Student: ____________________

Teacher: ____________________

BRIGIDINE COLLEGE RANDWICK

HSC MATHEMATICS

YEAR 12

ASSESSMENT TASK

JUNE 2008

(TIME: 45 MINUTES)

Directions to candidates:

- Write your **name** at the top of this question paper and each of the **3 sections** to be handed in.
- All 3 questions are to be attempted.
- All 3 questions are of equal value.
- All questions are to be answered on **separate pages** and will be collected **separately at the conclusion of this exam.**
- Pen should be used and all necessary working should be shown for every question.
- Full marks may not be awarded for careless or badly arranged work.

QUESTION 1 *(Start a new page)*

(a) Differentiate $4e^{2x}$ (1)

(b) Determine $\int e^{3x}\,dx$ (1)

(c)

(i) Sketch the curve $y = e^{-x}$. On your sketch shade the region bound by the curve, the x-axis, the line $x = 0$, and the line $x = 1$. (1)

(ii) Find the volume of the solid of revolution when the region in (i) is rotated about the x-axis. (3)

(d) The function $g(x)$ is defined over the domain $x \geq 0$ as $g(x) = \dfrac{A}{2 + e^{-x}}$, where A is a constant.

(i) At $x = 0$, $g(x)$ has a value of 3×10^5. Determine the value of A. (1)

(ii) What is the value of $g(x)$ when $x = 1$? Express your answer to the nearest integer. (1)

(iii) When x is very large what would you expect a close approximation to the value of $g(x)$ to be? Justify your answer. (2)

(iv) Find an expression for $g'(x)$. (2)

QUESTION 2 *(Start a new page)*

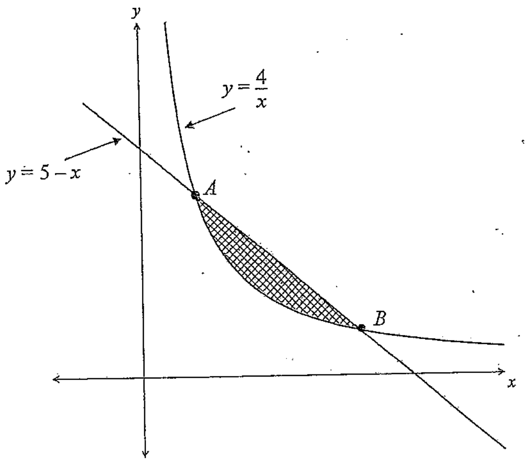
(a) Solve the equation $2\log_5 3 = \log_5 x - \log_5 6$. (2)

(b) Consider the function $f(x) = x - 3\log_e x$, where $1 \le x \le 7$.

(i) There is one turning point for $f(x)$. Find its coordinates and determine whether it is a maximum or minimum turning point. (3)

(ii) What is the maximum value of the function $f(x)$, where $1 \le x \le 7$. (2)

(c)



The diagram shows the graphs of $y=\frac{4}{x}$ and $y=5-x$. The graphs intersect at the points A and B as shown.

(i) Determine the coordinates of A and B. (2)

(ii) Calculate the area of the shaded region between $y=\frac{4}{x}$ and $y=5-x$. (3)

QUESTION 3 *(Start a new page)*

(a) Differentiate $\frac{\cos x}{x}$ (2)

(b) Solve the equation $2\cos x = \sqrt{3}$ where $0 \le x \le 2\pi$. (2)

(c)

(i) Graph $y = \cos x$, for $0 \le x \le 2\pi$. (1)

(ii) On your diagram shade the regions bounded by the curve $y = \cos x$, the x-axis, and the lines $x = 0$ and $x = 2\pi$. (3)

Calculate the total area of these regions.

(d)

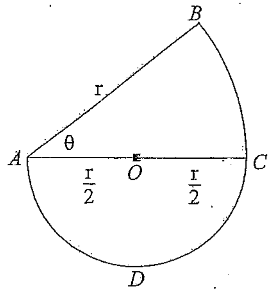


FIGURE NOT TO SCALE

The shape shown above is made up of the semi-circle ADC with centre O and radius $\frac{r}{2}$ together with the sector ABC of radius r, centre A and angle θ radians.

(i) What is the perimeter $ABCD$ of the shape in terms of r and θ? (1)

(ii) If the area of $ABCD$ is 1 square unit show the angle θ is given by $\theta = \frac{2}{r^2} - \frac{\pi}{4}$. (3)

Hence show the perimeter, P, is given by $P = \frac{2}{r} + r\left(1 + \frac{\pi}{4}\right)$.

END OF EXAMINATION PAPER

Mathematics yr 12 HSC
Marking Scheme June 2008

Q1a) (1 mark) $8e^{2x}$

b) (1 mark) $\frac{e^{3x}}{3} + c$

c) i) see solutions (MUST show $y=1$)

ii) (3 marks) $V = -\frac{\pi}{2}e^{-2} + \frac{\pi}{2}$

or $V = 0.4323\pi$

(2 marks) $V = \pi\left[\frac{-e^{-2x}}{2}\right]_0^1$

or $V = -\frac{1}{2}e^{-2} + \frac{1}{2}$ (forgot π)

(1 mark) $V = \pi\int_0^1 e^{-2x}\,dx$

or $V = \pi\int_0^1 (e^{-x})^2\,dx$

or $V = \pi\left[-e^{-1}+1\right]$ ← students forgot y^2

or any correct exponential integration (except $\int e^x dx$)

d) i) $A = 9\times10^5$ (1 mark)

ii) (1 mark) $g(x) = 380087$

or using/showing incorrect A but correctly solved.

iii) (2 marks) 450000

or $A \div 2$ = correctly solved.

(1 mark) saying $\lim_{x\to\infty} e^{-x} \Rightarrow 0$

iv) (2 marks) $A(2+e^{-x})^{-2}e^{-2}$

A or correct numeral from (i)

(1 mark) some correct working

Q2a) (2 marks) $x = 54$

(1 mark) $\log_5 3^2 = \log_5\left(\frac{x}{6}\right)$

b i) (3 marks) $(3, 3-3\ln 3)$ or $(3, -0.295)$ with correct test.

(2 marks) $x = 3$ (correctly solved)

or $f'(x) = 1-\frac{3}{x}$ and $f''(x) = 3x^{-2}$

(1 mark) $f'(x) = 1-\frac{3}{x}$

or solving incorrect 1st derivative and correctly testing

or wrong x correctly ~~sub~~ substituted for y.

or one mistake.

b ii) (2 marks) 1·2 or $7-3\ln 7$

(1 mark) $f''(x) = \frac{3}{x^2}$ always >0

or testing $x=7$ but wrong answer for y ~~and saying this~~

or $(7, 7-3\ln 7)$

3a) (2 marks) $\frac{-x\sin x - \cos x}{x^2}$

(1 mark) correct numerator or correct denominator using quotient rule.

b) (2 marks) $x = \frac{\pi}{6}, \frac{11\pi}{6}$

(1 mark) $x = \frac{\pi}{6}$ or correct 2nd value in 4th quad from incorrect initial x.

or $x = 30°$ and $330°$

c) i) (1 mark) correct cos curve with both axes fully labelled.

ii) (3 marks) 4 (correctly found)

(2 marks) $4\times\left[\sin x\right]_0^{\frac{\pi}{2}}$

or $\left.\sin x\right]_0^{\frac{\pi}{2}} + \left|\left.\sin x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right| + \left.\sin x\right]_{\frac{3\pi}{2}}^{2\pi}$

or $A = 2$ units^2 found from area in 1st + 4th quads.

(1 mark) $4\int_0^{\frac{\pi}{2}} \cos x\,dx$

or integrating to get $\sin x$.

c) i) (2 marks) $A(1,4)$ and $B(4,1)$

(1 mark) $A(1,4)$ or $B(4,1)$

or $A(4,1)$ and $B(1,4)$

or wrong pair of x's correctly substituted to get y.

ii) (3 marks) $7\frac{1}{2} - 4\ln 4$

(2 marks) $5x - \frac{x^2}{2} - 4\ln x$

Note: must have a ln in their correct calculations to get 2 marks

(1 mark) $\int_1^4 5 - x - \frac{4}{x}\,dx$

or "$\ln x$" in their integral with no other ln's.

d i) $P = r + r\theta + \frac{\pi r}{2}$

ii) (3 marks) correctly showing all steps to final given answer.

(2 marks) showing $\theta = \frac{2}{r^2} - \frac{\pi}{4}$ and then substituting into (i) answer

(1 mark) substituting $\theta = \frac{2}{r^2} - \frac{\pi}{4}$ into part (i) answer

or $\frac{1}{2}r^2\theta + \frac{1}{2}\pi\times\left(\frac{r}{2}\right)^2 = 1$

FORTITER ET SUAVITER

BRIGIDINE COLLEGE RANDWICK

HSC Mathematics Assessment Task
June, 2008

SUGGESTED SOLUTIONS

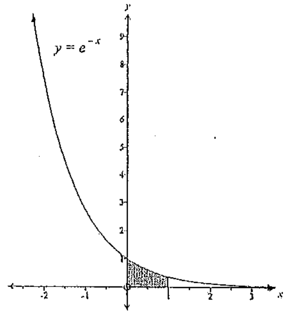
QUESTION 1

(a) $\dfrac{d\,4e^{2x}}{dx} = 4e^{2x}\dfrac{d\,2x}{dx}$

$= 8e^{2x}$

(b) $\displaystyle\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$

(c) (i)



(ii) $V = \pi\displaystyle\int y^2\,dx$

$= \pi\displaystyle\int_0^1 (e^{-x})^2\,dx$

$= \pi\displaystyle\int_0^1 e^{-2x}\,dx$

$= \pi\left[-\frac{1}{2}e^{-2x}\right]_0^1$

$= -\frac{\pi}{2}[e^{-2} - e^{0}]$

$= -\frac{\pi}{2}[e^{-2} - 1]$

$= \frac{\pi}{2}[1 - e^{-2}]$

$= \frac{\pi}{2}\left[1 - \frac{1}{e^2}\right]$

The volume of the solid of revolution is

$\frac{\pi}{2}\left[1 - \frac{1}{e^2}\right]$ unit^3.

(d) (i) $g(0) = 3 \times 10^5$

$\therefore \dfrac{A}{2 + e^0} = 3 \times 10^5$

$\dfrac{A}{3} = 3 \times 10^5$

$A = 9 \times 10^5$

(ii) From (i)

$g(x) = \dfrac{9 \times 10^5}{2 + e^{-x}}$

$g(1) = \dfrac{9 \times 10^5}{2 + e^{-1}}$

$= 380\,087$ (to nearest integer)

(iii) As x becomes very large, e^{-x} becomes very small (negligibly small) and $2 + e^{-x}$ approaches 2.

ie $\lim\limits_{x \to \infty}(2 + e^{-x}) = 2$

and $\lim\limits_{x \to \infty} g(x) = \lim\limits_{x \to \infty} \dfrac{9 \times 10^5}{2 + e^{-x}}$

$= \dfrac{9 \times 10^5}{2}$

$= 4{\cdot}5 \times 10^5$ or $450\,000$

(iv)

$g(x) = \dfrac{9 \times 10^5}{2 + e^{-x}}$

$= 9 \times 10^5\,(2 + e^{-x})^{-1}$

$g'(x) = 9 \times 10^5\,[(-1)(2 + e^{-x})^{-1-1}] \times \dfrac{d\,(2 + e^{-x})}{dx}$

$= 9 \times 10^5\,[(-1)(2 + e^{-x})^{-2}] \times (-e^{-x})$

$= 9 \times 10^5\,(2 + e^{-x})^{-2}e^{-x}$

$= \dfrac{9 \times 10^5}{e^{x}(2 + e^{-x})^2}$

QUESTION 2

(a) $2\log_e 3 = \log_5 x - \log_5 6$

$\log_5 3^2 = \log_5\left(\dfrac{x}{6}\right)$

$3^2 = \dfrac{x}{6}$

$\dfrac{x}{6} = 9$

$x = 54$

(b) $f(x) = x - 3\log_e x$, where $1 \le x \le 7$

(i) $f'(x) = 1 - 3 \div x$

For a turning point

$f'(x) = 0$

$1 - 3 \div x = 0$

$x - 3 = 0$

$x = 3$

$f''(x) = \dfrac{d(1 - 3x^{-1})}{dx}$

$= 3x^{-2}$

$f''(3) = \dfrac{3}{9} > 0 \Rightarrow$ Min turning point

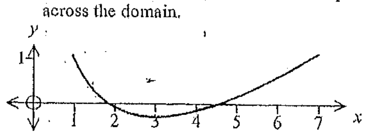
$f(3) = 3 - 3\log_e 3$

$= 3(1 - \log_e 3)$

The turning point is $(3,\ 3(1 - \log_e 3))$ and is a minimum

(ii) For $f(x) = x - 3\log_e x$, where $1 \le x \le 7$

- There is only one turning point, a minimum at $x = 3$;
- $f''(x) = \dfrac{3}{x^2}$ which is positive for all x in the domain $1 \le x \le 7 \Rightarrow f(x)$ is concave up across the domain.



Thus the maximum value of $f(x)$ must occur for either $x = 1$ or $x = 7$.

$f(1) = 1 - 3\log_e 1 \quad = 1$

$f(7) = 7 - 3\log_e 7 \quad \approx 1{\cdot}2 > f(1)$

$\therefore$ The maximum value of $f(x)$ in the domain $1 \le x \le 7$ is $f(7)$ i.e. $7 - 3\log_e 7 \approx 1{\cdot}2$

(c) (i) $y = \dfrac{4}{x}$ ①

$y = 5 - x$ ②

Sub for y from ② into ①

$5 - x = \dfrac{4}{x}$

$5x - x^2 = 4$

$x^2 - 5x + 4 = 0$

$(x - 1)(x - 4) = 0$

$x = 1$ or $x = 4$

$y = 4$ or $y = 1$

A is $(1, 4)$

B is $(4, 1)$

(ii)

Shaded Area $= \displaystyle\int_1^4 \left(5 - x - \frac{4}{x}\right)dx$

$= \left[5x - \dfrac{x^2}{2} - 4\log_e x\right]_1^4$

$= \left[5 \times 4 - \dfrac{4^2}{2} - 4\log_e 4\right]$

$- \left[5 \times 1 - \dfrac{1^2}{2} - 4\log_e 1\right]$

$= \left(7\tfrac{1}{2} - 4\log_e 4\right)$ unit^2

QUESTION 3

(a)

$\dfrac{d\left(\dfrac{\cos x}{x}\right)}{dx}$ $\qquad$ $\dfrac{d\left(\dfrac{u}{v}\right)}{dx} = \dfrac{u'v - uv'}{v^2}$

$= \dfrac{(-\sin x) \times x - 1 \times \cos x}{x^2}$ $\qquad$ $u = \cos x \quad u' = -\sin x$

$= -\dfrac{x\sin x + \cos x}{x^2}$ $\qquad$ $v = x \quad v' = 1$

(b)

$2\cos x = \sqrt{3}$ $\qquad 0 \le x \le 2\pi$

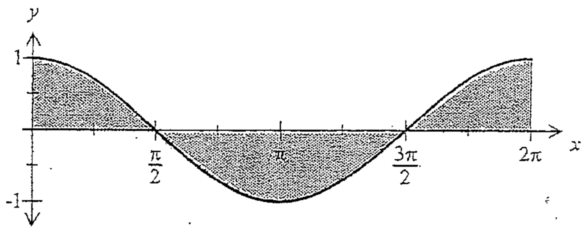
$\cos x = \dfrac{\sqrt{3}}{2}$ $\qquad x$ in 1st or 4th quadrants

As $\cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$

$x = \dfrac{\pi}{6},\ \left(2\pi - \dfrac{\pi}{6}\right)$

i.e. $x = \dfrac{\pi}{6},\ \dfrac{11\pi}{3}$

(c) (i)



Shaded Area

$$= \int_0^{\frac{\pi}{2}} \cos x \, dx + \left| \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \, dx \right|$$

(ii)

$$+ \int_{\frac{3\pi}{2}}^{2\pi} \cos x \, dx$$

$$= \left[\sin x \right]_0^{\frac{\pi}{2}} + \left| \left[\sin x \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \right| + \left[\sin x \right]_{\frac{3\pi}{2}}^{2\pi}$$

$$= \sin \frac{\pi}{2} - \sin 0 + \left| \sin \frac{3\pi}{2} - \sin \frac{\pi}{2} \right|$$

$$+ \sin 2\pi - \sin \frac{3\pi}{2}$$

$$= 1 + 0 + |-1 - 1| + 0 - -1$$

$$= 4 \text{ units}^2$$

(d) (i) Perimeter = AB + Arc BC + Arc CDA

$$= r + r\theta + \frac{r}{2} \times \pi$$

$$= r\left(1 + \theta + \frac{\pi}{2}\right)$$

(ii) Area $ABCD = 1$ unit^2

Also Area = Sector ABC + Semicircle ACD

$$= \frac{1}{2} r^2 + \frac{1}{2} \times \left(\frac{r}{2}\right)^2 \times \pi$$

$$= \frac{1}{2} r^2 + \frac{1}{8} r^2 \pi$$

$$= \frac{r^2}{2}\left(\theta + \frac{\pi}{4}\right)$$

So

$$\frac{r^2}{2}\left(\theta + \frac{\pi}{4}\right) = 1$$

$$\theta + \frac{\pi}{4} = \frac{2}{r^2}$$

$$\theta = \frac{2}{r^2} - \frac{\pi}{4}$$

Consequently

$$P = r\left(1 + \theta + \frac{\pi}{2}\right)$$

$$= r\left(1 + \frac{2}{r^2} - \frac{\pi}{r} + \frac{\pi}{2}\right)$$

$$= r\left(1 + \frac{2}{r^2} + \frac{\pi}{4}\right)$$

$$= r + \frac{2}{r} + \frac{\pi}{4} r$$

$$= \frac{2}{r} + r\left(1 + \frac{\pi}{4}\right)$$

*Alternate answer to question

$$A = 4 \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= 4 \left[\sin x \right]_0^{\frac{\pi}{2}}$$

$$= 4 \left[\sin \frac{\pi}{2} - \sin 0 \right]$$

$$= 4 \text{ units}^2$$