Centre Number

Student Number

CSSA

CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW

**2012**

**TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

# Mathematics Extension 2

Morning Session
Monday, 6 August 2012

**General Instructions**

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided on a separate sheet
- Show all necessary working for Questions 11-16
- Write your Centre Number and Student Number at the top of this page and page 7

**Total marks – 100**

**Section I** Pages 2–6

**10 marks**

- Attempt Questions 1–10
- Allow about 15 minutes for this section

**Section II** Pages 7–17

**90 marks**

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section





STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

## Section I
**10 marks**

**Attempt Questions 1 – 10**

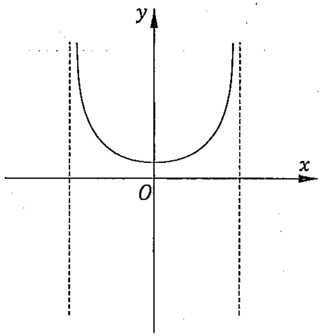
### Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

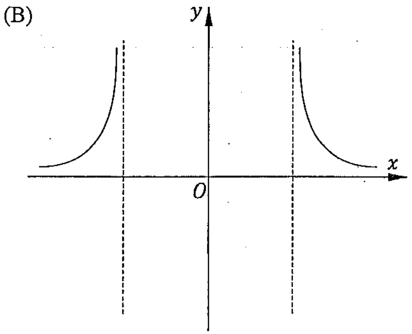
---

1 Which of the following best represents the graph of $y = \dfrac{1}{\sqrt{4-x^2}}$?

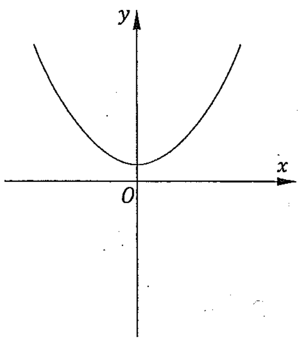
(A)



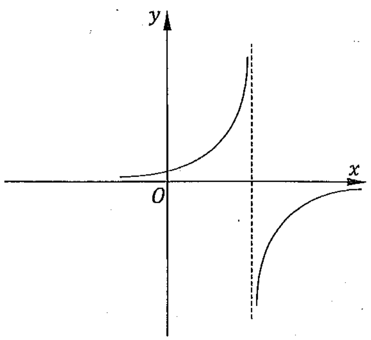

(B)




(C)




(D)






2 Let $z = a + ib$, where $a \neq 0$ and $b \neq 0$.

Which of the following statements is false?

(A) $z - \overline{z} = 2bi$

(B) $|z|^2 = |z||\overline{z}|$

(C) $|z| + |\overline{z}| = |z + \overline{z}|$

(D) $\arg(z) + \arg(\overline{z}) = 0$

3 $z$ satisfies the equation $\left|z - \sqrt{2} - i\sqrt{2}\right| = 1$. Find the minimum value of $\arg(z)$.

(A) $-75°$

(B) $-15°$

(C) $15°$

(D) $75°$

4 Find the coordinates of the foci of the hyperbola with equation $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$.

(A) $\left(\pm\sqrt{5}, 0\right)$

(B) $\left(\pm\sqrt{13}, 0\right)$

(C) $\left(\pm\dfrac{3\sqrt{5}}{2}, 0\right)$

(D) $\left(\pm\dfrac{3\sqrt{13}}{2}, 0\right)$

5 The substitution $t = \tan\frac{\theta}{2}$ is used to find $\int \frac{d\theta}{\cos\theta}$.

Which of the following gives the correct expression for the required integral?

(A) $\int \frac{1}{2(1-t^2)} dt$

(B) $\int \frac{2}{1-t^2} dt$

(C) $\int \frac{2t}{1-t^2} dt$

(D) $\int \frac{4t}{(1+t^2)^2} dt$

6 The equation $|z-3| + |z+3| = 10$ defines an ellipse.

Find the length of the semi-minor axis.

(A) 4

(B) 5

(C) 8

(D) 10



7 A particle of mass 1 kg is projected vertically upwards with speed $u$ from the origin $O$. The particle is subject to a constant gravitational force and a resistance which is proportional to half the square of its velocity $v$ ms$^{-1}$ (with $k$ the constant of proportionality).

Let $x$ be the displacement in metres of the particle above $O$ at time $t$ seconds after the particle is projected and let $g$ be the acceleration due to gravity.

Which of the following expressions gives the maximum height reached by the particle?

(A) $\int_u^0 \frac{dv}{-g + \frac{k}{2}v^2}$

(B) $\int_u^0 \frac{dv}{-g - \frac{k}{2}v^2}$

(C) $\int_u^0 \frac{v\,dv}{-g + \frac{k}{2}v^2}$

(D) $\int_u^0 \frac{v\,dv}{-g - \frac{k}{2}v^2}$

8 Let $\alpha, \beta$ and $\gamma$ be the zeros of the polynomial $x^3 + 5x - 3$.
Find the value of $\alpha^3 + \beta^3 + \gamma^3$.

(A) $-125$

(B) 0

(C) 9

(D) 34

9 In how many ways can 5 letters be chosen from the letters of the word ARRANGE?

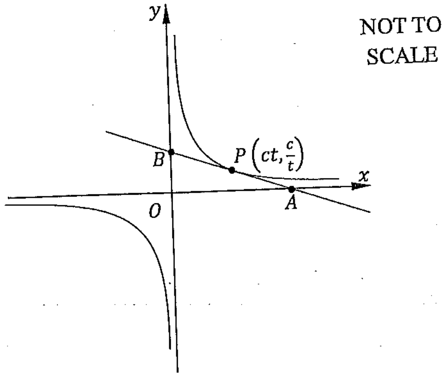
(A) 9

(B) 12

(C) 21

(D) 30

10 The equation of the tangent to the rectangular hyperbola $xy = c^2$ at $P\left(ct, \frac{c}{t}\right)$ is given by $x + t^2 y = 2ct$. The tangent cuts the $x$ and $y$ axes at $A$ and $B$ respectively.




Which of the following statements is false?

(A) $P$ is the centre of the circle that passes through $O$, $A$ and $B$.

(B) The area of $\Delta AOB$ is $2c^2$ square units.

(C) The distance $AB$ is $\sqrt{4c^2t^2 + \frac{4c^2}{t^2}}$.

(D) $AP > BP$.



CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW
2012 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

# Mathematics Extension 2

Centre Number

Student Number

## Section II
**90 marks**

**Attempt Questions 11–16**
**All questions are of equal value.**
**Allow about 2 hours and 45 minutes for this section.**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

**Question 11** (15 Marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{(\ln x)^2}{x}\,dx$. **1**

(b) Evaluate $\int_0^{\ln 2} x\,e^x\,dx$. **3**

(c) (i) Find real numbers $A$ and $B$ such that **2**

$$\frac{7x+1}{x^2-x-2} = \frac{A}{x+1} + \frac{B}{x-2}.$$

(ii) Hence, find $\int \frac{7x+1}{x^2-x-2}\,dx$. **2**

(d) Using a trigonometric substitution, or otherwise, find $\int \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}\,dx$. **3**

(e) Consider the region in the first quadrant bounded by the curves $y = x$ and $y = x^2$. This region is rotated about the vertical line $x = -1$. **4**

Use the method of cylindrical shells to find the volume of the solid formed.

**Question 12** (15 Marks) Use a SEPARATE writing booklet.

(a) Let $z = -1 + i$.

(i) Express $\frac{1}{z}$ in the form $a + ib$. **1**

(ii) Express $z$ in modulus-argument form. **1**

(iii) Hence, find $z^{10}$ in the form $a + ib$. **2**

(b) Draw a sketch of the locus specified by $|z - i| = |z + 3 - 2i|$ on an Argand diagram. **2**

(c) The polynomial $P(x) = x^4 - 2x^3 + 8x^2 - 8x + 16$ has $1 + i\sqrt{3}$ as a zero. **3**

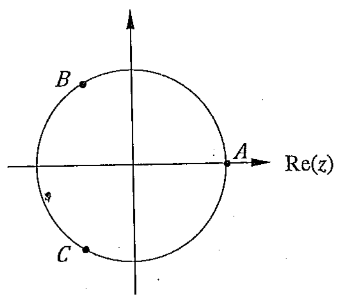
Express $P(x)$ as the product of two quadratic factors.

(d) The roots of $z^3 - 1 = 0$ are 1, $\omega$ and $\omega^2$ where $\omega$ is one of the complex roots.

(i) Find the value of $1 + \omega + \omega^2$. **1**

(ii) Explain why $z^2 + z + 1 = (z - \omega)(z - \omega^2)$. **2**

The Argand diagram below shows the points $A, B$ and $C$ which correspond to the roots 1, $\omega$ and $\omega^2$ of the equation $z^3 - 1 = 0$.




(iii) Copy this diagram into your writing booklet and show the vector $1 - \omega$ on your diagram, carefully indicating the direction of this vector. **1**

(iv) Hence, or otherwise, find the product of the lengths of the two chords $AB$ and $AC$. **2**



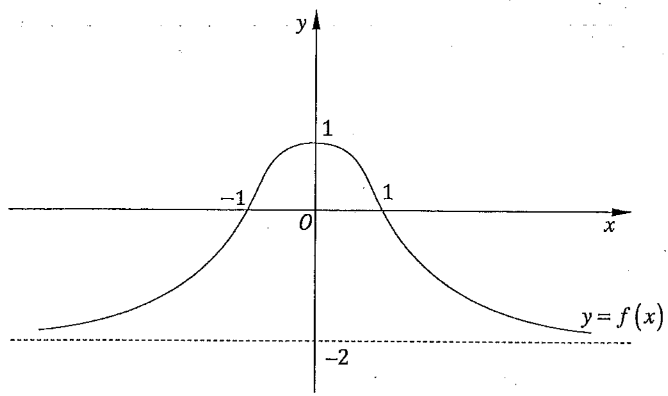
**Question 13** (15 Marks) Use a SEPARATE writing booklet.

(a) Consider the curve with equation $3x^2 + 3y^2 + 2xy = 24$.

(i) Show that $\frac{dy}{dx} = -\left(\frac{y + 3x}{3y + x}\right)$. **2**

(ii) Find the $x$-coordinates of the stationary points of the curve. **2**

(b) The diagram shows the graph of $y = f(x)$.
The line $y = -2$ is a horizontal asymptote and $f(x)$ is an even function.
$y = f(x)$ has $x$-intercepts at $(\pm 1, 0)$ and $y$-intercept at (0,1).




Draw separate one-third page diagrams of the graphs of the following, showing any horizontal or vertical asymptotes.

(i) $y = |f(x)|$ **1**

(ii) $y = [f(x)]^2$ **2**

(iii) $y = \ln f(x)$ **2**

**Question 13 continues on page 10**

Question 13 (continued)

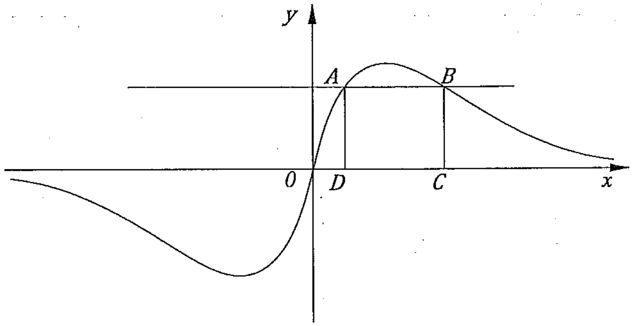
(c) $P(x) = x^4 + 3x^3 - 6x^2 - 28x + c$ has a zero of multiplicity 3. **3**

Find the value of $c$.

(d) Consider the curve $f(x) = \dfrac{2x}{x^2+1}$.

(i) Show that $f\left(\dfrac{1}{x}\right) = f(x)$, where $x \neq 0$. **1**

(ii) The diagram below shows the graph $y = f(x)$. **2**




A horizontal line intersects the curve in the first quadrant at $A$ and $B$. $C$ and $D$ lie on the $x$-axis directly beneath $B$ and $A$ respectively.

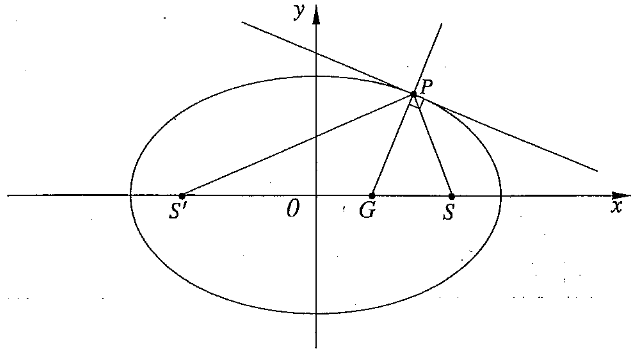
Find the coordinates of $C$ if $ABCD$ is a square.

**End of Question 13**



**Question 14** (15 Marks) Use a SEPARATE writing booklet.

(a) The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has foci $S(ae, 0)$ and $S'(-ae, 0)$, and directrices $x = \pm\dfrac{a}{e}$. $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse with the normal at $P$ meeting the $x$-axis at $G$.




(i) Using the focus/directrix definition of an ellipse show that **2**

$$\frac{PS}{PS'} = \frac{1 - e\cos\theta}{1 + e\cos\theta}.$$

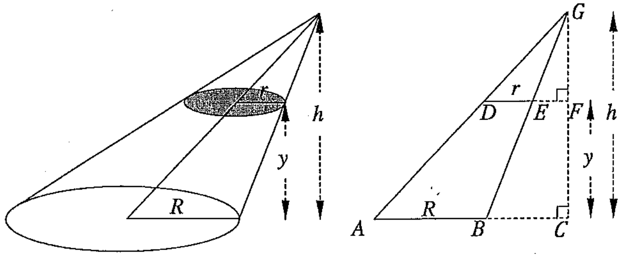
(ii) The equation of the normal at $P$ is given by **3**

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2. \text{ (Do NOT prove this.)}$$

Show that $\dfrac{GS}{GS'} = \dfrac{PS}{PS'}$ .

**Question 14 continues on page 12**

(b) The diagram shows an oblique cone of base radius $R$ and perpendicular height $h$.




A horizontal cross-section of the cone taken at height $y$, is a circle of radius $r$ shown shaded in the diagram.

(i) By considering the ratio of sides in two pairs of similar triangles, show that **1**

$$r = \left(\frac{h-y}{h}\right)R.$$

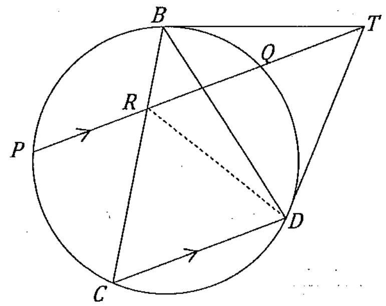
(ii) Show that the volume of the oblique cone is given by $\frac{1}{3}\pi R^2 h$. **3**

**Question 14 continues on page 13**





(c) The chords $PQ$ and $CD$ of a circle are parallel. The tangent at $D$ meets $PQ$ produced at $T$. $B$ is the point of contact of the other tangent from $T$ to the circle. $BC$ meets $PQ$ at $R$.




Copy or trace the diagram into your writing booklet.

(i) Prove $\angle BDT = \angle BRT$. **3**

Hence, state why $B$, $T$, $D$ and $R$ are concyclic points.

(ii) Prove $\angle BRT = \angle DRT$. **2**

(iii) Show that $\Delta RCD$ is isosceles. **1**

**End of Question 14**

**Question 15** (15 Marks) Use a SEPARATE writing booklet.

(a) A truck of mass $m$ kg moves at speed $v$ ms$^{-1}$ around a curve banked at angle $\theta$ and with radius $r$. The lateral force down the slope is $F$, the normal force is $N$ and $g$ is the acceleration due to gravity. **3**

By resolving forces in two directions show that $F = \frac{mv^2}{r}\cos\theta - mg\sin\theta$.

(b) (i) By considering the expansion of $(\cos\theta + i\sin\theta)^5$ and using de Moivre's theorem, show that **3**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

(ii) Hence, show that the roots of $16x^5 - 20x^3 + 5x = 0$ are **2**

$$x = 0, \pm\cos\frac{\pi}{10}, \pm\cos\frac{3\pi}{10}.$$

(iii) Hence, show that **2**

$$\cos^2\frac{\pi}{10} + \cos^2\frac{3\pi}{10} = \frac{5}{4}.$$

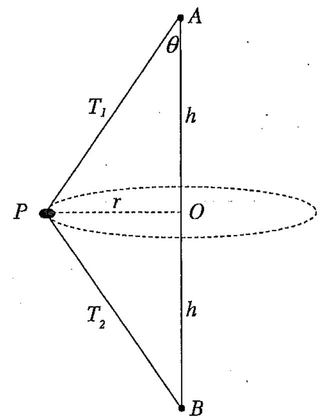
**Question 15 continues on page 15**





(c) Two points $A$ and $B$ are fixed with $A$ vertically above $B$. Two equal lengths of light inextensible string join $A$ and $B$ to a particle $P$ of mass $m$. The particle $P$ moves with angular velocity $\omega$ in a horizontal circle, centre $O$ and radius $r$, such that $AO = BO = h$.

This information is shown in the diagram below.




Let $T_1$ and $T_2$ be the tensions in $AP$ and $BP$ respectively. Let $\angle PAO = \theta$ and let $g$ be the acceleration due to gravity.

(i) By resolving forces vertically at $P$, show that **1**

$$(T_1 - T_2)\cos\theta = mg.$$

(ii) If $\omega = 2\sqrt{\frac{g}{h}}$, show that $T_1 : T_2 = 5 : 3$. **4**

**End of Question 15**

**Question 16** (15 Marks) Use a SEPARATE writing booklet.

(a) (i) The inequality $\frac{x+y}{2} \geq \sqrt{xy}$ is true for all $x \geq 0$ and $y \geq 0$, with equality when $x = y$. **1**

Use the inequality to show that the minimum value of the function $f(x) = Ae^{x} + Be^{-x}$ is $2\sqrt{AB}$, where $A>0$ and $B>0$ are constants.

(ii) The function $f(x) = Ae^{x} + Be^{-x}$ has line symmetry about the vertical line $x = c$. That is, we can write $f(x)$ in the form $f(x) = k\left[e^{x-c} + e^{-(x-c)}\right]$ for some values of $k$ and $c$. **2**

Using part (i), or otherwise, find $k$ and $c$ in terms of $A$ and $B$.

(b) (i) Show that $\frac{x^{2n-1}}{\sqrt{1-x^2}} - \frac{x^{2n+1}}{\sqrt{1-x^2}} = x^{2n-1}\sqrt{1-x^2}$. **1**

(ii) For every integer $n \geq 1$ let $I_{2n-1} = \int_0^1 \frac{x^{2n-1}}{\sqrt{1-x^2}}\,dx$. **3**

Using integration by parts and the result from part (i), show that for $n \geq 1$

$$I_{2n+1} = \left(\frac{2n}{2n+1}\right) I_{2n-1}.$$

(iii) Using part (ii), or otherwise, show that, **2**

$$I_{2n+1} = \frac{2^n \times n!}{1 \times 3 \times 5 \times \ldots \times (2n+1)}.$$

**Question 16 continues on page 17**





(iv) Using part (iii), or otherwise, show that **2**

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx + \int_0^1 \left[\sum_{n=1}^{\infty}\left(C_n \frac{x^{2n+1}}{\sqrt{1-x^2}}\right)\right] dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

where $C_n = \frac{1 \times 3 \times 5 \times \ldots \times (2n-1)}{(2n+1)2^n\, n!}$.

You may assume each infinite series has a limiting sum.

(v) The inverse sine function $\sin^{-1} x$ can be defined by the following series: **2**

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} C_n x^{2n+1}$$

where $C_n = \frac{1 \times 3 \times 5 \times \ldots \times (2n-1)}{(2n+1)2^n\, n!}$.

Using this definition of $\sin^{-1} x$ and the result from part (iv), show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}.$$

(vi) Hence, find the limiting value of $S$ if **2**

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots$$

**End of paper**

CSSA

# CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW

## 2012 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

## MATHEMATICS EXTENSION 2

**Section I**
**10 marks**

**Questions 1–10** (1 mark each)

**Question 1** (1 mark)
***Outcomes Assessed: E6***
***Targeted Performance Bands: E2***

| Solution | Answer | Mark |
|---|---|---|
| Asymptotes exist at $x = \pm 2$.<br>The graph exists for $x$ values $-2 < x < 2$.<br>The $y$-intercept is $\frac{1}{2}$. | A | 1 |

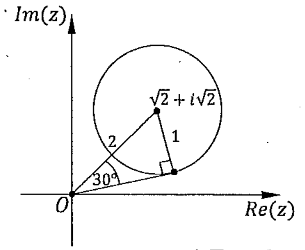
**Question 2** (1 mark)
***Outcomes Assessed: E3***
***Targeted Performance Bands: E3***

| Solution | Answer | Mark |
|---|---|---|
| Let $z = a + ib$, then $\bar{z} = a - ib$<br>$\lvert z\rvert + \lvert\bar{z}\rvert = 2\sqrt{a^2+b^2}$<br>$\lvert z+\bar{z}\rvert = \lvert 2a\rvert$<br>$\neq \lvert z\rvert + \lvert\bar{z}\rvert$ | C | 1 |



6400-2

**Question 3** (1 mark)
***Outcomes Assessed: E3***
***Targeted Performance Bands: E3, E4***




| Solution | Answer | Mark |
|---|---|---|
| Minimum value of arg $(z)$ $= \arg(\sqrt{2}+i\sqrt{2}) - 30°$<br>$= 45° - 30°$<br>$= 15°$ | C | 1 |

**Question 4** (1 mark)
***Outcomes Assessed: E3***
***Targeted Performance Bands: E2***

| Solution | Answer | Mark |
|---|---|---|
| $a = 3, b = 2.$<br>$b^2 = a^2(e^2-1)$<br>$e = \frac{\sqrt{13}}{3}$<br>The coordinates of the foci are $(\pm\sqrt{13}, 0)$. | B | 1 |

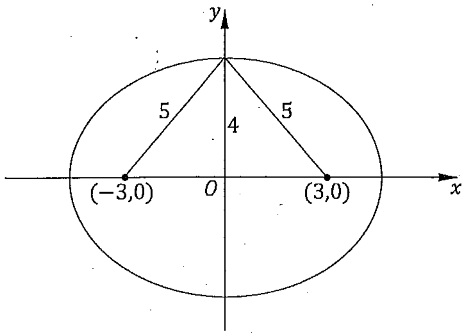
**Question 5** (1 mark)
***Outcomes Assessed: HE6, E8***
***Targeted Performance Bands: E2***

| Solution | Answer | Mark |
|---|---|---|
| $\int \frac{d\theta}{\cos\theta} = \int \frac{1+t^2}{1-t^2}.\frac{2}{1+t^2}dt$<br>$= \int \frac{2}{1-t^2}dt$ | B | 1 |



DISCLAIMER
The information contained in this document is intended for the professional assistance of teaching staff. It does not constitute advice to students. Further it is not the intention of CSSA to provide specific marking outcomes for all possible Trial HSC answers. Rather the purpose is to provide teachers with information so that they can better explore, understand and apply HSC marking requirements, as established by the NSW Board of Studies.
No guarantee or warranty is made or implied with respect to the application or use of CSSA Marking Guidelines in relation to any specific trial exam question or answer. The CSSA assumes no liability or responsibility for the accuracy, completeness or usefulness of any Marking Guidelines provided for the Trial HSC papers.

**Question 6** (1 mark)
***Outcomes Assessed: E3***
***Targeted Performance Bands: E3***

| Solution | Answer | Mark |
|---|---|---|
| <br>Length of the semi-minor axis is 4 units. | A | 1 |

**Question 7** (1 mark)
***Outcomes Assessed: E5***
***Targeted Performance Bands: E3 - E4***

| Solution | Answer | Mark |
|---|---|---|
| Particle is projected vertically upwards ($x$>0) and both the gravitational force ($1 \times g$) and resistance ($\frac{kv^2}{2}$) act downwards.<br>$\therefore 1 \times \ddot{x} = -1 \times g - \frac{kv^2}{2}$<br>$\therefore v\frac{dv}{dx} = -g - \frac{kv^2}{2}$<br>$\frac{dv}{dx} = \frac{-g - \frac{kv^2}{2}}{v}$<br>$\frac{dx}{dv} = \frac{v}{-g - \frac{kv^2}{2}}$<br>As the initial velocity is $u$ and velocity at the maximum height is 0.<br>$x = \int_u^0 \frac{v\,dv}{-g - \frac{k}{2}v^2}$ | D | 1 |





**Question 8** (1 mark)
***Outcomes Assessed: E4***
***Targeted Performance Bands: E3***

| Solution | Answer | Mark |
|---|---|---|
| $\alpha^3 = 3 - 5\alpha$<br>$\beta^3 = 3 - 5\beta$<br>$\gamma^3 = 3 - 5\gamma$<br>Therefore, $\alpha^3 + \beta^3 + \gamma^3 = 9 - 5(\alpha + \beta + \gamma)$<br>$= 9 - 5 \times 0$<br>$= 9.$ | C | 1 |

**Question 9** (1 mark)
***Outcomes Assessed: E2, PE3***
***Targeted Performance Bands: E3***

| Solution | Answer | Mark |
|---|---|---|
| 5 different letters can be chosen in $\binom{5}{5}$ ways.<br>2 like letters (RR or AA) and 3 different letters can be chosen in $2 \times \binom{4}{3}$ ways.<br>2 pairs of like letters (RR and AA) and 1 different letter can be chosen in $1 \times \binom{3}{1}$ ways.<br>Therefore, 5 letters can be chosen in 12 ways. | B | 1 |

**Question 10** (1 mark)
***Outcomes Assessed: E4***
***Targeted Performance Bands: E3***

| Solution | Answer | Mark |
|---|---|---|
| Coordinates are $A(2ct, 0), B(0, \frac{2c}{t})$ and $P(ct, \frac{c}{t})$.<br>Therefore, $P$ is the midpoint of $AB$ i.e. $AP = BP$.<br>Hence, $AP > BP$ is false. | D | 1 |

## Section II
**90 marks**

**Question 11** (15 marks)
(a) (1 mark)
***Outcomes assessed: HE6, E8***
***Targeted Performance Bands: E2***

| Criteria | Mark |
|---|---|
| • Correct primitive (+$c$ not essential) | 1 |

***Sample answer:***

$$\int \frac{(\ln x)^2}{x}dx = \frac{(\ln x)^3}{3} + c$$

(b) (3 marks)
***Outcomes assessed: E8***
***Targeted Performance Bands: E2 - E3***

| Criteria | Marks |
|---|---|
| • Correct simplified solution | 3 |
| • Correct second integral and substitution of limits | 2 |
| • Correct use of Integration by Parts | 1 |

***Sample answer:***

$$\int_0^{\ln 2} xe^x dx = [xe^x]_0^{\ln 2} - \int_0^{\ln 2} e^x dx$$

$$= 2\ln 2 - [e^x]_0^{\ln 2}$$
$$= 2\ln 2 - 1$$

(c) (i) (2 marks)
***Outcomes assessed: E8***
***Targeted Performance Bands: E2***

| Criteria | Marks |
|---|---|
| • Correctly finds $A$ and $B$ | 2 |
| • Makes some progress in partial fraction decomposition | 1 |

***Sample answer:***

$$\frac{7x+1}{x^2-x-2} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$= \frac{A(x-2)+B(x+1)}{(x+1)(x+2)}$$

Equating:
$$A + B = 7$$
$$B - 2A = 1$$

Therefore, $A = 2$ and $B = 5$.





(c) (ii) (2 marks)
***Outcomes assessed: E8***
***Targeted Performance Bands: E2***

| Criteria | Marks |
|---|---|
| • Correct integration of both partial fractions found in (i). No penalty for no absolute value or no +$c$ or no simplification of answer | 2 |
| • Some progress towards the answer | 1 |

***Sample answer:***

$$\int \frac{7x+1}{x^2-x-2}dx = \int \left(\frac{2}{x+1} + \frac{5}{x-2}\right)dx$$

$$= 2\ln|x+1| + 5\ln|x-2| + c$$

$$= \ln(x+1)^2|x-2|^5 + c$$

(d) (3 marks)
***Outcomes assessed: HE6, E8***
***Targeted Performance Bands: E2 - E3***

| Criteria | Marks |
|---|---|
| • Correct answer in the variable $x$, (no penalty for no +$c$) | 3 |
| • Substantial progress towards simplifying the integral | 2 |
| • Correct substitution e.g. $x = \tan\theta$ | 1 |

***Sample answer:***

Let $x = \tan\theta$

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}dx = \int \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{\frac{3}{2}}}$$

$$= \int \frac{\sec^2\theta\, d\theta}{\sec^3\theta}$$

$$= \int \cos\theta\, d\theta$$

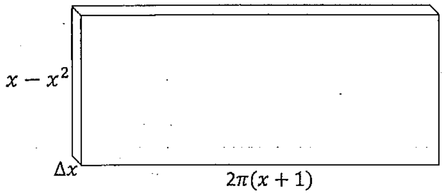
$$= \sin\theta + c$$

$$= \frac{x}{\sqrt{1+x^2}} + c$$

(e) (4 marks)
***Outcomes assessed: E7***
***Targeted Performance Bands: E3 - E4***

| Criteria | Marks |
|---|---|
| • Correct Solution | 4 |
| • Expression for the integral with correct limits | 3 |
| • Progress towards finding an expression for the volume of a shell | 2 |
| • Some progress towards finding an expression for the radius or height of a cylindrical shell | 1 |

***Sample answer:***




$$\Delta V = 2\pi(x+1)(x-x^2)\Delta x$$

$$\therefore V = \lim_{\Delta x \to 0} \sum_{x=0}^{1} 2\pi(x+1)(x-x^2)\Delta x$$

$$= 2\pi \int_0^1 (x - x^3)\, dx$$

$$= 2\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{\pi}{2}$$

Therefore, the volume is $\frac{\pi}{2}$ units$^3$.





**Question 12 (15 marks)**
(a) (i) (1 mark)
***Outcomes assessed: E3***
***Targeted Performance Bands: E2***

| Criteria | Mark |
|---|---|
| • Correct answer | 1 |

***Sample answer:***

$$\frac{1}{z} = \frac{1}{-1+i} \times \frac{-1-i}{-1-i}$$

$$= -\frac{1}{2} - \frac{1}{2}i$$

(a) (ii) (1 mark)
***Outcomes assessed: E3***
***Targeted Performance Bands: E2***

| Criteria | Mark |
|---|---|
| • Correct answer | 1 |

***Sample Answer:*** $|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$$\arg(z) = \frac{3\pi}{4}$$

$$\therefore z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

(a) (iii) (2 marks)
***Outcomes assessed: E3***
***Targeted Performance Bands: E2***

| Criteria | Mark |
|---|---|
| • Correct simplified answer | 2 |
| • Use of De Moivre's theorem | 1 |

***Sample answer***

$$z^{10} = \sqrt{2}^{10}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^{10}$$

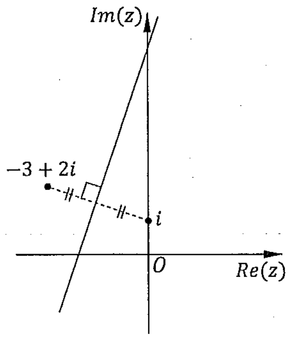
$$= 32\left(\cos\frac{15\pi}{2} + i\sin\frac{15\pi}{2}\right)$$

$$= -32i$$

(b) (2 marks)
***Outcomes assessed: E3***
***Targeted Performance Bands: E2***

| Criteria | Marks |
|---|---|
| • Entire correct diagram | 2 |
| • Some progress e.g. recognition that the locus is a line | 1 |

***Sample answer:***




(c) (3 marks)
***Outcomes assessed: E4***
***Targeted Performance Bands: E3***

| Criteria | Marks |
|---|---|
| • Correct answer | 3 |
| • Progression towards finding the factor $x^2 - 2x + 4$ | 2 |
| • Recognition that $1 - i\sqrt{3}$ is also a root | 1 |

***Sample answer:***
Since $1 + i\sqrt{3}$ is a root of $P(x)$, $1 - i\sqrt{3}$ is also a root of $P(x)$. (Roots occur in conjugate pairs)
The quadratic factor of $P(x)$ with roots $1 \pm i\sqrt{3}$ is $x^2 - 2x + 4$.
By division: $(x^4 - 2x^3 + 8x^2 - 8x + 16) \div (x^2 - 2x + 4) = (x^2 + 4)$
Therefore, $(x^4 - 2x^3 + 8x^2 - 8x + 16) = (x^2 - 2x + 4)(x^2 + 4)$

(d) (i) (1 mark)
***Outcomes assessed: E4***
***Targeted Performance Bands: E2***

| Criteria | Mark |
|---|---|
| • Correct answer | 1 |

***Sample answer:***
$$1 + \omega + \omega^2 = \frac{-b}{a}$$
$$= 0$$





(d) (ii) (2 marks)
***Outcomes assessed: E2, E4***
***Targeted Performance Bands: E3***

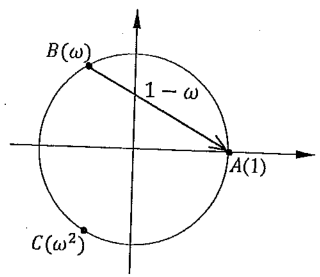
| Criteria | Marks |
|---|---|
| • Correct explanation | 2 |
| • Correct factorisation of $z^3 - 1$ | 1 |

***Sample answer:***
$z^3 - 1 = (z - 1)(z^2 + z + 1)$
Since $1, \omega$ and $\omega^2$ are roots of $z^3 - 1 = 0$, $z^3 - 1 = (z - 1)(z - \omega)(z - \omega^2)$
$\therefore (z - 1)(z^2 + z + 1) = (z - 1)(z - \omega)(z - \omega^2)$
Hence, $z^2 + z + 1 = (z - \omega)(z - \omega^2)$

(d) (iii) (1 mark)
***Outcomes assessed: E3***
***Targeted Performance Bands: E3***

| Criteria | Marks |
|---|---|
| • Correct vector | 1 |

***Sample answer:***




(d) (iv) (2 marks)
***Outcomes assessed: E2, E3***
***Targeted Performance Bands: E3 - E4***

| Criteria | Marks |
|---|---|
| • Correct argument and answer | 2 |
| • Recognition that the length of a vector is the modulus of a complex number | 1 |

***Sample answer:***
$$\text{Product of lengths of chords} = |1 - \omega| \times |1 - \omega^2|$$
$$= |(1 - \omega)(1 - \omega^2)|$$
$$= |1^2 + 1 + 1| \text{ (using part ii)}$$
$$= 3$$

**Question 13** (15 marks)
(a) (i) (2 marks)
***Outcomes assessed: E6***
***Targeted Performance Bands: E3***

| Criteria | Marks |
|---|---|
| • Correct solution. | 2 |
| • Demonstrates some knowledge of implicit differentiation | 1 |

***Sample answer:***

$$3x^2 + 3y^2 + 2xy = 24$$
$$6x + 6y.\frac{dy}{dx} + 2x.\frac{dy}{dx} + 2y = 0$$
$$\frac{dy}{dx}(6y + 2x) = -6x - 2y$$
$$\frac{dy}{dx} = -\left(\frac{y+3x}{x+3y}\right)$$

(a) (ii) (2 marks)
***Outcomes assessed: E6***
***Targeted Performance Bands: E2***

| Criteria | Marks |
|---|---|
| • Correct solution | 2 |
| • Some progress towards $x$ values or $y$ values | 1 |

***Sample answer:***

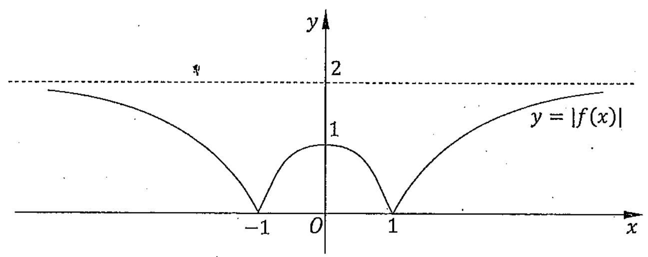
Stationary points occur when $\frac{dy}{dx} = 0$ i.e. $y = -3x$
Substitute into the equation.

$$3x^2 + 3(9x^2) + 2x(-3x) = 24$$
$$x^2 = 1$$
$$x = \pm 1$$

(b) (i) (1 mark)
***Outcomes assessed: E6***
***Targeted Performance Bands: E3***

| Criteria | Mark |
|---|---|
| • Correct graph | 1 |

***Sample answer:***



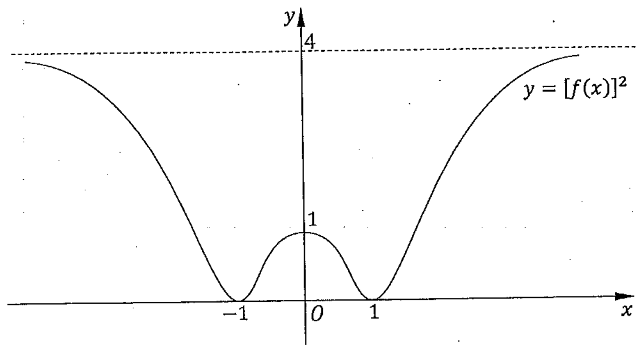





(b) (ii) (2 marks)
***Outcomes assessed: E6***
***Targeted Performance Bands: E3***

| Criteria | Marks |
|---|---|
| • Correct graph | 2 |
| • Significant progress towards correct graph | 1 |

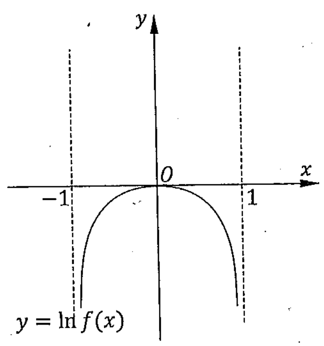
***Sample answer:***




(b) (iii) (2 marks)
***Outcomes assessed: E6***
***Targeted Performance Bands: E3 - E4***

| Criteria | Marks |
|---|---|
| • Correct graph | 2 |
| • Significant progress towards correct graph | 1 |

***Sample answer:***



(c) (3 marks)

***Outcomes assessed: E4***

***Targeted Performance Bands: E3***

| Criteria | Marks |
|---|---|
| • Correct solution | 3 |
| • Justifies that $x = -2$ is the zero of multiplicity 3 | 2 |
| • Displays knowledge that if $\alpha$ is a zero of multiplicity 3 then $P''(\alpha) = 0$ | 1 |

***Sample answer:***

$$P'(x) = 4x^3 + 9x^2 - 12x - 28$$
$$P''(x) = 12x^2 + 18x - 12$$
$$= 6(x + 2)(2x - 1)$$

$P''(x) = 0$ when $x = -2, \frac{1}{2}$

Now $P'\left(\frac{1}{2}\right) \neq 0$ and $P'(-2) = 0$, therefore, $x = -2$ is the zero of multiplicity 3.

$P(-2) = 0$. Hence $c = -24$.

(d) (i) (1 mark)

***Outcomes assessed: E2***

***Targeted Performance Bands: E2***

| Criteria | Mark |
|---|---|
| • Correct solution | 1 |

***Sample answer:***

$$f\left(\frac{1}{x}\right) = \frac{2 \times \frac{1}{x}}{\left(\frac{1}{x}\right)^2 + 1}$$

$$= \frac{\left(\frac{2}{x}\right)}{\left(\frac{1 + x^2}{x^2}\right)}$$

$$= \frac{2x}{1 + x^2}$$

$$= f(x)$$





(d) (ii) (2 marks)

***Outcomes assessed: E2, E9***

***Targeted Performance Bands: E3-E4***

| Criteria | Marks |
|---|---|
| • Correct answer | 2 |
| • Significant progress to correct answer | 1 |

***Sample answer:***

Let the coordinates of $C$ be $(x, 0)$.

$DA = CB = f(x)$ (since $ABCD$ is a square).

Therefore, using part (i), the coordinates of $D$ are $\left(\frac{1}{x}, 0\right)$

Given $ABCD$ is a square, $CD = CB = f(x)$

Therefore

$$x - \frac{1}{x} = \frac{2x}{x^2 + 1}$$

$$x^4 - 2x^2 - 1 = 0$$

$$x^2 = 1 + \sqrt{2} \text{ (since } x^2 > 0)$$

$$x = \sqrt{1 + \sqrt{2}} \text{ (since } x > 0)$$

$\therefore$ the coordinates of $C$ are $\left(\sqrt{1 + \sqrt{2}}, 0\right)$.

**Question14** (15 marks)
(a) (i) (2 marks)
***Outcomes assessed: E3***
***Targeted Performance Bands: E3***

| Criteria | Marks |
|---|---|
| • Shows the correct result | 2 |
| • Uses the correct focus/directrix definition of an ellipse | 1 |

***Sample answer***
Let $M$ and $M'$ be the feet of the perpendiculars from $P$ to the directrices $x = \frac{a}{e}$ and $x = -\frac{a}{e}$

$$\begin{aligned} PS &= e\,PM \\ &= e\left(\frac{a}{e} - a\cos\theta\right) \\ &= a\,(1 - e\cos\theta) \\ PS' &= e\,PM' \\ &= e\left(a\cos\theta + \frac{a}{e}\right) \\ &= a\,(1 + e\cos\theta) \end{aligned}$$

Therefore, $\frac{PS}{PS'} = \frac{1-e\cos\theta}{1+e\cos\theta}$

(a) (ii) (3 marks)
***Outcomes assessed: E4***
***Targeted Performance Bands: E3***

| Criteria | Marks |
|---|---|
| • Shows the correct result | 3 |
| • Significant progress to result | 2 |
| • Finds the coordinates of $G$ | 1 |

***Sample answer:*** The normal at $P$ meets the $x$-axis at $G$.

When $y = 0$, $x = \frac{(a^2-b^2)\cos\theta}{a}$

$= \frac{a^2e^2\cos\theta}{a}$ as $b^2 = a^2(1-e^2)$

$= ae^2\cos\theta$

Therefore $G$ has coordinates $(ae^2\cos\theta, 0)$.
$S$ has coordinates $(ae, 0)$, and $S'$ has coordinates $(-ae, 0)$

$$GS = ae - ae^2\cos\theta$$
$$GS' = ae + ae^2\cos\theta$$

$$\begin{aligned} \frac{GS}{GS'} &= \frac{ae-ae^2\cos\theta}{ae+ae^2\cos\theta} \\ &= \frac{1-e\cos\theta}{1+e\cos\theta} \\ &= \frac{PS}{PS'} \end{aligned}$$





(b) (i) (1 mark)
***Outcomes assessed: E7***
***Targeted Performance Bands: E3***

| Criteria | Mark |
|---|---|
| • Correctly shows the result | 1 |

***Sample answer:***

$\frac{r}{R} = \frac{GE}{GB}$ ($\Delta GDE$ is similar to $\Delta GAB$)

$\frac{GE}{GB} = \frac{h-y}{h}$ ($\Delta GEF$ is similar to $\Delta GBC$)

Therefore $\frac{r}{R} = \frac{h-y}{h}$

$r = \left(\frac{h-y}{h}\right).R$

(b) (ii) (3 marks)
***Outcomes assessed: E7***
***Targeted Performance Bands: E3***

| Criteria | Marks |
|---|---|
| • Shows the correct result | 3 |
| • Makes substantial progress towards required result | 2 |
| • Establishes a correct expression for the volume of a slice | 1 |

***Sample answer:*** Let the thickness of the slice be $\Delta y$.

Volume of Slice $= \pi\left(\frac{h-y}{h}\right)^2 R^2\Delta y$

$$\begin{aligned} \therefore \text{Total Volume} &= \lim_{\Delta y\to 0}\sum_{y=0}^{h}\pi\left(\frac{h-y}{h}\right)^2 R^2\Delta y \\ &= \pi\int_0^h\left(\frac{h-y}{h}\right)^2 R^2\,dy \\ &= \frac{-\pi R^2}{3h^2}\left[(h-y)^3\right]_0^h \\ &= \frac{1}{3}\pi R^2 h \end{aligned}$$

Therefore, the volume of the oblique cone is given by $\frac{1}{3}\pi R^2 h$ units$^3$.

(c) (i) (3 marks)
***Outcomes assessed: E2, E9***
***Targeted Performance Bands: E3-E4***

| Criteria | Marks |
|---|---|
| • Gives reason for concyclic points | 3 |
| • Proves the required pair of angles are equal | 2 |
| • Establishes 1 correct pair of equal angles leading to result | 1 |

***Sample answer:*** Let $\angle BDT = x°$

$\angle BCD = x°$ (angle between tangent and chord equals angle in alternate segment)

$\angle BRT = x°$ (corresponding angles, $PQ$ parallel to $CD$)

$\therefore \angle BDT = \angle BRT.$

Therefore, $B, T, D$ and $R$ are concyclic points as $\angle BRT$ and $\angle BDT$ are equal angles subtended by the interval $BT$.

(c) (ii) (2 marks)
***Outcomes assessed: E2, E9***
***Targeted Performance Bands: E2-E3***

| Criteria | Marks |
|---|---|
| • Correct proof | 2 |
| • Progress towards required result | 1 |

***Sample answer:***

$\angle TBD = \angle BCD = x°$ (angle between tangent and chord equals angle in alternate segment)

$\angle TBD = \angle DRT = x°$ (angle in the same segment of the circle passing through $B, T, D$ and $R$)

Therefore, $\angle BRT = \angle DRT = x°$

(c) (iii) (1 mark)
***Outcomes assessed: E2, E9***
***Targeted Performance Bands: E2 – E3***

| Criteria | Mark |
|---|---|
| • Correctly shows the result | 1 |

***Sample answer:*** $\angle CDR = \angle DRT = x°$ (alternate angles, $PQ$ parallel to $CD$)

$\angle DCR = \angle BCD = x°$ (from part (i))

Therefore, $\Delta RCD$ is an isosceles triangle (two angles equal).





**Question 15** (15 marks)
(a) (3 marks)
***Outcomes assessed: E5***
***Targeted Performance Bands: E3-E4***

| Criteria | Marks |
|---|---|
| • Shows the correct result | 3 |
| • Resolves forces correctly in TWO directions | 2 |
| • Resolves forces correctly in ONE direction | 1 |

***Sample answer:***
Resolving horizontally and vertically

$$F\cos\theta + N\sin\theta = \frac{mv^2}{r} \qquad \text{Eqn (i)}$$
$$N\cos\theta - mg - F\sin\theta = 0 \qquad \text{Eqn (ii)}$$

From Eqn (i): $F\cos\theta = \frac{mv^2}{r} - N\sin\theta$ Eqn (iii)

From Eqn (ii): $F\sin\theta = N\cos\theta - mg$ Eqn (iv)

Eqn (iii) × $\cos\theta$: $F\cos^2\theta = \frac{mv^2}{r}\cos\theta - N\sin\theta\cos\theta$

Eqn (iv) × $\sin\theta$: $F\sin^2\theta = N\cos\theta\sin\theta - mg\sin\theta$

$$F(\cos^2\theta + \sin^2\theta) = \frac{mv^2}{r}\cos\theta - mg\sin\theta$$
$$F = \frac{mv^2}{r}\cos\theta - mg\sin\theta$$

(b) (i) (3 marks)
***Outcomes assessed: E3, E4***
***Targeted Performance Bands: E3-E4***

| Criteria | Marks |
|---|---|
| • Shows the correct result | 3 |
| • Correctly equates TWO expressions | 2 |
| • Correctly expands binomial product | 1 |

***Sample answer:***

$(\cos\theta + i\sin\theta)^5$
$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

By De Moivre: $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

Equating real parts: $\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$



DISCLAIMER
The information contained in this document is intended for the professional assistance of teaching staff. It does not constitute advice to students. Further it is not the intention of CSSA to provide specific marking outcomes for all possible Trial HSC answers. Rather the purpose is to provide teachers with information so that they can better explore, understand and apply HSC marking requirements, as established by the NSW Board of Studies.
No guarantee or warranty is made or implied with respect to the application or use of CSSA Marking Guidelines in relation to any specific trial exam question or answer. The CSSA assumes no liability or responsibility for the accuracy, completeness or usefulness of any Marking Guidelines provided for the Trial HSC papers.

(b) (ii) (2 marks)
***Outcomes assessed: E3, E4***
***Targeted Performance Bands: E3-E4***

| Criteria | Marks |
|---|---|
| • Shows correctly the required roots | 2 |
| • Substantial progress towards the solution | 1 |

***Sample answer:***
Let $x = \cos\theta$: $16x^5 - 20x^3 + 5x = 0 \Longrightarrow 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$

$$\cos 5\theta = 0$$
$$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$$
$$\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$$
$$\therefore x = \cos\theta = \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{\pi}{2}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$$
$$= \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, 0, -\cos\frac{3\pi}{10}, -\cos\frac{\pi}{10}$$

Hence the roots of the equation are $x = 0, \pm\cos\frac{\pi}{10}, \pm\cos\frac{3\pi}{10}$.

(b) (iii) (2 marks)
***Outcomes assessed: E3, E4***
***Targeted Performance Bands: E3-E4***

| Criteria | Marks |
|---|---|
| • Shows correctly the required roots | 2 |
| • Substantial progress towards the solution | 1 |

***Sample answer:***

$$\sum\alpha\beta = \frac{-20}{16} = \frac{-5}{4}$$

$$\sum\alpha\beta = -\cos^2\frac{\pi}{10} + \cos\frac{\pi}{10}.\cos\frac{3\pi}{10} - \cos\frac{\pi}{10}.\cos\frac{3\pi}{10} - \cos\frac{\pi}{10}\cos\frac{3\pi}{10} + \cos\frac{\pi}{10}\cos\frac{3\pi}{10} - \cos^2\frac{3\pi}{10}$$
$$= -\left(\cos^2\frac{\pi}{10} + \cos^2\frac{3\pi}{10}\right)$$

Therefore $\left(\cos^2\frac{\pi}{10} + \cos^2\frac{3\pi}{10}\right) = \frac{5}{4}$





(c) (i) (1 mark)
***Outcomes assessed: E5***
***Targeted Performance Bands: E3***

| Criteria | Marks |
|---|---|
| • Shows the correct result | 1 |

***Sample answer:***
Resolving vertically: $T_1\cos\theta - mg - T_2\cos\theta = 0$
$$T_1\cos\theta - T_2\cos\theta = mg$$
$$\therefore (T_1 - T_2)\cos\theta = mg$$

(c) (ii) (4 marks)
***Outcomes assessed: E5***
***Targeted Performance Bands: E4***

| Criteria | Marks |
|---|---|
| • Correctly shows the required result | 4 |
| • Significant progress towards the result | 3 |
| • Attempts to solve simultaneously the correct equations | 2 |
| • Resolves forces horizontally | 1 |

***Sample answer:***
Resolving horizontally: $T_1\sin\theta + T_2\sin\theta = mr\omega^2$
$$(T_1 + T_2)\sin\theta = mr\omega^2$$

Using part (i) result:
$$\frac{(T_1+T_2)\sin\theta}{(T_1-T_2)\cos\theta} = \frac{mr\omega^2}{mg}$$

$r = h\tan\theta$ $\quad \frac{(T_1+T_2)\tan\theta}{(T_1-T_2)} = \frac{mh\tan\theta\omega^2}{mg}$
$$\therefore \frac{T_1+T_2}{T_1-T_2} = \frac{h\omega^2}{g}$$
$$= \frac{h\left(\frac{4g}{h}\right)}{g} \text{ since } \omega = 2\sqrt{\frac{g}{h}}$$
$$= 4$$

Hence, $\frac{T_1+T_2}{T_1-T_2} = 4$
$$T_1 + T_2 = 4T_1 - 4T_2$$
$$5T_2 = 3T_1$$
$$\frac{T_1}{T_2} = \frac{5}{3}$$
$$\therefore T_1 : T_2 = 5 : 3$$

**Question 16** (15 marks)
(a) (i) (1 mark)
***Outcomes assessed: E2, E9***
***Targeted Performance Bands: E3-E4***

| Criteria | Marks |
|---|---|
| • Correctly shows the result | 1 |

***Sample answer:***

$$\frac{Ae^x + Be^{-x}}{2} \geq \sqrt{Ae^x.Be^{-x}} \qquad \left(\text{using } \frac{x+y}{2} \geq \sqrt{xy}\right)$$

$$Ae^x + Be^{-x} \geq 2\sqrt{Ae^x.\frac{B}{e^x}}$$

$$Ae^x + Be^{-x} \geq 2\sqrt{AB}$$

$\therefore$ The minimum value of $f(x) = Ae^x + Be^{-x}$ is $2\sqrt{AB}$

(a) (ii) (2 marks)
***Outcomes assessed: E2, E9***
***Targeted Performance Bands: E4***

| Criteria | Marks |
|---|---|
| • Correct answers | 2 |
| • Significant progress towards the solution | 1 |

***Sample answer:***
If $f(x)$ is symmetrical about $x = c$, then $c$ must be the $x$-value corresponding to the minimum value of $f(x)$ which is $2\sqrt{AB}$ from part (i).

$\therefore$ The minimum value of $f(x)$ occurs when $x = c$ and is $2\sqrt{AB}$

$$f(c) = 2\sqrt{AB}$$

$$k\left[e^{c-c} + e^{-(c-c)}\right] = 2\sqrt{AB}$$

$$2k = 2\sqrt{AB}$$

$$\therefore\ k = \sqrt{AB}$$

Equality exists when $Ae^x = Be^{-x}$

$$\therefore e^{2x} = \frac{B}{A} \implies x = \frac{1}{2}\log_e\left(\frac{B}{A}\right)$$

$$\therefore c = \frac{1}{2}\log_e\left(\frac{B}{A}\right)$$

$$= \log_e\sqrt{\frac{B}{A}}$$





(b) (i) (1 Mark)
***Outcomes assessed: E2***
***Targeted Performance Bands: E2***

| Criteria | Mark |
|---|---|
| • Correctly shows the result | 1 |

***Sample answer:***

$$\text{LHS} = \frac{x^{2n-1} - x^{2n+1}}{\sqrt{1-x^2}}$$

$$= \frac{x^{2n-1}(1-x^2)}{\sqrt{1-x^2}}$$

$$= x^{2n-1}\sqrt{1-x^2}$$

$$= \text{RHS}$$

(b) (ii) (3 Marks)
***Outcomes assessed: E8***
***Targeted Performance Bands: E4***

| Criteria | Marks |
|---|---|
| • Correctly shows the result | 3 |
| • Significant progress towards the result | 2 |
| • Attempts to apply integration by parts or attempts to use the results from part (i) | 1 |

***Sample Answer:***

$$I_{2n+1} = \int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}dx$$

$$= \int_0^1 x^{2n} \times x(1-x^2)^{-\frac{1}{2}}dx$$

$$= \left[-x^{2n}\sqrt{1-x^2}\right]_0^1 + 2n\int_0^1 x^{2n-1}\sqrt{1-x^2}dx$$

$$= 2n\int_0^1\left(\frac{x^{2n-1}}{\sqrt{1-x^2}} - \frac{x^{2n+1}}{\sqrt{1-x^2}}\right)dx$$

$$= 2n[I_{2n-1} - I_{2n+1}]$$

$$I_{2n+1}(1+2n) = 2n\, I_{2n-1}$$

$$\therefore I_{2n+1} = \frac{2n}{2n+1}I_{2n-1}$$

(b) (iii) (2 Marks)

***Outcomes assessed: E2***

***Targeted Performance Bands: E3-E4***

| Criteria | Marks |
|---|---|
| • Correct solution | 2 |
| • Significant progress towards expression for $I_{2n+1}$. | 1 |

***Sample answer:***

$$I_{2n+1} = \left(\frac{2n}{2n+1}\right) I_{2n-1} \quad \text{from (ii)}$$

$$= \left(\frac{2n}{2n+1}\right)\left(\frac{2n-2}{2n-1}\right) I_{2n-3}$$

$$= \left(\frac{2n}{2n+1}\right)\left(\frac{2n-2}{2n-1}\right) \dots \frac{2}{3} . I_1$$

$$I_1 = \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$$

$$= \left[-\sqrt{1-x^2}\right]_0^1$$

$$= 1$$

$$\therefore I_{2n+1} = \left(\frac{2n}{2n+1}\right)\left(\frac{2n-2}{2n-1}\right) \dots \frac{2}{3}$$

$$= \frac{2^n n!}{1 \times 3 \times 5 \times \dots \times (2n-1) \times (2n+1)}$$





(b) (iv) (2 Marks)

***Outcomes assessed: E2, E9***

***Targeted Performance Bands: E4***

| Criteria | Marks |
|---|---|
| • Correctly shows the result | 2 |
| • Significant progress towards result | 1 |

***Sample answer:***

$$\int_0^1 \frac{x}{\sqrt{1-x^2}} dx + \int_0^1 \left[\sum_{n=1}^{\infty}\left(C_n \frac{x^{2n+1}}{\sqrt{1-x^2}}\right)\right] dx = 1 + C_1 I_3 + C_2 I_5 + \dots + C_n I_{2n+1} + \dots$$

$$\text{Now } C_n I_{2n+1} = \frac{1 \times 3 \times 5 \times \dots \times (2n-1)}{(2n+1)2^n n!} \times \frac{2^n n!}{1 \times 3 \times 5 \times \dots \times (2n+1)}$$

$$= \frac{1}{(2n+1)^2}$$

$$\therefore \text{LHS} = 1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

(b) (v) (2 Marks)
***Outcomes assessed: E2, E9*** (2 Marks)
***Targeted Performance Bands: E4***

| Criteria | Marks |
|---|---|
| • Correct answer to the limiting sum | 2 |
| • Progress towards correct result | 1 |

***Sample answer:***

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} C_n x^{2n+1}$$

$$\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^1 \frac{x + \sum_{n=1}^{\infty} C_n x^{2n+1}}{\sqrt{1-x^2}} dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

$$= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$

Now $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \left[\frac{1}{2}(\sin^{-1} x)^2\right]_0^1$

$$= \frac{\pi^2}{8}$$

$$\therefore 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$$





(b) (vi) (2 Marks)
***Outcomes assessed: E2, E9*** (2 Marks)
***Targeted Performance Bands: E4***

| Criteria | Marks |
|---|---|
| • Correct answer | 2 |
| • Progress towards correct result | 1 |

***Sample answer:***

Let $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots$$

$$= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

$$\therefore S = \frac{\pi^2}{8} + \frac{1}{4} S$$

$$\frac{3}{4} S = \frac{\pi^2}{8}$$

$$\therefore S = \frac{\pi^2}{6}$$