2}\cdot\frac{\sqrt3}{2} + \frac{1}{\sqrt2}\cdot\frac{1}{2}$

$= \frac{\sqrt3+1}{2\sqrt2}$ ✓ 6

$\frac{2x-3}{4x-5} + 2 \le 0$

$(2x-3)(4x-5) + 2(4x-5)^2 \le 0$ ✓

$8x^2 - 22x + 15 + 2[16x^2 - 40x + 25]$

$8x^2 - 22x + 15$

$32x^2 - 80x + 50$

$40x^2 - 102x + 65 \le 0$ ✓

$x = \frac{102 \pm \sqrt{(102)^2 - 4(40)65}}{80}$

$x = \frac{104}{80} = 1\frac{3}{10}$, $x = \frac{100}{80} = 1\frac14$ 4

$\frac54 < x \le \frac{13}{10}$

$\sqrt2 \cos\alpha + 1 = 0$

$\cos\alpha = -\frac{1}{\sqrt2}$ ✓ $\frac{3\pi}{4}$

$\alpha = 2n\pi \pm 135°$ ✓ 2

notation

Q2

a) i) using $\cos 2A = 2\cos^2 A - 1$ ✓

LHS $\frac{1+\cos 2A}{\sin 2A}$

$= \frac{2\cos^2 A}{2\sin A\cos A} = \frac{2\cos A}{2\sin A}$

$= \cot A$ ✓

ii) $\cot 15 = \frac{1+\cos 2(15)}{\sin 2(15)}$

$= \frac{1+\cos 30}{\sin 30} = \left[1+\frac{\sqrt3}{2}\right] \div \frac12$

$= \frac{2+\sqrt3}{2}\cdot 2 = 2+\sqrt3$ 4

b) $(-1,5)$ & $(6,-4)$ ratio 3:2

$\left[\frac{(3)(6)+(2)(-1)}{3+2}, \frac{(3)(-4)+(2)(5)}{3+2}\right]$

$\left(\frac{18-2}{5}, \frac{-12+10}{5}\right) = \left(\frac{16}{5}, -\frac{2}{5}\right)$ 2

c) $XB = XD$ Tangents from Exterior point ✓

$XA = XC$

$\hat{X}$ Common

∴ $\triangle BXD \parallel\!\mid \triangle AXC$ Common ∠ sides ratio 3

$\angle XBD = \angle XAC$ Corresp. parts

$AC \parallel BD$ ✓ Corresp ∠s

d) $y = \ln x$, $y' = \frac1x$ ✓ $x=1$, $y'=1$ | $x=2$, $y' = \frac12$

$\tan\alpha = \frac{1-\frac12}{1+\frac12} = \frac13$

∴ $\alpha = 18°26'$ ✓ 3

Ext 1 HY06 Yr12

Q3

a) * $n=1$

$\frac{1}{2\times5} = \frac{1}{2(3(1)+2)}$ ✓

* Add $\frac{1}{(3(n+1)-1)(3(n+1)+2)}$

i.e. $\frac{1}{(3n+2)(3n+5)}$ both sides

RHS

$\frac{n}{2(3n+2)} + \frac{1}{(3n+2)(3n+5)}$

$= \frac{n(3n+5)+2}{2(3n+2)(3n+5)}$ ✓

$= \frac{3n^2+5n+2}{2(3n+2)(3n+5)}$ ✓

$= \frac{(n+1)(3n+2)}{2(3n+2)(3n+5)}$

$= \frac{n+1}{2(3n+5)} \Rightarrow \frac{n+1}{2(3(n+1)+2)}$

Steps

& statement 3

b) $d = st$

$= \frac{450\text{ km}}{\text{hr}} \times \frac{1}{60}$

$= 7\frac12$ km ✓

Top view

$\tan 15 = \frac{h}{y}$ | $\tan 10 = \frac{h}{x}$

h as altitude

$(7\frac12)^2 + y^2 = x^2$

$\frac{225}{4} + \left[\frac{h^2}{\tan^2 15}\right] = \left[\frac{h^2}{\tan^2 10}\right]$

$\frac{225}{4} = h^2\left[\frac{1}{\tan^2 10} - \frac{1}{\tan^2 15}\right]$

$h^2 = \frac{225}{4} \div$ ⇑

$h^2 \doteqdot 3.08$

$h = 1756$ 4

c) $x = 2t+5$ | $y = 3t^2 - 2$

$t=1$

$x = 7$ | $y = 1$

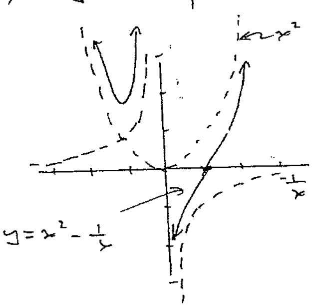
$\frac{dx}{dt} = 2$ ✓ | $\frac{dy}{dt} = 6t$ ✓ $\Rightarrow 6$

$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = 6\cdot\frac12 = 3$

∴ $y - 1 = 3(x-7)$

$y = 3x - 20$ ✓ 3

d) $y = x^2$ | $y = -\frac1x$



$y = x^2 - \frac1x$ 2

Q4

a) $\cos 2\square = 1 - 2\sin^2\square$

i) $2\sin^2\square = 1 - \cos 2\square$ ✓

$\frac{\theta}{2}$

∴ $2\sin^2\frac{\theta}{2} = 1 - \cos\theta$

ii) $\lim_{x\to0} \frac{1-\cos x}{x^2}$

$\lim_{x\to0} \frac{2\sin\frac{x}{2}\sin\frac{x}{2}}{2\frac{x}{2}\,2\frac{x}{2}}$ ✓

$= 2\cdot\frac12\cdot\frac12 = \frac12$ ✓ 3

5) $\int_0^1 \frac{x\,dx}{\sqrt{1+x}}$ $u = 1+x$, $du = dx$

$\int_1^2 (u-1)u^{-\frac12}\,du$ ✓

$\int_1^2 (u^{\frac12} - u^{-\frac12})\,du$

$= \frac23 u^{\frac32} - 2u^{\frac12} \Big|_1^2$ ✓

$= \left[\frac23 2\sqrt2 - 2\sqrt2\right] - \left[\frac23 - 2\right]$

$= \frac{-2\sqrt2}{3} - -\frac43$

$= \frac{4-2\sqrt2}{3}$ ✓ 3

c) $f(x) = \frac{x}{4-x^2}$

i) $f(-x) = \frac{-x}{4-(-x)^2} = \frac{-x}{4-x^2}$

$-f(x) = \frac{-x}{4-x^2}$ ✓

$f(-x) = -f(x)$ ∴ odd 1

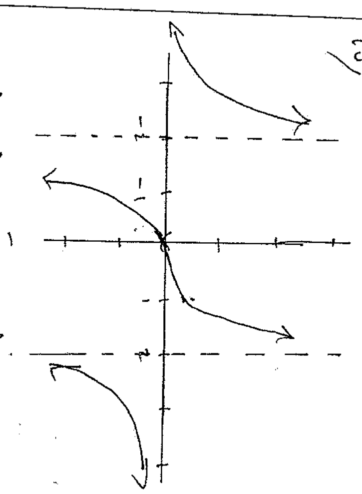
ii) $f'(x) = \frac{(4-x^2) - x(-2x)}{(4-x^2)^2}$ ✓

$= \frac{4+x^2}{(4-x^2)^2}$ $\frac{\text{pos}}{\text{pos}} > 0$ ✓ 2

iii) $f(x) = \frac{\frac{x}{x^2}}{\frac{4}{x^2} - \frac{x^2}{x^2}} = \frac{0}{\square - 1}$

$f(x)$ approaches 0 ✓ 1

iv)



$x = -2$ $x = 2$ 2