BRIGIDINE COLLEGE RANDWICK Name:____________________

YEAR 12 MATHEMATICS Teacher:____________________

HSC Task 12 June 2007

Time allowed: 45 minutes
Show all necessary working IN PEN
Marks may be deducted for careless or badly arranged work.

Question 1. (11 marks)

(a) Find the exact value of $\text{Cos}\,\frac{\pi}{6}$ — 1

(b) The 10^{th} term of an arithmetic sequence is 29 and the 15^{th} term is 44. — 3

(i) Find the value of the common difference and the first term

(ii) Find the sum of the first 75 terms

(c) Differentiate the following with respect to x:

(i) $x\tan 3x$ — 2

(ii) $\dfrac{x^2}{1+\cos x}$ — 2

(d) Find the equation of the tangent to the curve $y = \cos 2x$ at the point whose x-coordinate is $\frac{\pi}{6}$. — 3

Question 2. Start this question on a new page (12 marks)

(a) Find $\int 3\sec^2 x\,dx$ — 1

(b) (i) Sketch $y = -2\text{Sin}2x$ for $0 \le x \le 2\pi$ showing important features. — 2

(ii) Find the area bounded by the curve in part i, the x-axis, and $x = \frac{\pi}{4}$ to $x = \pi$ — 3

(c) Consider the series $1 + \sqrt{2} + 2 + \ldots\ldots\ldots$ etc

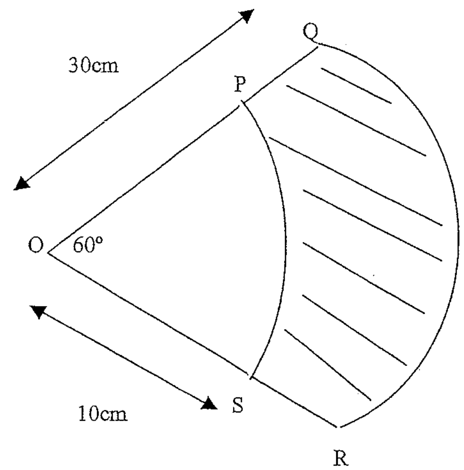
(i) Show that the series is geometric. — 1

(ii) Find the sum of the first 10 terms. (Leave your answer in exact form) — 1

(c) PS and QR are arcs of concentric circles with O as the centre.
Calculate in terms of π:

(i) The area of the shaded region PQRS — 2

(ii) The perimeter of the shaded region PQRS — 2



Question 3. Start this question on a new page (12 marks)

(a) Solve the equation $2\cos 2x = 1$ in the domain $0 \le x \le 2\pi$ — 3

(b) The first three terms of a geometric series are $(k + 3) + k + 4 + \ldots$

(i) Find the possible values for k. — 2

(ii) For what value of k does the series have a limiting sum? — 1

(iii) Find the limiting sum. — 2

(c) When Sarah was born her father deposited \$150 into a bank account earning 8% p.a. interest compounded annually. He decided to deposit \$150 into this account each time Sarah had a birthday and made his last payment on her seventeenth birthday.

(i) Show that the initial deposit of \$150 amounted to \$599.40 on her eighteenth birthday. — 2

(ii) Calculate the total amount that was in the account on Sarah's eighteenth birthday — 2

BRIGIDINE COLLEGE RANDWICK

YEAR 12 MATHEMATICS ASSESSMENT
TASK – June, 2007

SOLUTIONS

QUESTION 1

(a) $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

(b) $T_{10} = 29$
$T_{15} = 44$

(i) For an arithmetic sequence

$T_{10} = a(10-1)d$
$= a + 9d$
$T_{15} = a + 14d$

Thus

$a + 9d = 29$ ①
$a + 14d = 44$ ②
① – ②
$5d = 15$
$d = 3$

Substitute for d in ①

$a + 27 = 29$
$a = 2$

First term: 2
Common difference: 3

(ii) For an arithmetic sequence

$S_n = \frac{n}{2}[2a + (n-1)d]$

For $n = 75$

$S_{75} = \frac{75}{2}[2 \times 2 + (75-1) \times 3]$
$= 8475$

(c)

(i) $y = x\tan 3x \quad (=uv)$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= x\,d\frac{\tan 3x}{dx} + \tan 3x\frac{dx}{dx}$

$= 3x\sec^2 3x + \tan 3x$

(ii) $y = \frac{x^2}{1+\cos x} \quad (=\frac{u}{v})$

$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$

$u = x^2 \qquad \frac{dv}{dx} = 2x$

$v = 1 + \cos 2x \qquad \frac{dv}{dx} = -\sin x$

$\frac{dy}{dx} = \frac{2x(1+\cos x) - 2x(-\sin x)}{(1+\cos x)^2}$

(d) $y = \cos 2x$

$\frac{dy}{dx} = -2\sin 2x$

At $x = \frac{\pi}{6}$

$y = \cos\left(2 \times \frac{\pi}{6}\right)$

$= \cos\left(\frac{\pi}{3}\right)$

$= \frac{1}{2}$

$\frac{dy}{dx} = -2\sin\left(2 \times \frac{\pi}{6}\right)$

$= -2\sin\left(\frac{\pi}{3}\right)$

$= -2 \times \frac{\sqrt{3}}{2}$

$= -\sqrt{3}$

The required tangent has a gradient of $-\sqrt{3}$ and passes through the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$. Its equation is given by:

$y - y_1 = m(x - x_1)$

$y - \frac{1}{2} = -\sqrt{3}\left(x - \frac{\pi}{6}\right)$

$= -\sqrt{3}\,x + \sqrt{3}\,\frac{\pi}{6}$

$y = -\sqrt{3}\,x - \sqrt{3}\,\frac{\pi}{6} + \frac{1}{2}$

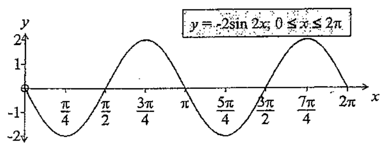
$y = -\sqrt{3}\,x + \left(\frac{1}{2} - \frac{\pi\sqrt{3}}{6}\right)$

QUESTION 2

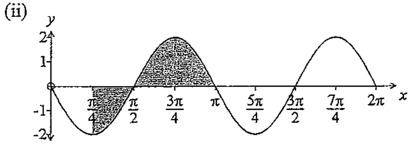
(a) $\int 3\sec^2 x\,dx = 3\tan x + C$

(b)

(i)



(ii)



Area $= \left|\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(-2\sin 2x)\,dx\right| + \int_{\frac{\pi}{2}}^{\pi}(-2\sin 2x)\,dx$

$= \left|[\cos 2x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right| + [\cos 2x]_{\frac{\pi}{2}}^{\pi}$

$= \left|\cos 2\times\frac{\pi}{2} - \cos 2\times\frac{\pi}{4}\right| + \cos 2\times\pi - \cos 2\times\frac{\pi}{2}$

$= \left|\cos\pi - \cos\frac{\pi}{2}\right| + \cos 2\pi - \cos\pi$

$= |-1-0| + 1 - (-1)$

$= 1+1+1$

$= 3\text{ unit}^2$

(b) Series: $1 + \sqrt{2} + 2 + \ldots$

(i) If the series is geometric then:

$\frac{T_2}{T_1} = \frac{T_3}{T_2}$ (= common ratio)

LHS $= \frac{\sqrt{2}}{1}$

$= \sqrt{2}$

RHS $= \frac{2}{\sqrt{2}}$

$= \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{2\sqrt{2}}{2}$

$= \sqrt{2}$

$=$ LHS

∴ The series is geometric with $a = 1$ and $r = \sqrt{2}$

(ii) For a geometric series:

$S_n = \frac{1(r^n - 1)}{r - 1}$

$S_{10} = \frac{1((\sqrt{2})^{10} - 1)}{1 - \sqrt{2}}$

$= \frac{\left(2^{\frac{1}{2}}\right)^{10} - 1}{\sqrt{2} - 1}$

$= \frac{2^5 - 1}{\sqrt{2} - 1}$

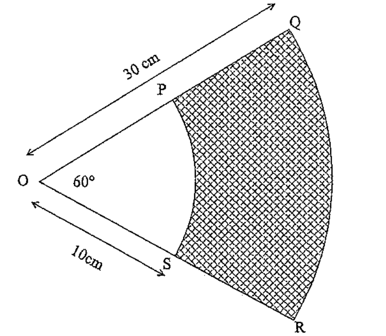
$= \frac{31}{\sqrt{2} - 1}$

$= \frac{31}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}$

$= \frac{31(\sqrt{2} + 1)}{2 - 1}$

$= 31(\sqrt{2} + 1)$

(d)



$\angle POS = 60^\circ = \frac{\pi}{3}$ randians

(i)

Area of a sector $= \frac{1}{2}r^2\theta$ (θ in radians)

Area PQRS = Area sector OQR – Area sector OPS

$= \frac{1}{2} \times 30^2 \times \frac{\pi}{3} - \frac{1}{2} \times 10^2 \times \frac{\pi}{3}$

$= \frac{1}{2} \times \frac{\pi}{3} \times (30^2 - 10^2)$

$= \frac{\pi}{6} \times 800$

$= \frac{400\pi}{3}\text{ cm}^2$

(ii)

Perimeter of PQRS

$=$ PQ + Arc QR + RS + Arc SP

$= (30-10) + 30 \times \frac{\pi}{3} + (30-10) + 10 \times \frac{\pi}{3}$

$= 20 + \frac{30\pi}{3} + 20 + \frac{10\pi}{3}$

$= \left(40 + \frac{40\pi}{3}\right)$ cm

QUESTION 3

(a) $2\cos 2x = 1; \quad 0 \le x \le 2\pi$

First solve for $2x$ in $0 \le 2x \le 4\pi$

$2\cos 2x = 1$

$\cos 2x = ½ \Rightarrow$ 1st and 4th quadrants

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

So

$$2x = \frac{\pi}{3}, \left(2\pi - \frac{\pi}{3}\right), \left(2\pi + \frac{\pi}{3}\right), \left(4\pi - \frac{\pi}{3}\right)$$

$$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

Now find x $0 \le x \le 2\pi$

$$2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

$(\div 2)$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

(b)

(i) $(k+3) + k + 4 + \ldots$ is a geometric series.

Thus

$$\frac{T_2}{T_1} = \frac{T_3}{T_2} \ (= r)$$

$$\frac{k}{k+3} = \frac{4}{k}$$

$$k^2 = 4(k+3)$$

$$k^2 = 4k + 12$$

$$k^2 - 4k - 12 = 0$$

$$(k+2)(k-6) = 0$$

$$k = -2,\ 6$$

(ii) For a limiting sum $|r| < 1$

For $k = -2$:

$r = \frac{4}{-2} = -2 \Rightarrow$ No limiting sum

For $k = 6$

$r = \frac{4}{6} = \frac{2}{3} \Rightarrow$ Limiting sum

The series has a limiting sum when $k = 6$.

(iii) $k = 6,\ r = \frac{2}{3}$

$$S = \frac{a}{1-r}$$

$$= \frac{6+3}{1 - \frac{2}{3}}$$

$$= \frac{9}{\frac{1}{3}}$$

$$= 27$$

The limiting sum is 27.

(c)

(i)

P = \$150

r = 8% pa

n = 18 yrs

$$A = P(1 + r^n)$$

$$= \$150 \times 1 \cdot 08^{18}$$

$$= \$599{\cdot}40$$

(ii)

Total amount in account

$$= (1{\cdot}08^{18} + 1\ 08^{17} + 1\ 08^{16} + \ldots + 1\ 08) \times \$150$$

$$= \$150 \times (1{\cdot}08 + 1{\cdot}08^2 + \ldots + 1{\cdot}08^{17} + 1{\cdot}08^{18})$$

Now

$$1{\cdot}08 + 1\ 08^2 + \ldots + 1{\cdot}08^{17} + 1{\cdot}08^{18}$$

is a geometric series with

$a = 1{\cdot}08$

$r = 1{\cdot}08$

$n = 18$

$$S_{18} = \frac{1{\cdot}08(1{\cdot}08^{18} - 1)}{1{\cdot}08 - 1}$$

$\therefore$ Total amount in account

$$= \$150 \times \frac{1{\cdot}08(1{\cdot}08^{18} - 1)}{1{\cdot}08 - 1}$$

$$= \$6066\ 94$$