Christian Brothers' High School Lewisham

Year 12 Term 1 2010

MATHEMATICS – EXTENSION 1
ASSESSMENT TASK

- All necessary working must be shown.
- Approved calculators may be used.
- Each question is of equal value.
- Begin each question in a new booklet.
- **Outcomes Assessed:** H1, H2, H3, H5, H8, H9, HE1, HE7
- Time allowed: 60 minutes

Question 1:

(i) Prove that $\log 2 + \log 4 + \log 8 + \log 16 + \ldots\ldots$ is an arithmetic series and find its sum to 60 terms. (Leave your answer in the form $k \log 2$, where k is a constant). [4 marks]

(ii) A cone of length h and radius r is generated by rotating the straight line $y = \frac{r}{h}x$ between $x = 0$ and $x = h$, about the *axis*. Show that the volume of the cone is given by $\frac{1}{3}\pi r^2 h$. [4 marks]

(iii) Solve for x to 3 significant figures if $10^{2x+1} = 5^{x-2}$. [4 marks]

Question 2:

(i) Find $\int x^2\sqrt{2+x^3}\,dx$ using the substitution $u = 2 + x^3$. [2 marks]

(ii) Prove by mathematical induction that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for $n = 1, 2, 3, \ldots\ldots$ [5 marks]

(iii) (a) Find the sum of the arithmetic series $1 + 3 + 5 + \ldots$ to n terms. [1 mark]

(b) Find the sum of the arithmetic series $2 + 4 + 6 + \ldots\ldots$ to n terms. [1 mark]

(c) Hence or otherwise find the sum of the first 100 terms of the series:
$1 - 2 + 3 - 4 + 5 - 6 + \ldots\ldots + (-1)^{n+1}n + \ldots\ldots$ [3 marks]

Question 3:

(i) Find the area of the region enclosed by the curves $f(x) = 4 - x^2$ and $g(x) = x^2 - 2x$ [4 marks]

(ii) Use the substitution $u = x^2 + 1$ to evaluate $\int_0^2 \frac{x}{(x^2+1)^3}dx$ [3 marks]

(iii) Use mathematical induction to prove that
$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \ldots\ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$ [5 marks]

for all positive integers n.

1. i) $\log 2 + \log 4 + \log 8 + \log 16 + \ldots$

$U_2 - U_1 = U_3 - U_2$

$\log 4 - \log 2 = \log 8 - \log 4$

$\log 2 = \log 2$

LHS = RHS

$\therefore$ Arithmetic series — 1

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ — 1

$S_{60} = \frac{60}{2}\left[2\log 2 + (60-1)\log 2\right]$ — 1

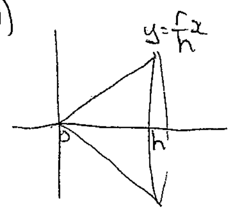
$= 30\left[2\log 2 + 59\log 2\right]$

$= 60\log 2 + 1770\log 2$

$= 1830\log 2$ — 1

4 marks

ii)



$V = \pi\int_0^h \left(\frac{r}{h}x\right)^2 dx$ — 1

$= \pi\int_0^h \frac{r^2}{h^2}x^2\,dx$

$= \pi\left[\frac{r^2}{3h^2}x^3\right]_0^h$ — 1

$= \frac{\pi r^2}{3h^2}\left[h^3 - 0\right]$ — 1

$= \frac{\pi r^2 h}{3}$

$= \frac{1}{3}\pi r^2 h$ — 1

4

i) $10^{2x+1} = 5^{x\mp 2}$

$\log_e 10^{2x+1} = \log_e 5^{x\mp 2}$

$(2x+1)\log_e 10 = (x\mp 2)\log_e 5$ — 1

$2x\log_e 10 + \log_e 10 = x\log_e 5 \mp 2\log_e 5$

$2x\log_e 10 - x\log_e 5 = -2\log_e 5 - \log_e 10$

$x(2\log_e 10 - \log_e 5) = -2\log_e 5 - \log_e 10$ — 1

$x = \dfrac{-2\log_e 5 - \log_e 10}{2\log_e 10 - \log_e 5}$ — 1

$= \cancel{0.30586536}\ -1.843108432$

$= \cancel{0.3059}\ -1.843$ — 1

2. i) $\int x^2\sqrt{2+x^3}\,dx$

Let $u = 2+x^3$

$\frac{du}{dx} = 3x^2$

$du = 3x^2\,dx$ 1

$\therefore \frac{1}{3}\int 3x^2\sqrt{2+x^3}\,dx = \frac{1}{3}\int u^{\frac{1}{2}}du$

$= \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}} + C$

$= \frac{2}{9}u^{\frac{3}{2}} + C$ 2

$= \frac{2}{9}\sqrt{(2+x^3)^3} + C$ 1

ii) RTP $n^3+(n+1)^3+(n+2)^3 = 9p$ where p is some integer

Step 1: Try $n=1$

$LHS = n^3+(n+1)^3+(n+2)^3$

$= 1^3+2^3+3^3$

$= 1+8+27$

$= 36$

$= 9\times 4$ where $p=4$

$\therefore$ true for $n=1$

Step 2: Assume true for $n=k$

ie. $k^3+(k+1)^3+(k+2)^3 = 9m$ where m is some integer 1

Step 3: R.T.P true for $n=k+1$

ie. $(k+1)^3+(k+2)^3+(k+3)^3 = 9Q$ where Q is an integer 1

Now $LHS = (k+1)^3+(k+2)^3+(k+3)^3$

$= 9m - k^3 + (k+3)^3$ since $(k+1)^3+(k+2)^3 = 9m-k^3$ (from Step 2)

$= 9m - k^3 + (k+3)(k+3)^2$

$= 9n - k^3 + (k+3)(k^2+6k+9)$

$= 9m - k^3 + k^3+6k^2+9k+3k^2+18k+27$ 1

$= 9m + 9k^2+27k+27$

$= 9(m+k^2+3k+3)$

$= 9Q$ where $Q = m+k^2+3k+3$ 1

$\therefore$ true for $n=k+1$

Step 4: If the statement is true for $n=1$, then it is true for $n=2$, If it is true for $n=2$, then it is true for $n=3$ and so on. $\therefore$ If true for $n=k$, then true for $n=k+1$ 1

$\therefore$ statement is true for $n=1,2,3,\ldots$

iii) a) $1+3+5+\ldots$

$S_n = \frac{n}{2}[2+(n-1)2]$

$= \frac{n}{2}[2+2n-2]$ 1

$= n^2$

b) $2+4+6+\ldots$

$S_n = \frac{n}{2}[4+(n-1)2]$

$= \frac{n}{2}[4+2n-2]$

$= \frac{n}{2}\cdot 2(1+n)$

$= n(1+n)$ 1

$= n^2+n$

c) $1-2+3-4+5-6+\ldots(-1)^{n+1}n+\ldots$

$(1+3+5+7+\ldots+99) + (-2-4-6-8\ldots-100)$

$S_{50} = \frac{50}{2}[1+99]$ $-$ $(2+4+6+\ldots+100)$ 2

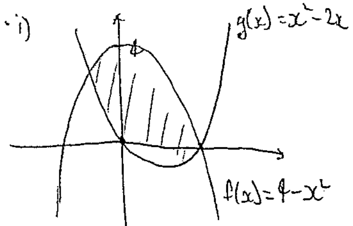
$= 2500$ $-$ $S_{50} = \frac{50}{2}[2+100]$

$= 2550$

$\therefore 2500 - 2550$ 1

$= -50$

3. i)



Solve simultaneously

$4-x^2=x^2-2x$

$0=2x^2-2x-4$

$0=x^2-x-2$

$0=(x-2)(x+1)$

$\therefore x=-1,2$ 1

$\therefore \text{Area}=\int_{-1}^{2}\left[(4-x^2)-(x^2-2x)\right]dx$ 1

$=\int_{-1}^{2}\left[4-x^2-x^2+2x\right]dx$

$=\int_{-1}^{2}\left[4-2x^2+2x\right]dx$

$=\left[4x-\frac{2x^3}{3}+x^2\right]_{-1}^{2}$ 1

$=\left[\left(8-\frac{16}{3}+4\right)-\left(-4+\frac{2}{3}+1\right)\right]$

$=6\frac{2}{3}+2\frac{1}{3}$

$=9\ u^2$ 1

ii) $\int_0^2 \frac{x}{(x^2+1)^3}\,dx$

Let $u=x^2+1$

$\frac{du}{dx}=2x$

$du=2x\,dx$

 when $x=2,\ u=5$

$x=0,\ u=1$ 1

$\therefore \int_0^2 \frac{x}{(x^2+1)^3}dx=\frac{1}{2}\int_0^2\frac{2x\,dx}{(x^2+1)^3}$ *

$=\frac{1}{2}\int_1^5\frac{du}{u^3}$

$=\frac{1}{2}\int_1^5 u^{-3}\,du$

$=\frac{1}{2}\left[-\frac{u^{-2}}{2}\right]_1^5$ 1

$=-\frac{1}{4}\left[\frac{1}{u^2}\right]_1^5$

$=-\frac{1}{4}\left[\frac{1}{25}-1\right]$

$=-\frac{1}{4}\times-\frac{24}{25}$

$=\frac{6}{25}.$ 1

iii) R.T.P $\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\dots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

Step 1: Try $n=1$

$LHS=\frac{1}{(2n-1)(2n+1)}$ $\quad RHS=\frac{n}{2n+1}$

$=\frac{1}{(2-1)(2+1)}$ $\quad =\frac{1}{2+1}$

$=\frac{1}{3}$ $\quad =\frac{1}{3}$

$LHS=RHS$

$\therefore$ true for $n=1$

Step 2: Assume true for $n=k$

ie. $\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\dots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$ 1

(7)

Step 3: R.T.P true for $n = k+1$

ie. $\frac{1}{1\times 3} + \frac{1}{3\times 5} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{[2(k+1)-1][2(k+1)+1]} = \frac{k+1}{2(k+1)+1}$

ie. $\frac{1}{1\times 3} + \frac{1}{3\times 5} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$ 1

LHS $= \frac{1}{1\times 3} + \frac{1}{3\times 5} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$ from Step 2

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$

$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$ 1

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$

$= \frac{k+1}{2k+3}$

$=$ RHS 1

$\therefore$ true for $n = k+1$

Step 4: If the statement is true for $n=1$, then its true for $n=2$.
If its true for $n=2$, then its true for $n=3$ and so on.
$\therefore$ if true for $n=k$, then true for $n=k+1$. 1
Therefore the statement is true for all positive integers n.