CHRISTIAN BROTHERS' HIGH SCHOOL, LEWISHAM

Year 12 Ext 2 MATHEMATICS
HSC ASSESSMENT TASK 2
2010

TIME ALLOWED: 90 MINUTES

OUTCOMES : E1, E2, E3, E4, E6, E9

TOPICS Tested : Complex Numbers, Graphs, Conics

<u>NAME:</u>

<u>TEACHER:</u>

INSTRUCTIONS:

Attempt all questions
Start each question on a new page
Calculators may be used
Write in blue or black pen only
Show all necessary working
Marks may be deducted for careless or badly arranged work

Question 1 Complex Numbers **25 marks**

a) If $z = 4+3i$, express z in mod-arg form (give argument to the nearest degree) and write down the value of $\arg(z^5)$. 2

b) $\dfrac{4+3i}{1+\sqrt{2}i} = a+ib$, for a,b real.

Find the exact value of a and b. 3

c) Find the equation and sketch the locus of z if

$|z-i| = \text{Im}(z)$ 2

d) Sketch the region in the Argand plane where the inequalities

$\frac{\pi}{4} \le \arg(z-i) \le \frac{3\pi}{4}$ and $|z-i| \le 2$

both hold simultaneously. 3

e) i) Write down the six complex sixth roots of unity in modulus-argument form. Sketch the roots on an Argand diagram and explain why they form a regular polygon and name the polygon 3

ii) Factorise $z^6 - 1$ completely into real factors. 2

f) i) Find the square roots of the complex number $-3+4i$ 2

ii) Find the roots of the quadratic equation $x^2 - (4-2i)x + (6-8i) = 0$ 2

g) If P represents the complex number z, where z satisfies

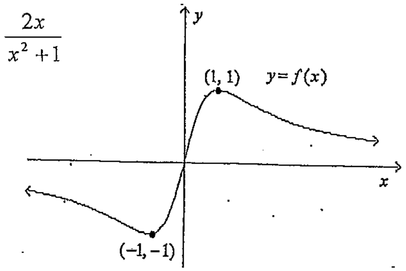
$|z-2| = 2$ and $0 \langle \arg z \langle \frac{\pi}{2}$

i) Show that $|z^2 - 2z| = 2|z|$ 3

ii) Find the value of k (a real number) if $\arg(z-2) = k\arg(z^2-2z)$ 3

Question 2 Graphs **25 marks**

a) The diagram below represents the curve $f(x)=\dfrac{2x}{x^2+1}$



Sketch the following on separate number planes, without using calculus 10

i) $y=f(-x)$

ii) $y=f(|x|)$

iii) $|y|=f(x)$?

iv) $y=f(2x)$

v) $y\times f(x)=1$

b) i) Sketch on the same axes the graphs

$$y=x+3 \quad \text{and} \quad y=2|x| \qquad 2$$

ii) Hence or otherwise solve for x :

$$2|x|<x+3 \qquad 2$$

iii) Sketch the curve $y=\dfrac{2|x|}{x+3}$ 3

c) Let $f(x)=\dfrac{3}{x-1}$

On separate diagrams sketch the following :

i) $y=f(|x|)$ 2

ii) $y^2=f(x)$? 3

iii) $y=e^{f(x)}$ 3

Question 3 Conics **25 marks**

a) $P(t,\frac{1}{t})$ is any point on the rectangular hyperbola $xy=1$. N is the foot of the perpendicular from P on to the X axis. The tangent at P meets the Y axis at M and the line through M parallel to the X axis meets the hyperbola at the point Q.

i) Represent this information on a diagram 1

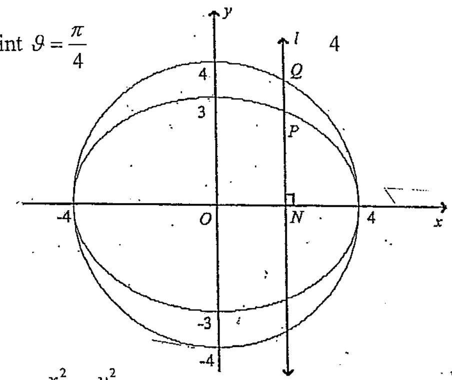
ii) Find the coordinates of M and Q 4

iii) Show that NQ is a tangent to the hyperbola. 3

b) Find the equation of the normal to the hyperbola

$x=2\sec\vartheta$, $y=\tan\vartheta$ at the point $\vartheta=\dfrac{\pi}{4}$ 4

c)



The diagram shows the ellipse, E, with equation $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$ and its auxiliary circle C. The coordinates of a point P on E are $(4\cos\vartheta, 3\sin\vartheta)$.

A straight line l, parallel to the y axis intersects the x axis at N and the curves E and C at the points P and Q respectively.

i) Find the eccentricity of E 1

ii) Write down the coordinates of N and Q 2

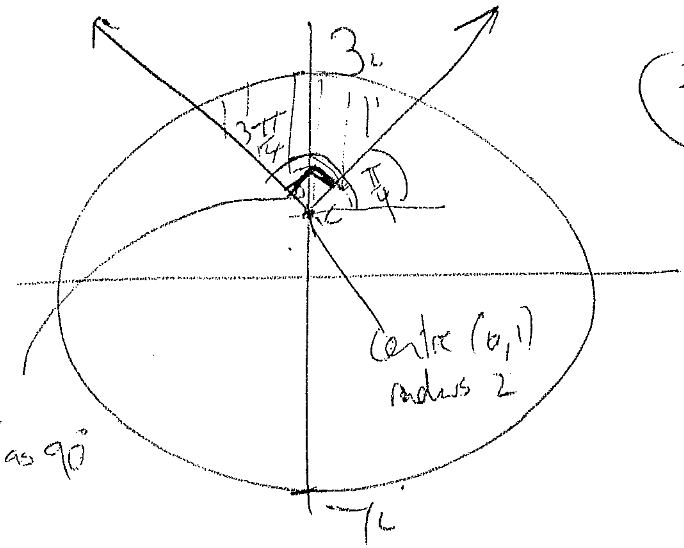
iii) Find the equations of the tangents at P and Q to the curves E and C respectively 6

iv) The tangents at P and Q intersect at R. Show that R lies on the x axis. 2

v) Prove that $ON.OR$ is independent of the positions of P and Q. 2

Ext2 2010 Mid year Soln's.

d) i)

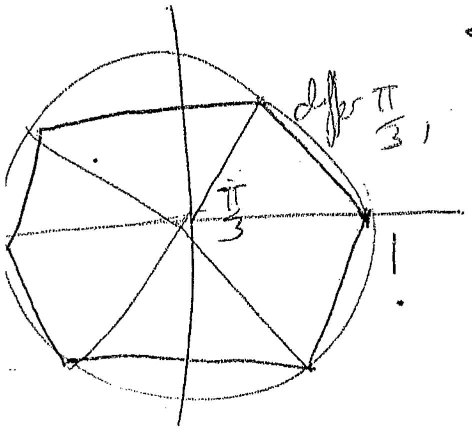


not if noted as 90°

(3)

ii) i) Roots are $\pm 1, \pm \text{cis}\frac{\pi}{3}, \pm \text{cis}\frac{2\pi}{3}, \cancel{\text{cis}\frac{4\pi}{3}}$

since moduli equal 1, and arguments differ $\frac{\pi}{3}$, form regular hexagon



ii) $z^6 - 1 = (z^3)^2 - 1$
$= (z^3+1)(z^3-1)$
$= (z+1)(z^2-z+1)(z-1)(z^2+z+1)$

iii) a) $|z| = \sqrt{16+9}$
$= 5$
$\arg z \doteqdot \tan^{-1}\left(\frac{3}{4}\right)$
$\doteqdot 37$
$\therefore z = 5\text{ cis } 37°$ (1) $5\text{cis} + \frac{37\pi}{180}$
$\therefore \arg(z^5) \doteqdot -176°$ (1) $= -\frac{44\pi}{45}$

b) $\frac{4+3i}{1+\sqrt{2}i} \times \frac{1-\sqrt{2}i}{1-\sqrt{2}i}$

$= \frac{4+3\sqrt{2}}{3} + i\frac{(3-4\sqrt{2})}{3}$ (1)

$a = \frac{4+3\sqrt{2}}{3}$ (1) $b = \frac{3-4\sqrt{2}}{3}$ (1)

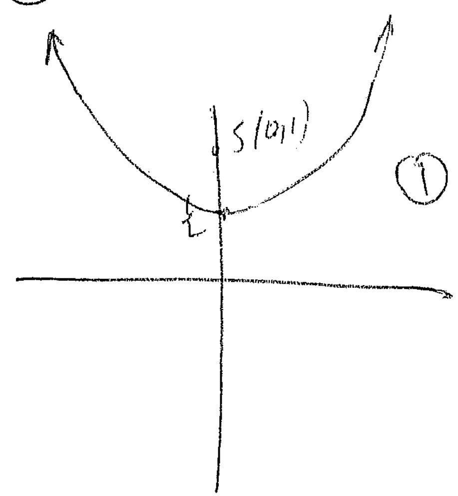
c) Let $z = x + iy$

$\therefore \sqrt{x^2+(y-1)^2} = y$

$x^2 + y^2 - 2y + 1 = y^2$

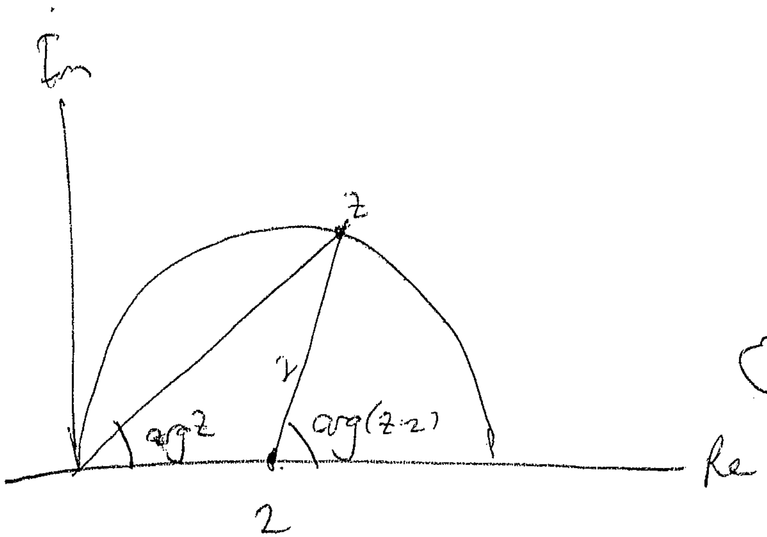
$x^2 = 2y - 1$

$x^2 = 4\left(\frac{1}{2}\right)\left(y - \frac{1}{2}\right)$ (1)



(1)

i)



(1)

)

$$|z^2 - 2z| = |z||z-2|$$
$$= 2|z|$$

(3)

ii)

$$\arg(z-2) = k(\arg(z^2 - 2z))$$
$$\arg(z-2) = k(\arg z + \arg(z-2))$$
$$\arg(z-2)(1-k) = k \arg z$$
$$2 \arg z(1-k) = k \arg z$$
$$2(1-k) = k$$
$$3k = 2$$
$$k = \frac{2}{3}$$

(3)

i) let $\sqrt{-3+4i} = x + iy$

$$-3 + 4i = x^2 - y^2 + 2xyi$$

$x^2 - y^2 = -3$
$2xy = 4$
solve simultaneously

(1)

$x^2 = -4$ $\quad$ $x^2 = 1$

No soln as x is real $\quad$ $x = \pm 1$

$x = 1$ $\quad$ $x = -1$
$y = 2$ $\quad$ $y = -2$

(1)

soln $\pm(1+2i)$

ii)

$$x^2 - (4-2i)x + (6-8i) = 0$$

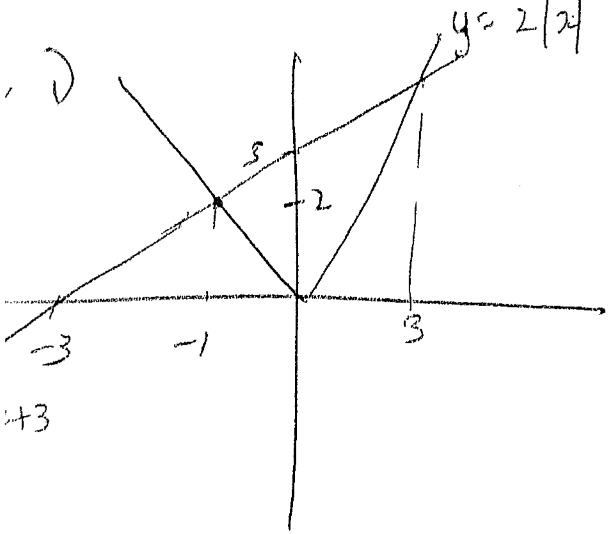
$$x = \frac{4 - 2i \pm \sqrt{(-(4-2i))^2 - 4(1)(6-8i)}}{2(1)}$$

(1)

$x = 3 + i$ or $x = +1 - 3i$

(1)

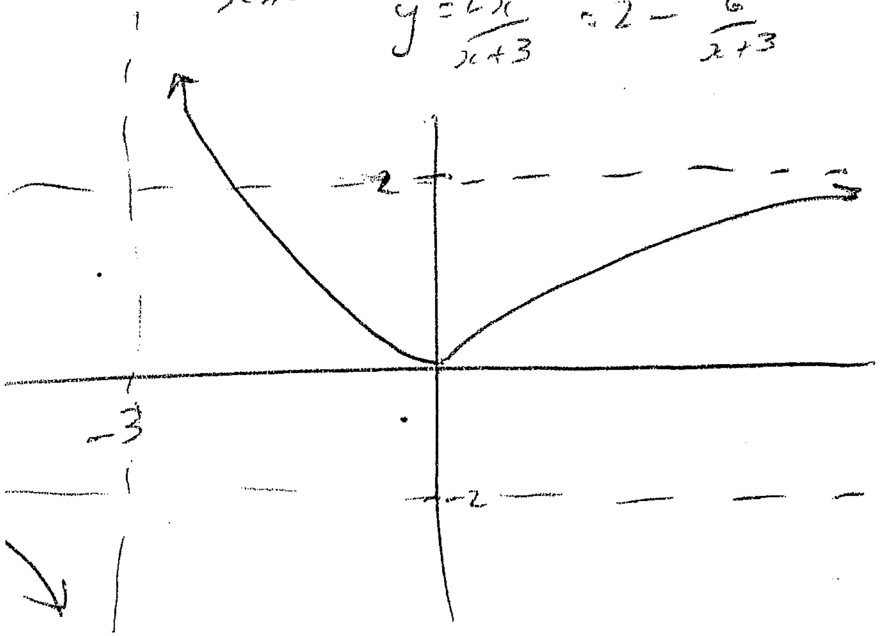
i)



ii) $-1 < x < 3$

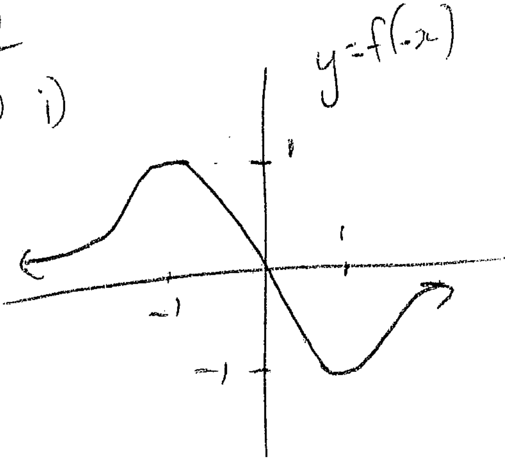
iii) If $x<0$, $y=\frac{-2x}{x+3} = -2+\frac{6}{x+3}$

$x \geq 0$ $\quad y=\frac{2x}{x+3} = 2-\frac{6}{x+3}$

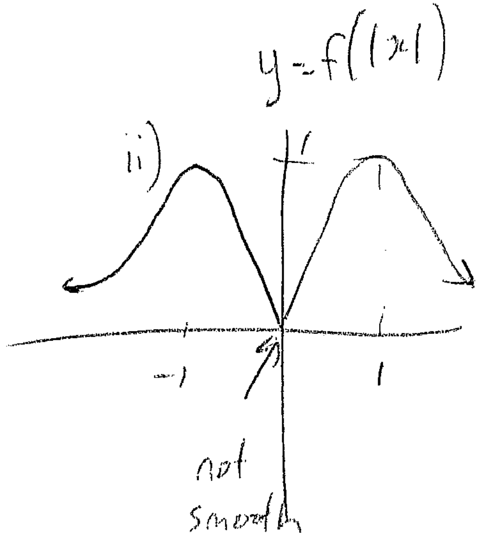


Q2

a) i)



ii)



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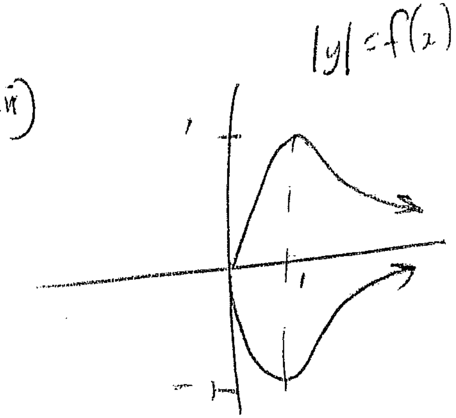
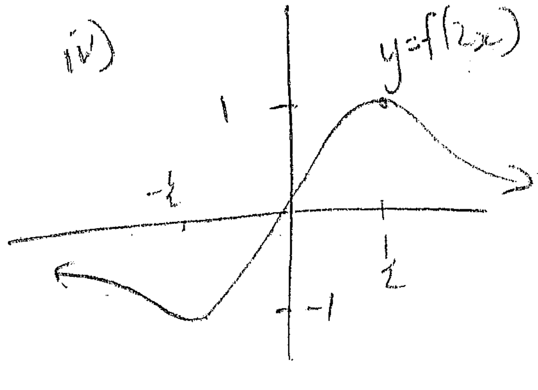
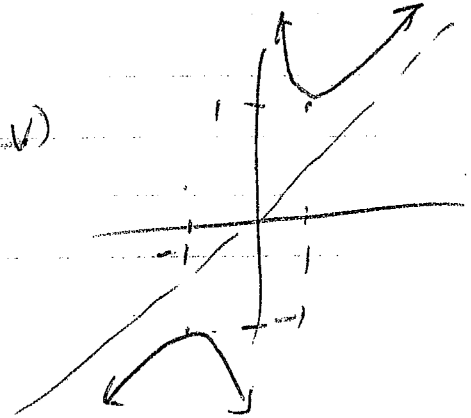


iv)

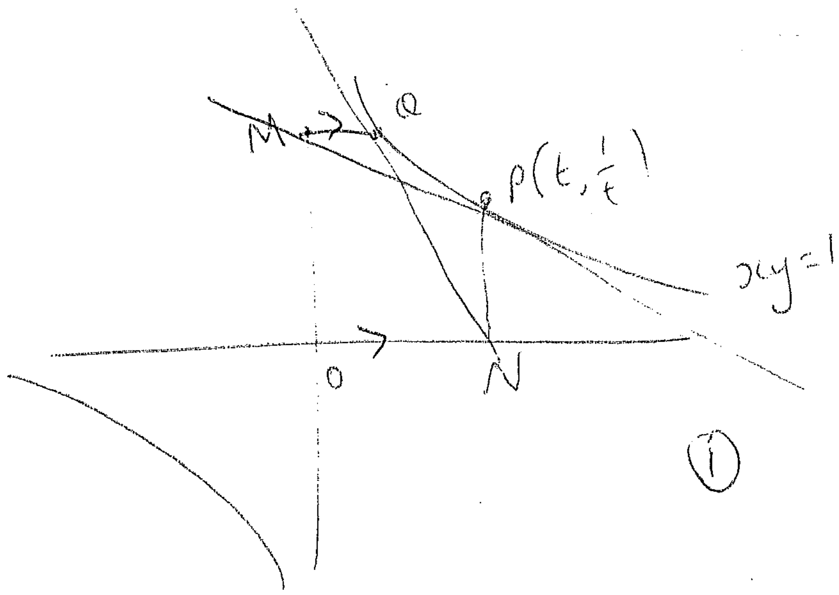


v)



x 3

a)



(i)

At P $x = t$ $y = \frac{1}{t}$

gradient at P $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{1}{t^2} \times 1 = -\frac{1}{t^2}$

tangent at P $y - \frac{1}{t} = -\frac{1}{t^2}(x - t)$

At M $x = 0$, $y - \frac{1}{t} = \frac{1}{t}$

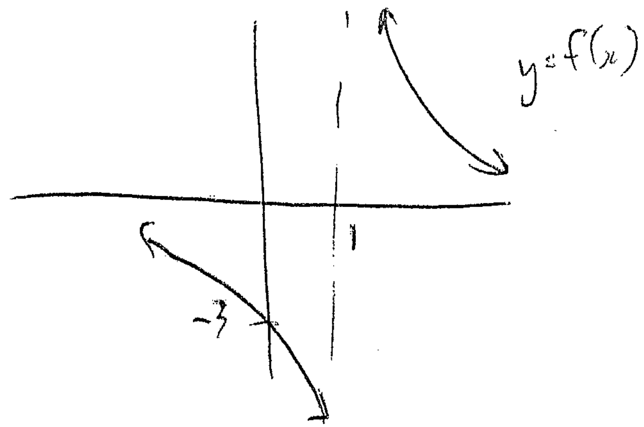
$y = \frac{2}{t}$

$M(0, \frac{2}{t})$ (2)

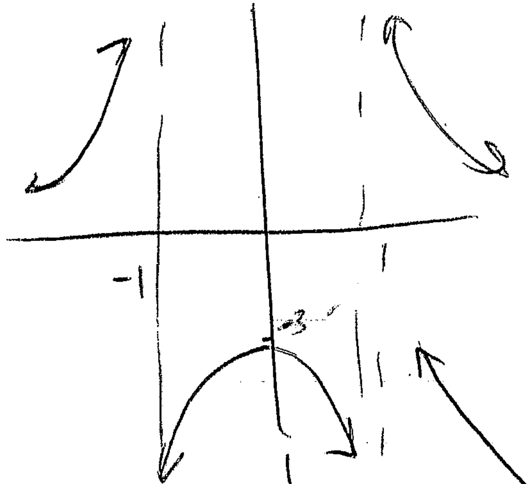
(2)

At Q $y = \frac{2}{t}$ $x = \frac{t}{2}$ as $xy = 1$ $Q(\frac{t}{2}, \frac{2}{t})$

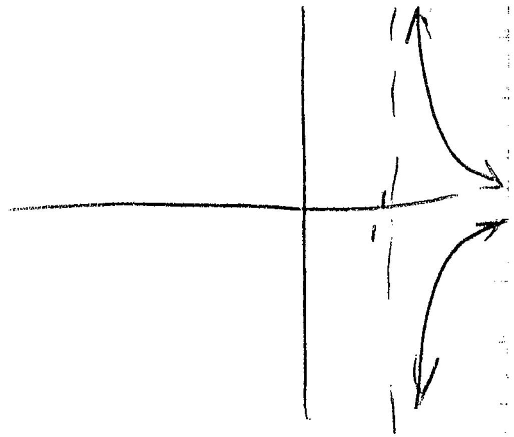
c)



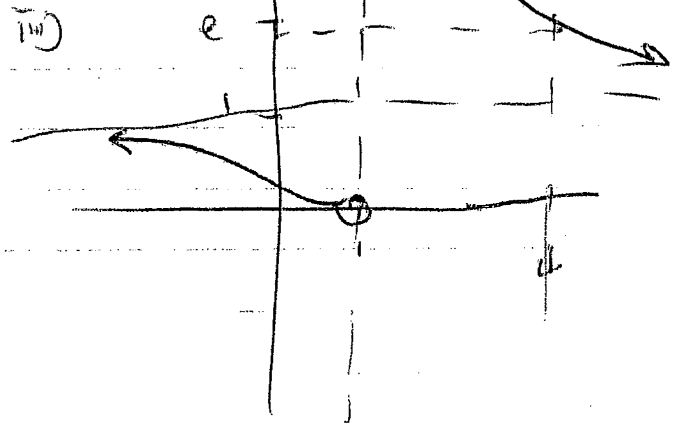
i) 2



ii) 3



iii)



3

i) $3x\cos\theta + 4y\sin\theta = 12$ ①

$x\cos\theta + y\sin\theta = 4$ ②

② × 3 $3\cos\theta + 4y\sin\theta = 12$

$3x\cos\theta + 3y\sin\theta = 12$

$y\sin\theta = 0 \quad \therefore y = 0$ ①

$3x\cos\theta = 12$

$x\cos\theta = 4$ ①

$R(4\sec\theta, 0)$ which lies on x-axis

i) $ON = |4\cos\theta| \quad OR = |4\sec\theta|$ ①

$\therefore ON \times OR = 16$ this is independent of θ, thus independent of P and Q ①

iii) $\frac{2x}{16} + \frac{2y}{9}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-9x}{16y} = \frac{-9(4\cos\theta)}{16(3\sin\theta)} = -\frac{3\cos\theta}{4\sin\theta}$ ①

$\therefore y - 3\sin\theta = \frac{-3\cos\theta}{4\sin\theta}(x - 4\cos\theta)$ ①

$\therefore 4y\sin\theta - 12\sin^2\theta = -3x\cos\theta + 12\cos^2\theta$

$\therefore 3x\cos\theta + 4y\sin\theta = 12$

$\frac{x\cos\theta}{4} + \frac{y\sin\theta}{3} = 1$ ①

ii) $x^2 + y^2 = 16$

$2x + 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{x}{y} = -\frac{4\cos\theta}{4\sin\theta} = -\frac{\cos\theta}{\sin\theta}$ ①

$y - 4\sin\theta = -\frac{\cos\theta}{\sin\theta}(x - 4\cos\theta)$ ①

$x\cos\theta + y\sin\theta = 4$ ①