Burwood Girls

PRELIMINARY HSC EXAMINATION

1999

MATHEMATICS
3 UNIT ADDITIONAL

Time Allowed - One and a half hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- Start each question on a new page.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board approved calculators may be used.
- Clearly mark each question (Question 1, Question 2, ... etc) and each question part.
- Write your Student Name/Number on every page.
- *This question paper must not be removed from the examination room.*

STUDENT NUMBER/NAME: ....

Question 1 (Start a new page) **Marks**

a. Find the exact value of tan 15°. **2**

b. The equation $x^2 - (1 - 2k)x + k + 3 = 0$ has consecutive roots. Find the value of k. **3**

c. Solve the inequality $\dfrac{x}{2x-1} \le 5$. **3**

d. i. In how many ways can 8 committee members be selected from 12 people? **4**

ii. Two people say that they will only serve as committee members if both are selected. Otherwise, neither will serve. In how many ways can this be done to satisfy BOTH these conditions?

Question 2 (Start a new page)

a. Find the equation of the line through the point of intersection of the lines $2x + 3y - 7 = 0$ and $x - 2y + 1 = 0$ and perpendicular to the line $y = 1 - 3x$. **3**

b. i. On the same axes, sketch the curves $y = x^2$ and $y = |x|$ **3**

ii. Hence, or otherwise, solve $x^2 < |x|$

c. The diagram shows the line interval AB *trisected* at P and Q. The coordinates of A are $(-6, -2)$ and of Q are $(2, 2)$. Find the coordinates of B. **2**

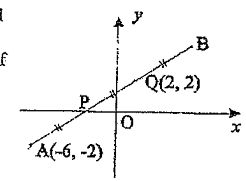


Diagram not to scale

$x^2 < |x|$

$x^4 < x^2$

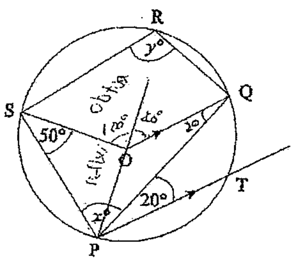
Question 2 is continued on the next page

Question 2 (continued)

Marks

d. In the diagram, O is the centre of the circle and $PT \parallel OQ$. Find the values of x and y, giving reasons. 4

Diagram not to scale



$x° = 70°$

Question 3 (Start a new page)

a. The lines $3x - y + 2 = 0$ and $mx - y - 1 = 0$ intersect at 45°. Find the possible value(s) of m. 3

b. For the parabola $y = x^2 - 4x - 1$, write down the coordinates of the focus and the equation of the directrix. 2

c. Find the equation of the tangent to the curve $y = 2x\sqrt{x+1}$ at the point where $x = 3$. 4

d. Make n the subject of the formula $S = \frac{n}{2}[2a + (n - 1)d]$. 3

Question 4 (Start a new page)

a. A man, M, is standing in a horizontal plane which also contains a tower, AB, which is 10 metres high, and a building, XY, which is 95 metres high. The man is 210 metres from the tower and his bearing from it is 42°. His bearing from the building is 110°. The bearing of the tower from the building is 150°. 4

Find the angle of elevation from the top of the tower to the top of the building.

Question 4 is continued on the next page

Question 4 (continued)

Marks

b. The polynomial, $P(x)$, is given by $P(x) = x^4 + 6x^3 + 5x^2 - 12x$. 5

i. Find $P(-4)$.

ii. Hence or otherwise find the factors of $P(x)$.

iii. Draw a neat sketch of $y = P(x)$ in the domain $-5 \le x \le 2$.

iv. Hence, or otherwise, solve $x^4 + 6x^3 + 5x^2 - 12x \le 0$

c. Sketch the inequality $y \le \sqrt{16 - x^2}$ 3

Question 5 (Start a new page)

a. i. In how many ways can the letters of the word MATHS be arranged? 3

ii. If it does not matter whether the "word" formed makes sense,

1. how many three letter "words" can be formed from the word MATHS if no repetitions are allowed?

2. how many three letter "words" can be formed if repetitions are allowed?

b. i. Show that $\frac{\sin^3\theta}{\cos\theta} + \sin\theta\cos\theta = \tan\theta$. 3

ii. Hence, or otherwise, solve $\frac{\sin^3\theta}{\cos\theta} = 1 - \sin\theta\cos\theta$ for $0° \le \theta \le 360°$

c. i. Find the Cartesian equation of the parabola $x = 4t$, $y = 2t^2$. 6

ii. Show that the equation of the chord of contact from the point T(3, −2) is $3x - 4y + 8 = 0$

iii. Find the coordinates of the points P and Q where the tangents from T touch the parabola.

iv. Find the gradients of the tangents at P and Q.

v. What conclusion can you draw about the chord of contact?

Question 6 (Start a new page) **Marks**

a. A is the point (5, 0) and O is the origin. Given that the point B(x, y) lies on the line $y = 1 - 3x$ and that OB is perpendicular to AB, find the coordinates of B. **4**

b. In the diagram, $\angle A > \angle B > 0$ and $\dfrac{\cos(A + B)}{\cos(A - B)} = \dfrac{4}{5}$, **8**

i. Show that $\tan A \,.\, \tan B = \dfrac{1}{9}$

ii. If $PR = QR$, show that

$9(\tan A + \tan B) = 8$

iii. Hence find A and B.

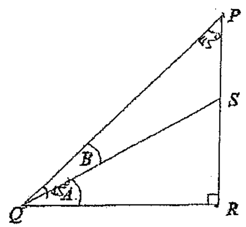


Diagram not to scale

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1.(a) $\tan 15$

$= \tan(\pi/4 - \pi/6)$

$= \dfrac{\tan \pi/4 - \tan \pi/6}{1 + \tan \pi/4 \tan \pi/6}$ ✓

$= \dfrac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$

$= \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ ✓

(b). $x^2 - (1-2k)x + k + 3$.

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$\alpha, \alpha+1$

$\Sigma_\alpha = 2\alpha + 1 = 1 - 2k$

$\alpha\beta = \alpha(\alpha+1) = k+3$

$k = \alpha(\alpha+1) - 3$.

$\therefore 2\alpha + 1 = 1 - 2(\alpha(\alpha+1) + 3)$

$2\alpha + 1 = 7 - 2\alpha^2 - 2\alpha$

$2\alpha^2 + 4\alpha - 6 = 0$. ✓

$\alpha^2 + 2\alpha - 3 = 0$.

$\alpha = 1, \alpha = -3$.

Let $\alpha = 1$: $-2 = k+3$ ✓, $k = -1$

Let $\alpha = -3$: $-5 = 1 - 2k$ ✓, $k = 3$.

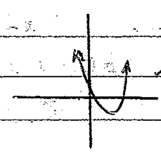
c). $\dfrac{x}{2x-1} \leq 5$ $\quad x \neq 1/2$.

$x(2x-1) \leq 5(2x-1)^2$

$2x^2 - x \leq 5(4x^2 - 4x + 1)$ ✓

$2x^2 - x \leq 20x^2 - 20x + 5$.

$18x^2 - 19x + 5 \geq 0$.



$(9x-5)(2x-1) \geq 0$. ✓

$x < 1/2 \cap x \geq 5/9$ ✓

d) (i) $\binom{12}{8} = 495$. ✓

(ii). Case 1 – Both in ~~+ Case 2 – Both out~~

$\binom{10}{6}$ ✓ ~~$\binom{10}{8}$~~

$= 255$.

2.(a). $2x + 3y - 7 = 0$. —①

$x - 2y + 1 = 0$. —② ✓

$x = 2y - 1$

$4y - 2 + 3y - 7 = 0$.

$7y - 9 = 0$.

$9/7 = y$ ✓

$\therefore x = 11/7$.

or use the 'k' method

i.e. Eqn. is given by

$2x + 3y - 7 + k(x - 2y + 1) = 0$

then use the gradient given.

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$y = 1 - 3x$

$m_1 = -3$, $m_2 = 1/3$

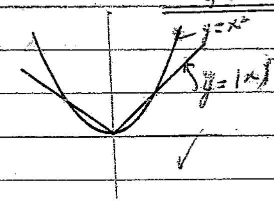
$(y - 9/7) = 1/3(x - 11/7)$

$y - 9/7 = \frac{x}{3} - 11/21$ ✓

$y = x/3 + \frac{16}{21}$ ✓

$21y = 7x + 16$.

(b). (i).



(ii). $-1 < x < 1$ ✓

(c). $A(-6, -2)$, $Q(2, 2)$

$P: x = -2, y = 0$.

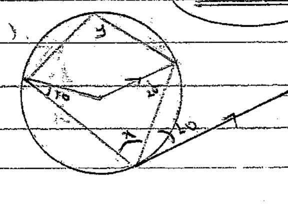
$= (-2, 0)$

$\therefore B(6, 4)$?

Using division of lines, i.e.

Let B be the pt of division of AQ externally in the ratio 3 : 1

(d).



Quad SOQP = 2π

$S\hat{O}Q = 2\pi - 2x$ (Double Angle at centre)

$\therefore 50 + 20 + x + ~~2\pi~~ - 2x = ~~2\pi~~$

$70 - x = 0$.

$x = 70$. ✓

$\therefore y = 110$ ✓ (cyclic quad)

3.(a) $y = x^2 - 4x - 1$

$y + 5 = x^2 - 4x + 4$

$y + 5 = (x-2)^2$ ✓ $a = \frac{1}{4}$

V: $(2, -5)$ ✓ F: $(2, -4\frac{3}{4})$ D: $y = -5\frac{1}{4}$ ✓

$y = 2x\sqrt{x+1}$

$x = 3, \quad y = 6 \times 2$
$= 12.$

$y = 2x\,(x+1)^{1/2} \qquad u = 2x \qquad v = (x+1)^{1/2}$

$y' = uv' + vu' \qquad u' = 2 \qquad v' = \frac{1}{2}(x+1)^{-1/2}$

$= 2(x+1)^{1/2} + \frac{x}{\sqrt{x+1}}$

at $x = 3$.

$y' = 2 \times 2 + \frac{3}{2}$

$= \cancel{5} \cdot 5\frac{1}{2}$

$y - 12 = 5\frac{1}{2}(x-3) \qquad 2y - 24 = 11x - 33$

$11x - 2y - 9 = 0$

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(a). $S = \frac{n}{2}[2a + (n-1)d]$

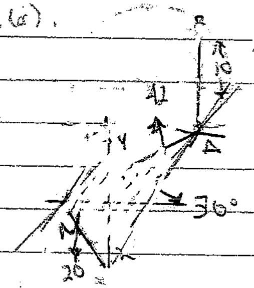
$2S = 2an + dn^2 - dn$

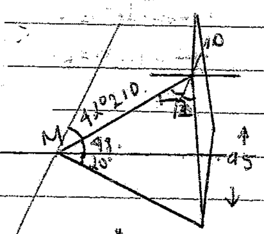
$= dn^2 - dn + 2an = 2S$

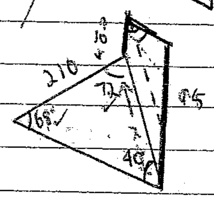
$dn^2 + n(2a-d) - 2S = 0$

$n = \frac{-(2a-d) \pm \sqrt{(2a-d)^2 + 8Sd}}{2d}$

4. (a).



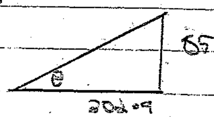




$\frac{210}{\sin 40} = \frac{x}{\sin 65}$

$\frac{210 \sin 65}{\sin 40} = x.$

$x = 302 \cdot 9$



$\therefore \theta = \tan^{-1}\left(\frac{85}{302\cdot 9}\right)$

$\theta = 15°10'$

(i) $P(x) = x^4 + 6x^3 + 5x^2 - 12x$

$P(-4) = 0.$

(ii) $P(0) = 0.$

$x(x+4)$ is a factor.

$$\begin{array}{r} x^2 + 2x - 3 \\ x^2+4x \overline{)\, x^4 + 6x^3 + 5x^2 - 12x} \\ x^4 + 4x^3 \\ \hline 2x^3 + 5x^2 \\ 2x^3 + 8x^2 \\ \hline -3x^2 - 12x \\ -3x^2 - 12x \\ \hline 0 \end{array}$$

$P(x) = (x^2+4x)(x^2+2x-3)$

$= x(x+4)(x+3)(x-1)$

Roots: $0, -4, -3, 1$

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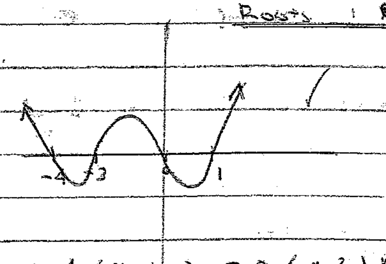
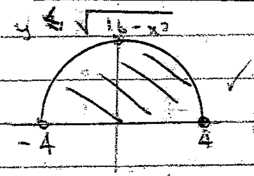
(iii).



(iv) $-4 \le x \le -3$ n $0 \le x \le 1$

(v) $y \le \sqrt{16 - x^2}$



$y \le \sqrt{16 - x^2}$

$y^2 \le 16 - x^2$

5.(a)(i) MATHS

$= 5! = 120$

(ii). $\binom{3}{?} \times 3! = 60$

(iii). 3 of the same + 2/1 + 1 of each $\boxed{5 \mid 5 \mid 5}$

$\binom{?}{?} \times \quad = \binom{?}{?} \times \binom{4}{1} \times 2! \times 1! + \binom{5}{3} \times 3! \quad = 125$

$= 105$ X

(b). LHS $\frac{\sin^3\theta}{\cos\theta} + \sin\theta\cos\theta = \frac{\sin^3\theta + \sin\theta\cos^2\theta}{\cos\theta}$ Let $t = \tan\frac{\theta}{2}$ $= \frac{\sin\theta}{\cos\theta}(\sin^2\theta + \cos^2\theta)$

$= \cancel{\sin^3\theta + \sin\theta\cos^2\theta}$ $\quad \frac{\left(\frac{2t}{1+t^2}\right)^3}{\frac{1-t^2}{1+t^2}} \quad + \frac{2t(1-t^2)}{(1+t^2)^2} \quad = \tan\theta = R.H.S.$

$$\frac{8t^3}{(1-t^2)(1+t^2)^2} + \frac{2t(1-t^2)}{1+t^2}$$

$$= \frac{8t^3 + 2t(1-t^2)^2}{(1-t^2)(1+t^2)^2}$$

$$= \frac{8t^3 + 2t - 4t^3 + 2t^5}{(1-t^2)(1+t^2)^2}$$

$$= \frac{t(2t^4 + 4t^2 + 2)}{(1-t^2)(1+t^2)^2}$$

Too long!

$$= \frac{2t(1+t^2)^2}{(1-t^2)(1+t^2)^2}$$

$$= \frac{2t}{1-t^2}$$

$$= \tan x$$

(ii) $\frac{\sin^2\theta}{\cos\theta} + \cos\theta\sin\theta = \tan\theta = 1.$

$\therefore \tan\theta = 1$

$\theta = \pi/4, \frac{5\pi}{4}$

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5) (i) $x = 4t,\ y = 2t^2$

$t = \frac{x}{4}$

$y = 2\left(\frac{x^2}{16}\right)$

$y = \frac{x^2}{8} \implies 8y = x^2 \implies a = 2.$

(ii). Chord of Contact : $xx_0 = 2a(y + y_0)$

$(3, -2) = x_0, y_0$

$3x = 4(y-2)$

$3x = 4y + 8$

$3x - 4y + 8 = 0.$

(iii) $3x - \frac{4x^2}{8} + 8 = 0.$

$6x - x^2 + 16 = 0.$

$x^2 - 6x - 16 = 0.$

$x = -2, 8.$

$y = 1/2, 8.$

$P(-2, 1/2),\ Q(8, 8)$

(iv). $8y = x^2$

$y = \frac{x^2}{8}$

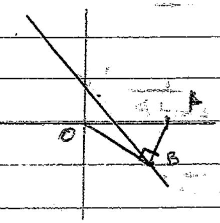
$\frac{dy}{dx} = \frac{x}{4}.$

At $x = -2$ $\quad x = 8$

$m = -1/2 \quad m = 2.$

(v). Focal Chord

6. (a).



$m_{AB} = \frac{y-0}{x-5} = \frac{y}{x-5}.$

$m_{OB} = \frac{y}{x}$

$m_{AB} \times m_{OB} = -1$

$\frac{y}{x-5} \times \frac{y}{x} = -1$

12

$y^2 = -(x^2 - 5)$

$y^2 = -x^2 + 5$

$(1 - 3x)^2 = -x^2 + 5.$

$1 - 6x + 9x^2 = -x^2 + 5.$

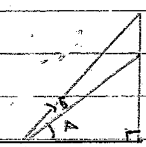
$10x^2 - 6x - 4 = 0.$

$x = 1, \frac{-2}{5}$

$y = -2, 11/5.$ $\quad \therefore B = (1, -2)$

$(1, -2)\ \&\ \left(\frac{-2}{5}, 11/5\right)$

(b).



$$\frac{\cos(A+B)}{\cos(A-B)} = \frac{\cos A\cos B - \sin A\sin B}{\cos A\cos B + \sin A\sin B} = \frac{4}{5}.$$

$5\cos A\cos B - 5\sin A\sin B = 4\cos A\cos B + 4\sin A\sin B$

$\cos A\cos B = 9\sin A\sin B$

$\frac{1}{9} = \frac{\sin A\sin B}{\cos A\cos B}$

$= \tan A\tan B$

(ii). $A+B=45$.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1.$$

$$\frac{\tan A + \tan B}{1 - \frac{1}{9}} = 1$$

$$9(\tan A + \tan B) = 8.$$

(iii). $\tan A = \dfrac{1}{9\tan B}$

$$9\left(\frac{1}{9\tan B} + \tan B\right) = 8$$

$$\frac{1}{\tan B} + 9\tan B = 8$$

$$9\tan^2 B - 8\tan B + 1 = 0.$$

~~Let tan B = x~~

$$\tan B = \frac{+8 \pm \sqrt{64-36}}{18}.$$

$$= 0.738 \text{ or } 0.1564$$

$B = 36°25'$ or. $8°35'$

$A = 8°35'$ or $36°28'$